BALTIC BATTLE
SURVIVAL KIT

A Toolbox of Tricks and Techniques collected by
His Excellency Princess Qimh Xantcha

Fourth Edition 2012

Contents

1 Algebra

Algebra, the art of symbolic reasoning, is at the heart of mathematics. While the working mathematician can get along pretty well without Number Theory, only know the most elementary Combinatorics, and be completely lost at Geometry, there is not much he can accomplish without invoking Algebra.

1.1 The Art of Symbolism

> I've seen the Professor work lots of sums on the blackboard, and he claimed anything could be done with x's and y's and a's, and such things, by mixing them up with plenty of plusses and minuses and equals, and so forth. But he never said anything, so far as I can remember, about counting up to the odd number of seventeen by the even numbers of twos.
>
> – Baum, *The Marvelous Land of Oz*

Some problems are just about being clever, and playing around with symbols. . .

1. *Baltic Way 1990.* Let $*$ denote an operation, assigning a real number $a * b$ to each pair of real numbers (a, b) (e.g., $a * b = a + b^2 - 17$). Devise an equation which is true (for all possible values of variables) provided the operation $*$ is commutative or associative and which can be false otherwise.

2. *Baltic Way 1992.* Given that $a^2 + b^2 + (a + b)^2 = c^2 + d^2 + (c + d)^2$, prove that $a^4 + b^4 + (a + b)^4 = c^4 + d^4 + (c + d)^4$.

1.2 Helvetet i en Algebraisk Eqvation

> [...] je donnerais bien cent sous au mathématicien qui me démontrerait par une équation algébrique l'existence de l'enfer.
>
> – Honoré de Balzac, *La Peau de chagrin*

Ah! Nothing like solving a classical equation. At least there's a problem type we all recognise from school. But while the set-up may be familiar, the equations encountered in competitions are usually a far cry from those found in school textbooks.

1. *Skolornas Matematiktävling 1995.* Bestäm alla reella lösningar till ekvationen
$$\left(\sqrt{2+\sqrt{3}}\right)^x + \left(\sqrt{2-\sqrt{3}}\right)^x = 2^x.$$
[*Rewrite the equation in the simpler form $a^x + b^x = 1$. This equation has one obvious solution and no further — why not?*]

2. *Baltic Way 1994.* Let $a \circ b = a + b - ab$. Find all triples (x, y, z) of integers such that $(x \circ y) \circ z + (y \circ z) \circ x + (z \circ x) \circ y = 0$.

3. *Baltic Way 1993.* Solve the system of equations in integers:
$$\begin{cases} z^x = y^{2x} \\ 2^z = 4^x \\ x + y + z = 20. \end{cases}$$
[*The problem is curiously phrased. Why not put $z = y^2$ for the first equation? Why restrict attention to integers? — This is not number theory, and the assumption does not simplify the problem.*]

4. *Baltic Way 1992.* Find all integers satisfying the equation $2^x \cdot (4 - x) = 2x + 4$. [*Apparently* $\frac{2x+4}{4-x} > 0 \ldots$]

5. *Baltic Way 1995 (classified as Number Theory).* Find all triples (x, y, z) of positive integers satisfying the system of equations
$$\begin{cases} x^2 = 2(y + z) \\ x^6 = y^6 + z^6 + 31(y^2 + z^2). \end{cases}$$
[*(1) It is clear that $x > y, z$ (why?). (2) What can you do with that information?*]

1.3 Fearful Symmetry

Tyger Tyger, burning bright,
In the forests of the night:
What immortal hand or eye,
Could frame thy fearful symmetry?

— Blake, *The Tyger*

Systems of equations are like single ones, only more. Hence, the methods of the preceding section apply.

However, if a system of equations is *symmetrical* — it consists of n more or less identical equations — there is an obvious trick: Add all of them together (or perhaps all but one of them).

1. *Baltic Way 1993.* Solve the system of equations:

$$\begin{cases} x^5 = y + y^5 \\ y^5 = z + z^5 \\ z^5 = t + t^5 \\ t^5 = x + x^5. \end{cases}$$

2. *Baltic Way 1995.* The real numbers a, b and c satisfy the inequalities $|a| \geqslant |b+c|$, $|b| \geqslant |c+a|$ and $|c| \geqslant |a+b|$. Prove that $a+b+c=0$. [*Absolute values are absolutely horrendous and should never have been invented, according to the venerable Dr Hans-Uno Bengtsson. But one can get rid of them — how?*]

1.4 The Call of Clhculhus

Clhculhus fhtagn

Calculus assumes an interesting position in the math competition pantheon. On an international level, it is universally banned and abhorred; possibly because the school curricula of other countries concentrate on algebra, and not calculus, as do the Nordic countries. On the other hand, it sometimes comes in handy, and here Scandinavians are for once at an advantage!

It may be remarked that the early days of the Baltic Way Contest actually featured some pure calculus problems, though this seems to have fallen out of fashion in later years. Now the four branches algebra, number theory, combinatorics and geometry are firmly rooted, and any attempt to include calculus is seen as a sacrilege. We have known Germany in particular to be quite militant about this.

For one thing, note the following fact: If a is a multiple root of the polynomial equation $p(x) = 0$, then a is also a root of the equation $p'(x) = 0$. The converse also holds: If a is a single root of $p(x) = 0$, then $p'(a) \neq 0$.

1. *Skolornas Matematiktävling 1988.* Låt $P(x)$ vara ett tredjegradspolynom med exakt tre olika reella nollställen. Hur många reella rötter

har ekvationen

$$(P'(x))^2 - 2P(x)P''(x) = 0?$$

2. *Baltic Way 1991.* Let a, b, c, d, e be distinct real numbers. Prove that the equation

$$\begin{aligned}&(x-a)(x-b)(x-c)(x-d)\\&+(x-a)(x-b)(x-c)(x-e)\\&+(x-a)(x-b)(x-d)(x-e)\\&+(x-a)(x-c)(x-d)(x-e)\\&+(x-b)(x-c)(x-d)(x-e)=0\end{aligned}$$

has 4 distinct real solutions.

3. *Baltic Way 1992.* A polynomial $f(x) = x^3 + ax^2 + bx + c$ is such that $b < 0$ and $ab = 9c$. Prove that the polynomial has three different real roots.

1.5 Evil under the sin...

Why do you so seldom see mathematicians at the beach?
– Because they use the sin, and not the sun, to get their tan.

Trigonometry is always a trustworthy old friend, and not only in geometry. Some impossibly difficult algebra problems rely on encoding some innocent-looking equation in trigonometric formulæ.

1. *Skolornas Matematiktävling 1994.* Cirkeln C har medelpunkt i origo och radien 1. Kan det finnas två punkter på C, på avstånd 1 ifrån varandra, vilkas koordinater är rationella?

2. *Baltic Way 1995.* Prove that $\sin^3 18^\circ + \sin^2 18^\circ = 1/8$. [*This is a warm-up exercise, really. Use a full frontal attack with every trigonometric formula you could think of!*]

3. *Baltic Way 1990.* Let $a_0 > 0$, $c > 0$ and

$$a_{n+1} = \frac{a_n + c}{1 - a_n c}, \qquad n = 0, 1, \ldots.$$

Is it possible that the first 1990 terms $a_0, a_1, \ldots, a_{1989}$ are all positive but $a_{1990} < 0$? [*Here's the one I prepared for earlier. The recursion formula bears a haunting familiarity to the addition formula for the tangent:* $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$.]

