Properties of the Beurling generalized primes

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Abstract

In this paper, we prove a generalization of Mertens' theorem to Beurling primes, namely that $\lim_{x\to\infty} \frac{1}{\ln x} \prod_{p\leq x} \left(1-p^{-1}\right)^{-1} = Ae^{\gamma}$, where γ is Euler's constant and Ax is the asymptotic number of generalized integers less than x. Thus the limit $M = \lim_{x\to\infty} \left(\sum_{p\leq x} p^{-1} - \ln(\ln x)\right)$ exists. We also show that this limit coincides with $\lim_{\alpha\to 0^+} \left(\sum_p p^{-1}(\ln p)^{-\alpha} - 1/\alpha\right)$; for ordinary primes this claim is called Meissel's theorem. Finally, we will discuss a problem posed by Beurling, namely how small $|N(x)-[x]|$ can be made for a Beurling prime number system $Q \neq P$, where P is the rational primes. We prove that for each $c>0$ there exists a Q such that $|N(x)-[x]| < c\ln x$ and conjecture that this is the best possible bound.

Keywords. Analytic number theory, Zeta functions, Beurling primes, Mertens' theorem, Beurling's conjecture

1 Introduction

Beurling [1] considered sequences $1 < p_1 \leq p_2 \leq p_3 \leq ...$ of real numbers with $p_n \to \infty$ and such that the multiplicative semigroup had $N(x) \approx Ax$ elements less or equal to x, counting with multiplicities. He showed that if $N(x) = Ax + O\left(x/(\ln x)^{\eta}\right)$, where $\eta > 3/2$, then the number of $p_n \leq x$ (henceforth called $\pi(x)$) is equal to $x/\ln x + o\left(x/\ln x\right)$. Since the sequence p_n satisfies the prime number theorem, it is called a Beurling prime number system. Beurling also showed that $\eta > 3/2$ is necessary in the sense that there is a "continuous analog" of a prime number system, such that $\eta = 3/2$ and where the prime number theorem does not hold. This idea has later been used by Diamond [2] to produce a Beurling prime system with this property. The obvious interpretation of the theorem of Beurling is that the theorem does not rely on the additive structure of the natural numbers. This idea was known before Beurling, and used by Landau [9] in 1903 in his proof of the prime ideal theorem, stating

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that given a fixed algebraic number field, the number of prime ideals in the ring of integers with norm less than x is asymptotically $x/\ln x$. The proof uses the multiplicative structure of the norms and the result of Weber [12] that the number of ideals with norm less than x is $Ax + O(x^\theta)$.

Our ambition is to generalize a theorem of Mertens [10] to Beurling systems, giving a simple formula for calculating A given only the Beurling primes p. Unless otherwise stated we will always assume that $N(x) = Ax + o(x)$, where A is a real number larger than 0.

Theorem 1.1. *If P is a generalized prime number system for which $N(x) = Ax + o(x)$ then*

$$\lim_{x\to\infty} \frac{1}{\ln x} \prod_{p\le x} \left(1 - \frac{1}{p}\right)^{-1} = Ae^\gamma.$$

Remark. γ is Euler's constant defined by $\gamma = \lim_{n\to\infty} \sum_{k=1}^{n} k^{-1} - \ln n = 0.5772...$

Let us introduce the function $\Pi(x) = \sum_{n=1}^{\infty} (1/n)\pi\left(x^{1/n}\right)$. Under the assumption $\pi(x) = o(x)$ (which follows from $N(x) = Ax + o(x)$, see the proof of Lemma 2.3 for details) one can obtain that

$$\int_1^x \frac{d\Pi(t)}{t} + \sum_{p\le x} \ln\left(1 - \frac{1}{p}\right) = o(1)$$

as $x \to \infty$. Using this we see that if $N(x) = Ax + o(x)$ then

$$\lim_{x\to\infty} \int_1^x \frac{d\Pi(t)}{t} - \ln\ln x = \ln A + \gamma.$$

As a special case of Theorem 1.1, we of course get a formula for the asymptotic number of ideals with norms less than x in the ring of integers of a fixed algebraic number field.