4. *Baltic Way 1994.* Find the largest value of the expression

$$xy + x\sqrt{1-y^2} + y\sqrt{1-x^2} - \sqrt{(1-x^2)(1-y^2)}.$$

[*x and $\sqrt{1-x^2}$ together definitely look suspicious, and* always *call for a substitution $x = \sin\alpha$ and $\sqrt{1-x^2} = \cos\alpha$.*]

5. *Nordic Mathematical Contest 1989.* Låt S vara mängden av alla tal t i det slutna intervallet $[-1, 1]$ sådana att talföljden $x_0, x_1, x_2, \ldots$, definierad genom $x_{n+1} = 2x_n^2 - 1$, $x_0 = t$, har egenskapen att det finns ett positivt heltal N sådant att $x_n = 1$ för alla $n \geqslant N$. Visa att S har oändligt många element.

1.6 Softcore Inequalities

Liberté, inegalité, fraternité...

The following problems are about estimating quantities.

1. *Baltic Way 1990.* Integers $1, 2, \ldots, n$ are written (in some order) on the circumference of a circle. What is the smallest possible sum of moduli of the differences of neighbouring numbers? [*You know the Triangle Inequality, don't you: $|a| + |b| \leqslant |a+b|$?*]

2. *Baltic Way 1991.* Let A, B, C be the angles of an acute-angled triangle. Prove the inequality

$$\sin A + \sin B > \cos A + \cos B + \cos C.$$

3. *Baltic Way 1992.* Let $a = \sqrt[1992]{1992}$. Which number is greater:

$$\left.a^{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}\right\}1992$$

or 1992? [*Try first replacing 1992 by a smaller number, like 2, to see what* ought *to be the greater number. Then prove your assertion.*]

1.7 Hardcore Inequalities

Liberté, inegalité, fraternité — ou la Mort!

Inequalities are stapleware in mathematics competitions. They have come to be established as one of only two really standardised problem types; the other being functional equations.

There are literally hundreds and hundreds of known inequalities out there for the budding student to memorise and master. It is possible that for each of these, there exists a competition problem that it actually solves, but we would not swear to it. For the Baltic Way Contest, two[1] inequalities will crack most problems, once applied correctly.

The first of these is AM–GM — the inequality of *arithmetic* and *geometric* mean. In its simplest form, it states

$$\frac{a+b}{2} \geqslant \sqrt{ab},$$

for positive numbers a and b, with equality if and only if $a = b$. (Proof: Square both sides.) There are several generalisations of it, though. Firstly, we may consider any number n of variables $a_1, \ldots, a_n$, again assumed positive:

$$\frac{a_1 + \cdots + a_n}{n} \geqslant \sqrt[n]{a_1 \ldots a_n}$$

with equality if and only if $a_1 = \cdots = a_n$.

Secondly, these are, by no means, the only means. Given numbers $a_1, \ldots, a_n$, we may also form their *harmonic* mean

$$\frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}},$$

and their *quadratic* mean

$$\sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}}.$$

We then have the glorious chain of inequalities known as QM–AM–GM–HM:

$$\sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}} \geqslant \frac{a_1 + \cdots + a_n}{n} \geqslant \sqrt[n]{a_1 \ldots a_n} \geqslant \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}},$$

with equality if and only if bla, bla, bla.

The second inequality the student should be desperate to learn, is the *Cauchy–Schwarz Inequality*. It states that, given two sequences $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ of positive numbers, we have

$$(a_1 b_1 + \cdots + a_n b_n)^2 \leqslant \left(a_1^2 + \cdots + a_n^2\right)\left(b_1^2 + \cdots + b_n^2\right)$$

In this case, equality holds if and only if the sequences are *proportional* to each other (this includes the case when one or both of them are identically zero).

[1] A third inequality will also fit into the Baltic Way canon, namely *Jensen's Inequality*. The reader is encouraged to look it up.

1. *Baltic Way 1994 (classified as Geometry).* Let α, β, γ be the angles of a triangle opposite to its sides with lengths a, b and c, respectively. Prove the inequality

$$a \cdot \left(\frac{1}{\beta} + \frac{1}{\gamma}\right) + b \cdot \left(\frac{1}{\gamma} + \frac{1}{\alpha}\right) + c \cdot \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) \geqslant 2 \cdot \left(\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma}\right).$$

[*Split it up into parts; prove the inequality*

$$\frac{a}{\beta} + \frac{b}{\alpha} \geqslant \frac{a}{\alpha} + \frac{b}{\beta}.$$

No advanced inequalities are required here.]

2. *Baltic Way 1991.* For any positive numbers a, b, c prove the inequalities

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geqslant \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \geqslant \frac{9}{a+b+c}.$$

3. *Baltic Way 1992.* Prove that for any positive x_1, x_2, $\ldots$, x_n and y_1, y_2, $\ldots$, y_n the inequality

$$\sum_{i=1}^{n} \frac{1}{x_i y_i} \geqslant \frac{4n^2}{\sum_{i=1}^{n}(x_i + y_i)^2}$$

holds.

4. *Baltic Way 1995.* Prove that for positive a, b, c, d

$$\frac{a+c}{a+b} + \frac{b+d}{b+c} + \frac{c+a}{c+d} + \frac{d+b}{d+a} \geqslant 4.$$

[*Split into two parts; first prove*

$$\frac{a+c}{a+b} + \frac{c+a}{c+d}$$

*is greater than something (*not 2*).*]

5. *Baltic Way 1996.* For which positive real numbers a, b does the inequality

$$x_1 \cdot x_2 + x_2 \cdot x_3 + \cdots + x_{n-1} \cdot x_n + x_n \cdot x_1 \geqslant x_1^a \cdot x_2^b \cdot x_3^a + x_2^a \cdot x_3^b \cdot x_4^a + \cdots + x_n^a \cdot x_1^b \cdot x_2^a$$

hold for all integers $n > 2$ and positive real numbers $x_1, x_2, \ldots, x_n$? [*First substitute "nice" values for the x_k to establish what values a and b must have. Then prove the resulting inequality.*]

6. *Nordic Mathematical Contest 1987.* Låt a, b, c vara positiva reella tal. Bevisa att

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \leqslant \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}.$$

Equation	Solution
$f(x) + f(y) = f(x + y)$	$f(z) = az$
$f(x)f(y) = f(x + y)$	$f(z) = e^{az}$
$f(x) + f(y) = f(xy)$	$f(z) = a \ln z$
$f(x)f(y) = f(xy)$	$f(z) = z^a$

Table 1: Functional Equations and their solutions $f\colon \mathbf{R} \to \mathbf{R}$, assuming continuity or monotony.

1.8 Functional Food

att vara en funktion av allt som inte fungerar
att vara något annat eller inte vara alls

– Erik Lindgren, *mannen utan väg XXVIII*

Functional equations are staple foods at the olympiads, being the only truly standardised problem type, save for inequalities. Some noteworthy points:

I. Standard method: Extract information by trying special values for the variables: $x = 0$, $y = 1$, $x = y$, $y = \frac{1}{x}$, and so on.

II. Frequently, it is possible to determine the function f on all *rational* numbers. If the function f is assumed *continuous*, the matter will be settled, for two continuous functions that are equal on all rational numbers must be equal everywhere. This fortuitous assumption is not always provided, however, and an extension to all real numbers must then be secured through more ingenious means.

III. The classical equations occurring in Table 1 have only the unique solutions indicated, assuming that f be either *continuous* or *monotonous*.

1. *Baltic Way 1996.* Consider the functions f defined on the set of integers such that
$$f(x) = f(x^2 + x + 1),$$
for all integers x. Find
   (a) all even functions,
   (b) all odd functions of this kind.

2. *Baltic Way 1997.* Determine all functions f from the real numbers to the real numbers, different from the zero function, such that

$f(x)f(y) = f(x-y)$ for all real numbers x and y. [*Try* $y = 0,\ x,\ \frac{x}{2}$, *in turn.*]

3. *Baltic Way 1995.* Find all real-valued functions f defined on the set of all non-zero real numbers such that:

   (i) $f(1) = 1$,

   (ii) $f\left(\dfrac{1}{x+y}\right) = f\left(\dfrac{1}{x}\right) + f\left(\dfrac{1}{y}\right)$ for all non-zero x, y, $x+y$,

   (iii) $(x+y)f(x+y) = xyf(x)f(y)$ for all non-zero x, y, $x+y$.

   [*Try* $x = y$.]

4. *Baltic Way 1998.* Let $\mathbf{R}$ be the set of all real numbers. Find all functions $f\colon \mathbf{R} \to \mathbf{R}$ satisfying for all $x, y \in \mathbf{R}$ the equation $f(x) + f(y) = f(f(x)f(y))$. [*Play around with an* arbitrary $f(a) = b$.]

2 Number Theory

Number Theory is the only discipline of Olympiad Mathematics that actually constitutes an active area of research. Alas, there is no ongoing research in Geometry as Euclid knew it, and the problems that the working combinatorist or algebraist devotes his life to, are usually quite unlike the recreational puzzles here appearing in the Algebra and Combinatorics sections. On the other hand, Fermat's Great Theorem

$$x^n + y^n \neq z^n, \quad n \geqslant 3,$$

looks innocent enough and certainly indistinguishable from genuine competition problems (but those are usually required to be soluble in somewhat less than 400 years).

Number Theory thus forms a branch of serious mathematics, and all of its theorems may be invoked when solving a particular problem. This includes Fermat's Great Theorem, by the way, though we know not of a single instance where this would have provided or simplified a solution! We do know of a singular problem where the Twin Prime Conjecture played a part — see below.

But while certain classical results occasionally come to the rescue — see below for Fermat's Little Theorem — number-theoretical problems often suffer from a striking uniformity in their methods of attack. Hence the rather few sections of this chapter.

2.1 Be More Than Just a Number

Primtalen

Euklides har visat med full evidens
att primtalens antal är utan gräns,
o jubel, o glädjekälla!
Nu kan skolbarnet sitta förnöjt i sin sal
och tänka sig större och större tal
som alla är originella.
Men detta är icke vad skolbarnen gör;
de tycks icke ha nödig förståelse för
vad det har med dem att beställa.

– Alf Henriksson

Lots of fun can be had just playing with numbers...

1. *Baltic Way 1992.* Let p and q be two consecutive odd prime numbers. Prove that $p + q$ is a product of at least three positive integers greater than 1 (not necessarily different). [*Just a Quicky.*]

2. *Baltic Way 1993.* Let's call a positive integer "interesting" if it is a product of two (distinct or equal) prime numbers. What is the greatest number of consecutive positive integers all of which are "interesting"?

3. *Baltic Way 1994.* Let $p > 2$ be a prime number and $1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{(p-1)^3} = \frac{m}{n}$ where m and n are relatively prime. Show that m is a multiple of p. [*Use symmetry. The sum cannot be calculated, but try adding the first and the last term.*]

4. *Baltic Way 1995.* The positive integers a, b, c are pairwise relatively prime, a and c are odd and the numbers satisfy the equation $a^2 + b^2 = c^2$. Prove that $b + c$ is a square of an integer.

5. *Baltic Way 1996.* Let a and k be positive integers such that $a^2 + k$ divides $(a - 1)a(a + 1)$. Prove that $k \geqslant a$.

6. *Baltic Way 1996.* Let a, b, c d be positive integers such that $ab = cd$. Prove that $a + b + c + d$ is not prime. [*This is a quicky, though it may still be tricky. One solution may seem counter-intuitive. Multiply the given number by a, and see what happens. The more standard technique is to factor out greatest common divisors: Write $a = a'r$ and $c = c'r$, where now $\gcd(a', c') = 1$, and similarly for the pair b and d.*]

7. *Baltic Way 1999.* Does there exist a finite sequence of integers c_1, c_2, ..., c_n such that all the numbers $a + c_1$, $a + c_2$, ..., $a + c_n$ are primes for more than one but not infinitely many different integers a? [*The Twin Prime Conjecture will help you here.*]

8. *Nordic Mathematical Contest 1990.* Låt m, n och p vara positiva udda heltal. Visa att talet
$$\sum_{k=1}^{(n-1)^p} k^m$$
är delbart med n.

2.2 Divide and Conquer

> Oss emellan anser jag det för ett pedanteri, ty hvartill tjenar att säga 1/3 R:dr? kan man icke så gerna säga 16 sk., så slipper man både nämnare och täljare. Att addera 1/2 R:dr till 3/4 R:dr, som låter så konstigt, är rätt och slätt bara att lägga ihop 24 sk. med 36 sk., det gör:
>
> $$\begin{array}{r} 24 \\ 36 \\ \hline 60 \end{array}$$
>
> Således 60 skilling, som man bringar till riksdaler, genom att dividera med 48; blir 1 R:dr 12 sk. jemt, och så behöfs ingen bråkräkning i hushållslifvet, utan endast hela tal. Men yttra för Guds skull icke detta för herrarne på Gräseholm! De skulle blott göra narr af dig, ty karlar lägga alltid vigt på vetenskaper, som bringa fruntimmer i bryderi. Jag nekar också icke, att bråkkalkyler kunna vara behagliga och bra i Astronomi och Artilleri och mera, men icke för oss. Det slår icke felt, att Constance kan lefva och blifva gift sådant förutan. Att göra till lika nämnare! ha ha ha! och maximus dimunis covisor!
>
> — Almqvist, *Amalia Hillner*

The key to solving number-theoretical problems is divisibility. The argument goes thus:

From the equation, we see that a must be even; hence we can write $a = 2a'$. A factor 2 can be divided away, and now we see that b must be odd. Therefore, we may write $b = 2b' + 1$, and substitute back. We expand and simplify, and — oh dear! — c must be divisible by 7! Write $c = 7c'$, ...