The theorem does not look particularly surprising in itself, but let us remark that at least one consequence is a bit peculiar. There is a well known heuristic argument, using Mertens' theorem together with the sieve of Eratosthenes to estimate the number of primes less than x (see [5] for details). The argument produces the incorrect asymptotics $2e^{-\gamma}x/\ln x \approx 1.12x/\ln x$ for ordinary primes, and this calculation is valid for all Beurling prime systems and gives the same asymptotics. In other words, the inconsistency between the sieve of Eratosthenes and the prime number theorem seems to be caused by a general property of all Beurling prime systems.

In the third chapter we will discuss a problem posed by Beurling [3] concerning the smallest possible non-zero size of $|N(x) - [x]|$. We conjecture the following:

Conjecture 1.2. *If P is a Beurling prime system different from the set of rational primes, then*

$$\limsup_{x\to\infty} \frac{|N(x) - [x]|}{\ln x} > 0.$$

We will prove this for some Beurling systems and give some indications of why one should expect this. If the conjecture is true, it is sharp in the following sense:

Theorem 1.3. *For every $c > 0$ there exists a generalized prime number system other than the rational primes for which*

$$|N(x) - [x]| < c \ln x$$

for all $x \geq 1$.

2 Mertens' theorem

In this chapter we will prove Theorem 1.1. The proof uses integration theory for Riemann integrals, and all integrals should be thought of as Riemann integrals, rather than Lebesgue integrals. For instance, Lemma 2.4 is not true for Lebesgue integrals. The proof will also make frequent use of the following lemma, which can be found in many text-books (see for instance [6]) and which is just a special case of integration by parts for Stieltjes integrals:

Lemma 2.1. *Suppose that $\lambda_n, n = 1, 2, \ldots$ is a nondecreasing sequence such that $\lambda_n \to \infty$ when $n \to \infty$. Define $C(t) = \sum_{\lambda_n \leq t} c_n$ and let $\phi(t)$ be a function which is defined and has a continuous derivative for $t \geq \lambda_1$. Then*

$$\sum_{\lambda_n \leq x} c_n \phi(\lambda_n) = C(x)\phi(x) - \int_{\lambda_1}^{x} C(t)\phi'(t)\, dt.$$

Let us define what we mean by the Beurling integers corresponding to the Beurling primes p_n :

Definition 2.1. The Beurling integers are all commutative monomials in the variables p_n and the value of the Beurling integer is the corresponding product of the values of p_n. Furthermore, let $N(x)$ denote the number of Beurling integers such that their value is less than or equal to x.

Let us stress that we think of two Beurling integers (two monomials) as different, even though their values (seen as real numbers) are the same. We can also think of the Beurling integers as real numbers with multiplicities, but we will adopt this equivalent formulation.

Definition 2.2. Given a Beurling prime number system P we define the function $\zeta_P(s)$ for $\mathrm{Re}(s) > 1$ by

$$\zeta_P(s) = \sum_{i=1}^{\infty} \frac{1}{n_i^s},$$

where n_i are the values of all Beurling integers.

Since the Beurling integers obey unique factorization by construction, the zeta function has the usual Euler product

$$\zeta_P(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Looking at Theorem 1.1 it is easy to realize that the theorem is related to the behavior of $\zeta_P(s)$ near $s = 1$. Let us therefore study this behavior:

Lemma 2.2. *If P is a generalized prime number system for which $N(x) = Ax + o(x)$ then*

$$\lim_{s\to 1^+}(s-1)\zeta_P(s) = A.$$

Proof. Let $R(x) = N(x) - Ax = o(x)$. Choose X so that $|R(x)| < \epsilon x/2$ for $x > X$, and $s > 1$ so that

$$s - 1 < \frac{\epsilon}{2\int_1^X \frac{|R(t)|}{t^2}\,dt}.$$

We have

$$\begin{aligned}\left|(s-1)\int_1^\infty \frac{R(t)}{t^{s+1}}\,dt\right| &\le \left|(s-1)\int_1^X \frac{R(t)}{t^{s+1}}\,dt\right| + \left|(s-1)\int_X^\infty \frac{R(t)}{t^{s+1}}\,dt\right| \\ &\le \frac{\epsilon}{2} + (s-1)\int_X^\infty \frac{\epsilon}{2t^s}\,dt < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

and therefore

$$\lim_{s\to 1^+}(s-1)\int_1^\infty \frac{R(t)}{t^{s+1}}\,dt = 0.$$

Using Lemma 2.1 we have

$$\begin{aligned}\lim_{s\to 1^+}(s-1)\zeta_P(s) &= \lim_{s\to 1^+}(s-1)s\int_1^\infty \frac{N(t)}{t^{s+1}}\,dt \\ &= \lim_{s\to 1^+}(s-1)s\left[\int_1^\infty \frac{A}{t^s}\,dt + \int_1^\infty \frac{R(t)}{t^{s+1}}\,dt\right] \\ &= \lim_{s\to 1^+}\left[sA + (s-1)\int_1^\infty \frac{R(t)}{t^{s+1}}\,dt\right] = A.\end{aligned}$$

□

To prove Theorem 1.1 we need some estimate on the density of the set of primes. It would for instance be enough to have the prime number theorem, but due to the work done by Beurling [1] and Diamond [2], which we mentioned in the introduction, we know that this is too much to hope for. Instead of counting each prime with weight 1 we will count each prime with the weight $\ln p/p$. This gives us a weaker asymptotic formula, but this asymptotic formula will be enough. The proof of the lemma uses the estimate $\pi(x) = o(x)$. Even though this may be thought of as well known to experts on Beurling primes, we have still added a proof of this claim for the readers' convenience.

Lemma 2.3. *If P is a generalized prime number system then*

$$\lim_{x\to\infty}\frac{1}{\ln x}\sum_{p\le x}\frac{\ln p}{p} = 1.$$

Proof. Let $N(x) = Ax + R(x)$, where $R(x) = o(x)$. If x_n denote the values of all Beurling integers, we define $T(x)$ to be

$$T(x) = \ln\left(\prod_{x_n\le x} x_n\right) = \sum_{x_n\le x}\ln x_n = N(x)\ln x - \int_1^x \frac{N(t)}{t}\,dt = Ax\ln x + o(x\ln x).$$

Counting the times $\exp\{T(x)\}$ is divisible by the primes p we get the estimate

$$T(x) = \sum_{p\le x}\left(N\left(\frac{x}{p}\right) + N\left(\frac{x}{p^2}\right) + ...\right)\ln p = \sum_{p\le x} N\left(\frac{x}{p}\right)\ln p + O(x).$$

It follows that

$$\begin{aligned} Ax\sum_{p\le x}\frac{\ln p}{p} &= \sum_{p\le x} N\left(\frac{x}{p}\right)\ln p + \sum_{p\le x}\left(\frac{Ax}{p} - N\left(\frac{x}{p}\right)\right)\ln p \\ &= Ax\ln x - \sum_{p\le x} R\left(\frac{x}{p}\right)\ln p + o(x\ln x) \end{aligned}$$

and therefore we are done if we can show that

$$\sum_{p\le x} R\left(\frac{x}{p}\right)\ln p = o\left(x\sum_{p\le x}\frac{\ln p}{p}\right) + o(x\ln x). \tag{1}$$