And so on. Surpringly many problems yield to this naïve method of attack.

1. *Skolornas Matematiktävling 1995.* Är det möjligt att något av heltalen x och y är delbart med 3 om
$$x^2 - y^2 = 1995?$$

2. *Baltic Way 1990.* Prove that none of the numbers

$$F_n = 2^{2^n} + 1, \qquad n = 0, 1, \ldots,$$

is a cube of an integer.

3. *Baltic Way 1993.* Prove that for any odd positive integer n, $n^{12} - n^8 - n^4 + 1$ is divisible by 2^9.

4. *Baltic Way 1994.* Find all pairs of positive integers (a, b) such that $2^a + 3^b$ is the square of an integer.

2.3 Modular Arithmetic

Faut-il qu'avec les soins qu'on prend incessamment
On ne te puisse apprendre à parler congrûment?

– Molière, *Les Femmes sçavantes*

The method of the preceding section can be refined, and turned into *modular arithmetic*, with which we assume the reader is familiar. Occasionally, *Fermat's Little Theorem*[2] is useful:

$$a^{p-1} \equiv 1 \mod p,$$

where p is a prime not dividing a. As a consequence,

$$a^p \equiv a \mod p$$

for any prime p and any a.

1. *Skolornas Matematiktävling 1992.* Är $(19^{92} - 91^{29})/90$ ett naturligt tal?

2. *Baltic Way 1991.* Prove that there are no positive integers n and $m > 1$ such that $102^{1991} + 103^{1991} = n^m$. [*The formula* $x^p + y^p = (x + y)(x^{p-1} - x^{p-2}y + \cdots + y^{p-1}$, *for odd* p, *will prove useful.*]

3. *Baltic Way 1996.* A sequence of integers $a_1, a_2, \ldots,$ is such that $a_1 = 1$, $a_2 = 2$ and for $n \geqslant 1$

$$a_{n+2} = \begin{cases} 5a_{n+1} - 3a_n & \text{if } a_n \cdot a_{n+1} \text{ is even,} \\ a_{n+1} - a_n & \text{if } a_n \cdot a_{n+1} \text{ is odd.} \end{cases}$$

Prove that $a_n \neq 0$. for all n.

[2] The reader may also want to learn its generalisation to composite numbers p, called *Euler's Theorem*.

2.4 “Maximus Dimunis Covisor”

> Oscar hade den godheten, att under sin geometri med lilla Constance tillika bese några exempel, som jag uträknat på försök. Han smålog här och der: på ett ställe hade det varit mig komplett ogörligt att bringa bråken till lika nämnare, ty det är af hela kalkylen det bedröfligaste. Han for öfver mina siffror med en lätt hand, nändes icke stryka ut dem, men visade mig på en särskild papperslapp huru man rätt skall begagna sin “Minimus communis Dividuus”, och huru man med dess tillhjelp får lika nämnare som en dans. Ett ljus uppgick för min själ! Jag kan icke beskrifva huru lycklig jag kände mig: det gifs icke många fruntimmer i Sverige, som nu kunna hvad jag kan.
>
> — Almqvist, *Amalia Hillner*

Given two positive integers a and b, their greatest common divisor $\textsc{gcd}(a,b)$ is the greatest number that divides both a and b. Also, the least common multiple $\textsc{lcm}(a,b)$ is the least number that both a and b divide. We then have the theorem

$$\textsc{gcd}(a,b)\textsc{lcm}(a,b) = ab.$$

(Proof: Consider the prime factors of a and b.)

Frequently useful is *Bézout's Identity*, which states that, given two relatively prime integers a and b, it is possible to find integers s and t so that

$$sa + tb = 1.$$

More generally, it is always possible to find integers s and t so that $sa + tb = \textsc{gcd}(a,b)$.

1. *Baltic Way 1990.* Let m and n be positive integers. Prove that $25m + 3n$ is divisible by 83 if and only if $3m + 7n$ is divisible by 83. [*Try finding a suitable linear combination of these numbers.*]

2. *Baltic Way 1996.* Consider the sequence

$$\begin{aligned} x_1 &= 19, \\ x_2 &= 95, \\ x_{n+2} &= \operatorname{lcm}(x_{n+1}, x_n) + x_n, \end{aligned}$$

   for $n > 1$, where $\operatorname{lcm}(a,b)$ means the least common multiple of a and b. Find the greatest common divisor of x_{1995} and x_{1996}.

2.5 Neo-Constructivism

> So Pooh rose and sat down and said "Thank you," which is the proper thing to say when you have been made a Knight, and he went into a dream again, in which he and Sir Pump and Sir Brazil and Factors lived together with a horse, and were faithful knights (all except Factors, who looked after the horse) to Good King Christopher Robin...
>
> — Milne, *The House at Pooh Corner*

A common problem type is to prove there are infinitely many numbers having a certain property. There are two ways to go about this. One is to provide an explicit construction — to actually exhibit a collection of infinitely many numbers with the desired property. The other is to shew how to, given one number possessing the property, construct another. It must then be verified that this process will actually yield infinitely many different numbers — and also, of course, that a *first* number with the property can be constructed!

1. *Baltic Way 1990.* Prove that the equation $x^2 - 7y^2 = 1$ has infinitely many solutions in natural numbers. [*Try the second method suggested above.*]

2. *Baltic Way 1990.* Do there exist 1990 relatively prime numbers such that all possible sums of two or more of these numbers are composite numbers? [*To guarantee that something is composite, try numbers of the form $k + N!$, where $k < N$.*]

3. *Baltic Way 1992.* Denote by $d(n)$ the number of all positive divisors of a positive integer n (including 1 and n). Prove that there are infinitely many n such that $\frac{n}{d(n)}$ is an integer. [*Try an especially simple form for n.*]

4. *Baltic Way 1994.* Prove that for any integer $a \geqslant 5$ there exist integers b and c, $c \geqslant b \geqslant a$, such that a, b, c are the lengths of the sides of a right-angled triangle.

5. *Baltic Way 1996.* Let n and k be integers, $1 < k \leqslant n$. Find an integer b and a set A of n integers satisfying the following conditions:

   (i) No product of $k - 1$ distinct elements of A is divisible by b.

   (ii) Every product of k distinct elements of A is divisible by b.

   (iii) For all distinct a, a' in A, a does not divide a'.

3 Combinatorics

Combinatorics is the art of counting. Here we find the most fanciful problem formulations — sometimes embroidered into little novellas that even the non-mathematician can (sometimes) understand. The solutions are usually very elementary, but clever.

Nordic kids tend to be very good at Combinatorics. They may not know as many advanced inequalities as their Russian fellows, but they are no less clever. As a consequence, Scandinavian ambassadors try to opt for difficult Combinatorics problems (and easy problems of any other kind).

3.1 Elementary Principles of Counting

> "Hallo, Rabbit. Fourteen, wasn't it?"
>
> "What was?"
>
> "My pots of honey what I was counting."
>
> "Fourteen, that's right."
>
> "Are you sure?"
>
> "No," said Rabbit. "Does it matter?"
>
> "I just like to know," said Pooh humbly. "So as I can say to myself: 'I've got fourteen pots of honey left.' Or fifteen, as the case may be. It's sort of comforting."
>
> "Well, let's call it sixteen," said Rabbit.
>
> — Milne, *The House at Pooh Corner*

First some problems requiring just some ingenuity. . .