From Lemma 2.2 and the Euler product, we may deduce that $\prod_p \left(1-\frac{1}{p}\right)^{-1}$ is divergent. Let $p_1, p_2, ..., p_r$ be the first Beurling primes and let $H_r(x)$ be the number of generalized integers less than x which are not divisible by p_j for any $j = 1, 2, ..., r$. Using the inclusion-exclusion principle we may write

$$\begin{aligned} H_r(x) &= N(x) - \sum_{j=1}^{r} N\left(\frac{x}{p_j}\right) + \sum_{j=2}^{r}\sum_{k=1}^{j-1} N\left(\frac{x}{p_j p_k}\right) + ... + (-1)^r N\left(\frac{x}{p_1 p_2 ... p_r}\right) \\ &= Ax\prod_{j=1}^{r}\left(1 - \frac{1}{p_j}\right) + o_r(x) \end{aligned}$$

and if we first choose r large enough and then x large enough this is less than ϵx. But the Beurling primes $p_{r+1}, p_{r+2}, ..., p_{\pi(x)}$ are less than x and not divisible by any p_j with $j = 1, 2, ..., r$ and this gives the estimate

$$\frac{\pi(x)}{x} \le \frac{H_r(x)}{x} + \frac{r}{x}.$$

This shows that $\pi(x)/x = o(1)$ and we now have the tools needed to prove equation (1). Take $\epsilon > 0$. Since $R(x) = o(x)$ there exists an x_0 such that $|R(x)| \le \epsilon x$ for all $x \ge x_0$. $R(x)$ is of course bounded on the interval $[1, x_0]$ and we assume that $|R(x)| < C$ for $x \le x_0$. We also assume $C\pi(x) \le \epsilon x$. This gives us

$$\begin{aligned} \sum_{p\le x} R\left(\frac{x}{p}\right)\ln p &= \sum_{x/x_0 \le p\le x} R\left(\frac{x}{p}\right)\ln p + \sum_{p< x/x_0} R\left(\frac{x}{p}\right)\ln p \\ &\le C\sum_{x/x_0\le p\le x}\ln p + \epsilon x\sum_{p<x/x_0}\frac{\ln p}{p} \le C\pi(x)\ln x + \epsilon x\sum_{p\le x}\frac{\ln p}{p} \\ &\le \epsilon x\sum_{p\le x}\frac{\ln p}{p} + \epsilon x\ln x. \end{aligned}$$

$\square$

Using the previous lemmata it is possible to follow the proof of Mertens' theorem from [7] closely:

Proof of Theorem 1.1. Take $s > 1$, $\phi(t) = \ln(1 - t^{-s})$ and let $x \to \infty$ in Lemma 2.1 to get

$$\begin{aligned}
\ln \zeta_P(s) &= -\sum_p \ln(1-p^{-s}) = s\int_{p_1}^{\infty} \frac{\pi(t)}{t(t^s-1)}\,dt \\
&= s\int_{p_1}^{\infty} \left(\pi(t) - \frac{t}{\ln t}\right) t^{-s-1}\,dt + s\int_{p_1}^{\infty} \frac{\pi(t)}{t^{s+1}\,(t^s-1)}\,dt + s\int_{p_1}^{\infty} \frac{t^{-s}}{\ln t}\,dt \\
&= s\int_{p_1}^{\infty} \left(\pi(t) - \frac{t}{\ln t}\right) \frac{1}{t^2} t^{-(s-1)}\,dt + \int_{p_1}^{\infty} \frac{\pi(t)}{t^2\,(t-1)}\,dt + \int_{p_1}^{\infty} \frac{t^{-s}}{\ln t} dt + o(1)
\end{aligned}$$

as $s \to 1^+$. The third integral can be simplified using the change of variables $u = (s-1)\ln t$, followed by integration by parts. The resulting formula is

$$\int_{p_1}^{\infty} \frac{t^{-s}}{\ln t} dt = \ln\left(\frac{1}{s-1}\right) - \ln\ln p_1 + \int_0^{\infty} e^{-u} \ln u du + o(1).$$