1. *Baltic Way 1991.* All positive integers from 1 to 1 000 000 are divided into two groups consisting of numbers with odd or even sums of digits respectively. Which group contains more numbers?

2. *Baltic Way 1993.* On each face of two dice some positive integer is written. The two dice are thrown and the numbers on the top faces are added. Determine whether one can select the integers on the faces so that the possible sums are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, all equally likely? (Each sum must occur in exactly three of the possible thirty-six outcomes. How could the sum 2 arise?)

3. *Baltic Way 1995.* In how many ways can the set of integers $\{1, 2, \ldots, 1995\}$ be partitioned into three nonempty sets so that none of these sets contains two consecutive integers? (Build it from the bottom up!)

3.2 The Odd One Out

J'inventai la couleur des voyelles! – *A* noir, *E* blanc, *I* rouge, *O* bleu, *U* vert.

– Rimbaud, *Une Saison en Enfer*

The classical Combinatorics argument goes by *colouring*. We want to prove the non-existence of a certain construction. To do this, we colour things black and white, and then count the objects of each colour. Or we might label the objects ' ± 1, and then add things together. Either way, we get an equation with no roots. Therefore, the construction is impossible.

A famous example asks us to prove the impossibility of covering a chessboard with 1×2 dominoes, after two opposite corner cells have been removed. This serves as a very didactical example, for the squares of the chessboard are already coloured in black and white, with 32 black and 32 white cells. Plainly, a domino will cover one white and one black cell, so for the construction to be possible, there must be an equal number of black and white squares. But the two opposite corner have the same colour; this results in 32 black and 30 white squares (or, as it may be, 30 black and 32 white squares). Hence the construction is impossible. Q.E.D.

1. *Skolornas Matematiktävling 1991.* I en regelbunden 3982-hörning indelas hörnen i par och de båda punkterna i varje par förbinds med en rät linje. Visa att de 1991 sträckor som erhålls inte alla kan ha olika längd. [*Colour the vertices alternately black and white.*]

2. *Baltic Way 1991.* The vertices of a convex 1991-gon are enumerated with integers from 1 to 1991. Each side and diagonal of the 1991-gon is coloured either red or blue. Prove that, for an arbitrary renumeration of vertices, one can find integers k and l such that the line connecting vertices with numbers k and l before the renumeration has the same colour as the line between the vertices having these numbers after the renumeration. [*If all the segments swap colours, then there has to be an equal number of red and blue segments...*]

3. *Baltic Way 1995.* There are n fleas on an infinite sheet of triangulated paper. Initially the fleas are in different small triangles, all of which are inside some equilateral triangle consisting of n^2 small triangles. Once a second each flea jumps from its original triangle to one of the three small triangles having a common vertex but no common side with it. For which natural numbers n does there exist an initial configuration such that after a finite number of jumps all the n fleas can meet in a single small triangle? [*Any flea ca'n't relocate just anywhere. Which triangles can it reach?*]

4. *Baltic Way 1998.* Is it possible to cover a 13×13 chessboard with forty-two tiles of size 4×1 so that only the central square of the chessboard remains uncovered? (It is assumed that each tile covers exactly four squares of the chessboard, and the tiles do not overlap.) [*No, and there are actually two colourings of the chessboard that demonstrate this: one bichromatic and one tetrachromatic.*]

3.3 The Pigeon-Hole Principle

> [. . .]
> Et, dans tous les Romans où j'ay jetté les yeux,
> Je n'ay rien rencontré de plus ingénieux.
>
> — Molière, *Les Femmes sçavantes*

The essence of the Pigeon-Hole Principle is: *Cram sufficiently many objects into a sufficiently small space, and there will be over-crowding somewhere.* This is admittedly very vague. What "over-crowding" means depends of course on the particular problem, but, in general, look out for the following hints:

- You are presented with an impossibly complicated situation with "much too many free variables". An explicit formula or case-by-case analysis is out of the question.
- Yet, you are to shew that somewhere, something interesting happens.

1. *Skolornas Matematiktävling 1988.* På ett schackbräde kallas två rutor närliggande om de har en kant eller ett hörn gemensamt. De 64 rutorna på ett schackbräde numreras i godtycklig ordning med talen från 1 till 64. Visa att det alltid finns två närliggande rutor vilkas nummer har en positiv differens som är högst 16. [*Compare what was outlined above. We don't know* how *the numbers have been distributed, and yet somewhere two numbers will come "close together" — simply because there isn't too much space. Split the board into* 16 *squares of size* 2×2*. These are the pigeon-holes — but where are the pigeons?*]

2. *Skolornas Matematiktävling 1988.* På ytan av en damm, som har formen av en cirkel med radien 5 meter, simmar 6 ankungar. Visa att i varje ögonblick två av ankungarna simmar på ett avstånd av högst 5 meter. [*Again: We have six ducks thrown into a small pond, which means that* somewhere*, they must be swimming close to one other. Try dividing the pond into sectors.*]

3. *Baltic Way 1991 (classified as Number Theory).* There are 20 cats priced from \$12 to \$15 and 20 sacks priced from 10 cents to \$1 for sale (all prices are different). Prove that each of two boys, John and Peter, can buy a cat in a sack paying the same amount of money.

4. *Baltic Way 1991.* An equilateral triangle is divided into 25 congruent triangles enumerated with numbers from 1 to 25. Prove that one can find two triangles having a common side and with the difference of the numbers assigned to them greater than 3. [*The trick from Problem 1 doesn't seem to work here. The triangle cannot be nicely subdivided into cells and, moreover, one must shew that some difference is* large, *rather than small. Consider the numbers 1 and 25, which differ by 24. In a sequence of eight numbers, beginning with 1 and ending in 25, what can be said about the seven consecutive differences?*]

5. *Baltic Way 1994 (classified as Number Theory).* Prove that any irreducible fraction $\frac{p}{q}$, where p and q are positive integers and q is odd, is equal to a fraction $\frac{n}{2^k-1}$ for some positive integers n and k. [*There is, in fact, a number-theoretical solution. Consider binary expansions.*]

3.4 The Useful Arithmetic

> "What age were you when you went to Lowood? "
>
> "About ten."
>
> "And you stayed there eight years: and are now, then, eighteen?"
>
> I assented.
>
> "Arithmetic, you see, is useful: without its aid, I should hardly have been able to guess your age. [. . .]"
>
> — Brontë, *Jane Eyre*

Many Combinatorics problems require us to count something in a clever way. (Confer Section 3.2.) The crux is rooting out *what* is supposed to be counted and *how*.

1. *Baltic Way 1990.* Positive integers 1, 2, . . . , 100, 101 are written in the cells of a 101×101 square grid so that each number is repeated 101 times. Prove that there exists either a column or a row containing at least 11 different numbers. [*Consider the number of rows and columns containing a certain number. Sum over all rows and columns.*]

2. *Baltic Way 1992.* All faces of a convex polyhedron are parallelograms. Can the polyhedron have exactly 1992 faces? [*Consider closed loops of faces running round the polyhedron, all common edges parallel. Two such loops must have exactly two faces in common. (The problem was ill formulated; and this statement may not quite follow from the hypotheses as they stand, depending on how degenerate the polyhedra may be.) Now count things, and arrive at an equation with no solutions.*]

3. *Baltic Way 1993.* An equilateral triangle ABC is divided into 100 congruent equilateral triangles. What is the greatest number of vertices of small triangles that can be chosen so that no two of them lie on a line that is parallel to any of the sides of the triangle ABC? [*Consider the distances from each chosen vertex to the edges of the triangle ABC. Sum over the chosen vertices.*]

4. *Baltic Way 1994.* The Wonder Island Intelligence Service has 16 spies in Tartu. Each of them watches on some of his colleagues. It is known that if spy A watches on spy B then B does not watch on A. Moreover, any 10 spies can be numbered in such a way that the first spy watches on the second, the second watches on the third, ..., the tenth watches on the first. Prove that any 11 spies can also be numbered in a similar manner. [*Consider, for each spy, the number of colleagues spied on, spying on him, and remaining neutral, respectively.*]

3.5 Game on! Choose your tactics.

> Rivalen, med en stor och öfverlägsen mine, svarade intet, men började trumfa för att visa sin styrka. Då föll olyckligtvis öfverstens sista trumfhacka, och han bröt ut: "Såhå – såhå verkligen? vi skola således trumfa ut hvarann, vi, som spela ihop? mins herrn icke, att begge fruntimmerna äro renonce, hvarföre då för fan trumfa? Tricken går ifrån oss! Skönt! Och dermed roberten! ty de stå redan på åtta, och hafva två honnörer. Jaha – de gå ut – åh det är magnifikt!"
>
> – Almqvist, *Araminta May*

Finite-length games without draws are always determined. One of the players must have a winning strategy. (Tacit assumption: the players should both possess "perfect information" on the game-play – that means Poker and Old Maid ("Svarte Petter") don't count.)

For suppose Player 1 has no winning strategy. Then, with each move he makes, Player 2 will have a choice on how to play *not to lose.* Let him follow this strategy. After Player 2 has laboured long enough with this simple aim of not losing, the game will be over, because it is of finite length. Since he didn't lose and a draw is impossible, he must have won the game. Hence his strategy was winning.

1. *Baltic Way 1990.* In two piles there are 72 and 30 sweets respectively. Two students take, one after another, some sweets from one of the piles. Each time the number of sweets taken from a pile must be an integer multiple of the number of sweets in the other pile. Is it the beginner of the game or his adversary who can always assure taking the last sweet from one of the piles? [*Recall what was said above: One of the players* must *have a winning strategy. Observe that opening by taking 30 sweets forces the opponent to respond by the same move, leading to the same situation as if the first player had opened by taking 60,* except that the rôles have been reversed. *Now if one of the players had a winning strategy, what would happen if his adversary were to play by the same design?*]

2. *Baltic Way 1993.* An equilateral triangle is divided into n^2 congruent equilateral triangles. A spider stands at one of the vertices, a fly at another. Alternately each of them moves to a neighbouring vertex. Prove that the spider can always catch the fly.

3. *Baltic Way 1995.* Consider the following two person game. A number of pebbles are situated on the table. Two players make their moves alternately. A move consists of taking off the table x pebbles where x is the square of any positive integer. The player who is unable to make a move loses. Prove that there are infinitely many initial situations in which the second player can win no matter how his opponent plays. [*Argue by contradiction. Suppose the first player had a winning strategy for all large initial numbers of pebbles. What would happen if the second player were to play by the same stratagem? Naturally, that he be able to do so requires there being sufficiently many pebbles left after the first move that the strategy applies. But perhaps that can be arranged for? Choose an initial situation where the first move ca'n't reduce the number of pebbles by too much.*]

4. *Baltic Way 1996.* On an infinite checkerboard, two players alternately mark one unmarked cell. One of them uses $\times$, the other $\circ$. The first who fills a 2×2 square with his symbols wins. Can the player who starts always win? [*In order to stop the first player, one must devise a procedure to systematically destroy any square he could attempt to form…*]

4 Geometry

And lastly, we have arrived at Geometry, the very subject which embodied Mathematics to the Greeks. Mathematics, to them, *was* Geometry (even though Euclid managed to squeeze quite a bit of Number Theory into his *Elements*).

Put bluntly, in any Baltic Way (or IMO) team, there is usually exactly one kid who loves indulging in problems involving triangles and circles (the more, the merrier), and four (or five) others who are very good at Mathematics in general, but strive to avoid Geometry at any cost. (The author falls into the latter category.)

Unlike the situation for Number Theory, where grand results like Fermat's Great Theorem are of absolutely no assistance, there is a plethora of obscure theorems of Euclidean Geometry the student ought to be conversant with. Knowing these results undeniably makes life simpler.

4.1 The Elements

> Baron Oscar gör sin sak ganska väl. Han lär Constance hvad en triangel vill säga här i verlden; han visar henne hvilken "lutning" en linie måste hafva till en annan, för att deraf må kunna blifva spetsig vinkel, samt huru en rät vinkel ser ut. Om jag icke ännu kan inse nyttan häraf för oss fruntimmer, så har jag dock ej erfarit någon skada; tvertom, Constance ritar alltid med större precision den dagen hon studerat geometri och fixerat baron Oscars plancher: och jag sjelf, hvem skulle hafva trott det? har börjat fatta tycke för parallela linier, alternatanglar och qvadrater. Jag sätter flere gånger ifrån mig bågen och gör mig till en elev i sällskap med Constance. Baron Oscar demonstrerar gång efter gång allt bättre, lifligare och — jag bekänner, att geometrien är icke det sämsta.
>
> — Almqvist, *Amalia Hillner*

Some problems can be solved exclusively through "Euclidean" tools, which in this setting does not refer to ruler and compass, but congruence and similarity.

We state these general guidelines for solving Geometry problems:

I. Calculate all angles, or introduce variables for some, and express the others in these. Some problems are solved entirely by ambitious angle chasing.

II. Equal angles, and equal or proportional lengths, indicate that congruent or similar triangles are lurking in the neighbourhood.

III. Do you recall the *Angle Bisector Theorem*? The parts in which an angle bisector divides the opposite side, are proportional to the adjacent sides.

Now try these appetisers...

1. *Baltic Way 1990.* Let $ABCD$ be a quadrangle, $|AD| = |BC|$, $\angle A + \angle B = 120^\circ$ and let P be a point exterior to the quadrangle such that P and A lie at opposite sides of the line DC and the triangle DPC is equilateral. Prove that the triangle APB is also equilateral.

2. *Baltic Way 1995.* In the triangle ABC, let l be the bisector of the external angle at C. The line through the midpoint O of the segment AB parallel to l meets the line AC at E. Determine $|CE|$, if $|AC| = 7$ and $|CB| = 4$. [*It is very tempting to draw the line through O parallel to BC...*]

3. *Baltic Way 1996.* Let α be the angle between two lines containing the diagonals of a regular 1996-gon, and let $\beta \neq 0$ be another such angle. Prove that α/β is a rational number.

4.2 "Algebraic" Geometry

> Bort med papper / och bleck; bort böker / Cirklar[4] / och pännor;
> Skulle du smitta dijn hand / dijn' Adlige miölk-hwijte finger /
> Skulle de fläckias i bleck; huru wille du Frustugun wittia?
>
> — Stiernhielm, *Hercules*

Some problems are only artifically geometrical. Once variables have been assigned to the lengths and areas involved, the problem transforms nicely into a set of algebraical equations. For some problems, there is even no need to introduce trigonometry (which will be treated in the next section)! We have designated these "algebraic" geometry. (The reason for enclosing the word "algebraic" within quotation marks is that there already exists a discipline of mathematics by the name Algebraic Geometry — extremely abstract, and as far removed from Euclidean Geometry as is conceivable. It deals with something called "schemes", and we are quite certain they are evil ones.)

1. *Baltic Way 1991.* Let two circles C_1 and C_2 (with radii r_1 and r_2) touch each other externally, and let l be their common tangent. A third circle C_3 (with radius $r_3 < \min(r_1, r_2)$) is externally tangent to the two given circles and tangent to the line l. Prove that

$$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}.$$

[4] passare

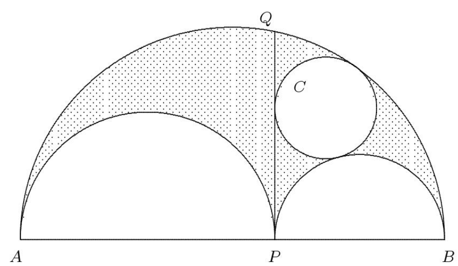

FIGURE 1: Three half-circles.

2. *Baltic Way 1991.* Let the coordinate planes have the reflection property. A beam falls onto one of them. How does the final direction of the beam after reflecting from all three coordinate planes depend on its initial direction?

3. *Baltic Way 1996.* In the figure below (Figure 1), you see three half-circles. The circle C is tangent to two of the half-circles and to the line PQ perpendicular to the diameter AB. The area of the shaded region is 39π, and the area of the circle C is 9π. Find the length of the diameter AB.

4.3 Cosplay

Je trône dans l'azur comme un sphinx incompris;
J'unis un cœur de neige à la blancheur des cygnes;
Je hais le mouvement qui déplace les lignes,
Et jamais je ne pleure et jamais je ne ris.

– Baudelaire, *La Beauté*

Trigonometry tends to lead to a mess of formulæ, and is arguably a clumsy way to solve a problem when a smooth Euclidean proof is available. None the less, for those who do not feel comfortable with classical geometrical reasoning (as stated above, approximately 4/5 of the population), it is comfortable to know that many problems which are highly graphical in nature can be transformed into a set of re-assuring (analytical) equations.

1. *Skolornas Matematiktävling 1988.* På periferin av en cirkel ligger punkterna P, Q och R så att triangeln PQR är liksidig. S är en godtycklig punkt på cirkelperiferin. Betrakta längderna av sträckorna PS, RS och QS. Visa att en av dem är summan av de två övriga. [*This one is very slick with trigonometry, but it is instructive to try to find a synthetic solution. There is a trick involved,*

which it might be worth knowing. To shew $|RS| = |PS| + |QS|$, choose *a point X on RS with $|XS| = |PS|$.]*

2. *Baltic Way 1992.* Quadrangle $ABCD$ is inscribed in a circle with radius 1 in such a way that one diagonal, AC, is a diameter of the circle, while the other diagonal, BD, is as long as AB. The diagonals intersect in P. It is known that the length of PC is $\frac{2}{5}$. How long is the side CD?

3. *Baltic Way 1994.* Does there exist a triangle such that the lengths of all its sides and altitudes are integers and its perimeter is equal to 1995?

4.4 Triangle Trivia

> Hjernan ännu i mig wrides
> när jag tänker på Euclides
> ock på de Trianglarna
> a b c – ock c d a –
> Swetten ur min Panna gnides,
> Wärre än på Golgatha.
>
> – Bellman, *Lefnadsbeskrifning*

Some situations call for heavier machinery. A full account of the many properties of triangles is far beyond the scope of these notes, but the following basic facts concerning a triangle ABC with sides a, b, c should be borne in mind.

I. The medians of a triangle are concurrent, and meet in a point G called the *centroid* (centre of gravity).

II. The altitudes of a triangle are concurrent, and meet in a point H called the *orthocenter*.

III. The perpendicular bisectors of the sides of the triangle are concurrent, and meet in a point O called the *circumcentre*. It is the centre of the circumscribed circle, whose radius is called the *circumradius* R.

IV. The angle bisectors are concurrent, and meet in a point I called the *incentre*. It is the centre of the inscribed circle, whose radius is called the *inradius* r.

V. The points O, G, H are collinear, in this order, and $2|GO| = |GH|$. The line they lie upon is called the *Euler line* of the triangle.

VI. The *Law of Sines* states, in its extended form, that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

VII. Let p denote the semiperimeter of the triangle, so that $2p = a+b+c$. According to *Heron's formula*, the area of the triangle is

$$|ABC| = \sqrt{p(p-a)(p-b)(p-c)}.$$

VIII. The area of the triangle is also given by

$$|ABC| = pr = \frac{abc}{4R}.$$

While these facts are not always needed to solve a problem, they can often considerably simplify matters.

1. *Skolornas Matematiktävling 1992.* I en triangel är sidorna a, b och c, där $c \geqslant a$ och $c \geqslant b$. Omskrivna cirkelns radie är R. Visa att om $a^2 + b^2 = 2Rc$ så är triangeln rätvinklig.

2. *Baltic Way 1992.* Show that in a non-obtuse triangle the perimeter of the triangle is always greater than two times the diameter of the circumcircle.

3. *Baltic Way 1992.* Let $a \leqslant b \leqslant c$ be the sides of a right triangle, and let $2p$ be its perimeter. Show that $p(p-c) = (p-a)(p-b) = S$ (the area of the triangle).

4. *Baltic Way 1996.* Let $ABCD$ be a cyclic convex quadrilateral and let r_a, r_b, r_c, r_d be the radii of the circles inscribed in the triangles BCD, ACD, ABD, ABC respectively. Prove that $r_a + r_c = r_b + r_d$.

5. *Nordic Mathematical Contest 1992.* Visa att av alla trianglar, vars inskrivna cirkel har radien 1, har den liksidiga triangeln den minsta *omkretsen*.

4.5 Circular Reasoning

> Their Houses are very ill built, the Walls bevil without one right Angle in any Apartment; and this Defect ariseth from the Contempt they bear to practical Geometry; which they despise as vulgar and mechanick, those Instructions they give being too refined for the Intellectuals of their Workmen; which occasions perpetual Mistakes. And although they are dexterous enough upon a Piece of Paper in the Management of the Rule, the Pencil, and the Divider, yet in the common Actions and Behaviour of Life, I have not seen a more clumsy,

> awkward, and unhandy People, nor so slow and perplexed in their Conceptions upon all other Subjects, except those of Mathematicks and Musick. They are very bad Reasoners, and vehemently given to Opposition, unless when they happen to be of the right Opinion, which is seldom their Case.
>
> — Swift, *Gulliver's Travels into Several Remote Nations of the World*

When it comes to circles, the following facts should be known and remembered:

I. The *Inscribed Angle Theorem*: A peripheral angle subtended by a circle arc is exactly half the central angle subtended by the same arc.

II. Two immediate corollaries: All peripheral angles subtended by the same arc are equal. The peripheral angle subtended by a half-circle is right.

III. The *Chordal Theorem*: If two chords divide each other into the parts a, b, and c, d, then

$$ab = cd.$$

Observe: This is true even if the point of intersection lies on or outside the circle!

IV. As a corollary: Given a point P. Draw any line through P, intersecting the circle in points A and B. The product

$$PA \cdot PB$$

is a constant, independent of the line. It is called the *power of the point P* with respect to the circle.

V. A *cyclic* quadrilateral is one having all its vertices on a circle, which is the case if and only if opposite angles sum to 180°. This is the customary procedure to shew four points are concyclic — prove that opposite angles are supplementary.

VI. *Ptolemy's Theorem*: If the sides of a cyclic quadrilateral be labelled a, b, c, d, and its diagonals x and y; then

$$xy = ac + bd.$$

The converse is equally valid: if this relation be true, then the quadrilateral is cyclic.

And, speaking of quadrilaterals, we remind the reader that connecting the midpoints of an *arbitrary* quadrilateral results in a parallelogram, called the *Varignon parallelogram* (this seems to have been discovered surprisingly late; the first published proof appears in 1731).

1. *Skolornas Matematiktävling 1995.* På en cirkel med medelpunkt O och radien r är punkterna A, B, C, D (i denna ordning) utplacerade på följande sätt. Kordorna AB, BC och CD har samma längd, s. Talet s är sådant att längden av kordan AD är $s + r$. Man vet också att $s < r$. Bestäm vinklarna i fyrhörningen $ABCD$.

2. *Baltic Way 1994.* Let NS and EW be two perpendicular diameters of a circle $\mathcal{C}$. A line l touches $\mathcal{C}$ at point S. Let A and B be two points on $\mathcal{C}$, symmetric with respect to the diameter EW. Denote the intersection points of l with the lines NA and NB by A' and B', respectively. Show that $|SA'| \cdot |SB'| = |SN|^2$.

3. *Baltic Way 1993.* A convex quadrangle $ABCD$ is inscribed in a circle with the centre O. The angles $\angle AOB$, $\angle BOC$, $\angle COD$ and $\angle DOA$, taken in some order, are of the same size as the angles of quadrangle $ABCD$. Prove that $ABCD$ is a square.

4. *Baltic Way 1995.* Let M be the midpoint of the side AC of a triangle ABC and let H be the foot point of the altitude from B. Let P and Q be the orthogonal projections of A and C on the bisector of angle B. Prove that the four points M, H, P and Q lie on the same circle.

5. *Nordic Mathematical Contest 1993.* En sexhörning är inskriven i en cirkel med radien r. Två av dess sidor har längd 1, två har längd 2 och de återstående två har längd 3. Visa att r är rot till ekvationen

$$2r^3 - 7r - 3 = 0.$$

4.6 Squeeze

> “The fact is,” said Rabbit, “you’re stuck.”
>
> “It all comes,” said Pooh crossly, “of not having front doors big enough.”
>
> “It all comes,” said Rabbit sternly, “of eating too much. I though at the time,” said Rabbit, “only I didn’t like to say anything,” said Rabbit, “that one of us was eating too much,” said Rabbit, “and I knew it wasn’t *me*,” he said.
>
> — Milne, *Winnie-the-Pooh*

From time to time there appear packing problems, in which the smallest enclosure of some shape (circle, square,. . .) that can contain a given

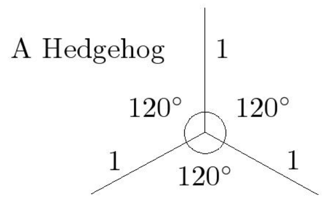

FIGURE 2: A Hedgehog.

assortment of objects is sought. First, of course, the minimal solution must be found. Usually it is the symmetric, or "obvious" one. Then, it must be proved that this gives indeed the minimum, a feat which might be achieved through the Pigeon-Hole Principle.

1. *Baltic Way 1991.* Is it possible to put two tetrahedra of volume $\frac{1}{2}$ without intersection into a sphere with radius 1? [*One of them has to contain the centre of the sphere.*]

2. *Baltic Way 1994.* Find the smallest number a such that a square of side a can contain five disks of radius 1 so that no two of the disks have a common interior point. [*The symmetrical arrangement will most probably give the optimum. Use the Pigeon-Hole Principle to prove it.*]

3. *Baltic Way 1995 (classified as Combinatorics).* The Wonder Island is inhabited by Hedgehogs. Each Hedgehog consists of three segments of unit length having a common endpoint, with all three angles between them equal to 120° (see Figure 2). Given that all Hedgehogs are lying flat on the island and no two of them touch each other, prove that there is a finite number of Hedgehogs on Wonder Island. [*Prove that the centra of two Hedgehogs have to be a certain distance apart.*]

4.7 The Unusual Suspects

PIQUANT FACTS FOR SIMILES. "There were originally but three Muses – Melete, Mneme, Aœde – meditation, memory and singing." You may make a great deal of that little fact if properly worked. You see it is not generally known, and looks *recherché*. You must be careful and give the thing with a downright improviso air.

– Poe, *How to Write a Blackwood Article*

A charming feature of Baltic Way, probably unique in the world, is the inclusion of highly original, one-of-a-kind problems. After all, in a set of twenty problems, there is some room for innovation. Nowhere is this as striking as in the Geometry department, where some truly unexampled problems have been presented. As a contrast, it is extremely rare to find an IMO geometry problem which does not follow the usual points–lines–circles pattern.

1. *Baltic Way 1990.* Two equal triangles are inscribed into an ellipse. Are they necessarily symmetrical with respect either to the axes or to the centre of the ellipse? [*Instead of inscribing two triangles into the same ellips, inscribe the same triangle into two equal ellipses.*]

2. *Baltic Way 1993.* Let's consider three pairwise non-parallel straight lines in the plane. Three points are moving along these lines with different non-zero velocities, one on each line (we consider the movement as having taken place for infinite time and continuing infinitely in the future). Is it possible to determine these straight lines, the velocities of each moving point and their positions at some "zero" moment in such a way that the points never were, are or will be collinear? [*Yes, it is possible. As usual, aim for symmetry. Try lines located to form an equilateral triangle.*]

3. *Baltic Way 1995.* Prove that if both coordinates of every vertex of a convex pentagon are integers, then the area of this pentagon is not less than $\frac{5}{2}$.

4. *Baltic Way 1999.* Prove that for any four points in the plane, no three of which are collinear, there exists a circle such that three of the four points are on the circumference and the fourth point is either on the circumference or inside the circle.