However, it is well known (see for instance [4]) that $\int_0^\infty e^{-u} \ln u du = \Gamma'(1) = -\gamma$, thus

$$\begin{aligned}
\ln \zeta_P(s) = s\int_{p_1}^{\infty} \left(\pi(t) - \frac{t}{\ln t}\right) \frac{1}{t^2} t^{-(s-1)}\,dt + \int_{p_1}^{\infty} \frac{\pi(t)}{t^2\,(t-1)}\,dt + \ln\left(\frac{1}{s-1}\right) \\
- \ln(\ln p_1) - \gamma + o(1).
\end{aligned}$$

From this it follows that $I(s-1) = \int_{p_1}^\infty (\pi(t) - t/\ln t)\, t^{-2} t^{-(s-1)}\,dt$ can be estimated using Lemma 2.2 as

$$\begin{aligned}
sI(s-1) &= \ln \zeta_P(s) - \int_{p_1}^{\infty} \frac{\pi(t)}{t^2\,(t-1)}\,dt - \ln\left(\frac{1}{s-1}\right) + \ln(\ln p_1) + \gamma + o(1) \\
&= \ln A - \int_{p_1}^{\infty} \frac{\pi(t)}{t^2\,(t-1)}\,dt + \ln(\ln p_1) + \gamma + o(1).
\end{aligned}$$

To prove that $I(0)$ is convergent we want to use a Tauberian theorem, Theorem 434 in [7], stating that:

Lemma 2.4. *Let $a > 1$ and assume that the integral*

$$I(\delta) = \int_a^{\infty} x^{-\delta} f(x)\,dx$$

is convergent for all $\delta > 0$, that $I(\delta) \to l$ when $\delta \to 0^+$ and that $\int_a^x \ln(t) f(t)\,dt = o(\ln x)$, when $x \to \infty$. Then $I(0)$ is convergent as a generalized Riemann integral and $I(0) = \int_a^\infty f(x)\,dx = l$.

The only thing we needed to verify in order to use the Lemma 2.4 is that for $f(t) = (\pi(t) - t/\ln t)t^{-2}$ we have $\int_{p_1}^x \ln t f(t)\,dt = o(\ln x)$. Using that $\pi(x) = o(x)$ together with Lemma 2.3 and Lemma 2.1 we see that

$$\int_{p_1}^{x} \ln t f(t)\,dt = \int_{p_1}^{x} \frac{\pi(t)\ln t}{t^2}\,dt - \int_{p_1}^{x} \frac{dt}{t} = \sum_{p \le x} \frac{\ln p}{p} + o(\ln x) - \ln x = o(\ln x),$$

which means that

$$I(0) = \ln A - \int_{p_1}^{\infty} \frac{\pi(t)}{t^2\,(t-1)}\,dt + \ln(\ln p_1) + \gamma.$$

Once again using Lemma 2.1 we get

$$\begin{aligned}
&\lim_{x\to\infty} \Big[\sum_{p\le x} \ln\left(1-\frac{1}{p}\right) + \ln(\ln x)\Big] \\
&\qquad = \lim_{x\to\infty} \Big[-\frac{\pi(x)}{x} - \int_{p_1}^{x} \left(\pi(t) - \frac{t}{\ln t}\right) t^{-2}\,dt \\
&\qquad - \int_{p_1}^{x} \frac{\pi(t)}{t^2\,(t-1)}\,dt - \int_{p_1}^{x} \frac{dt}{t\ln t} + \ln(\ln x)\Big] \\
&\qquad = -I(0) - \int_{p_1}^{\infty} \frac{\pi(t)}{t^2\,(t-1)}\,dt + \ln(\ln p_1) = -\ln A - \gamma.
\end{aligned}$$

After changing signs and exponentiating we obtain

$$\lim_{x\to\infty} \frac{1}{\ln x} \prod_{p\le x} \left(1-\frac{1}{p}\right)^{-1} = Ae^{\gamma}.$$

□

By taking logarithms in Theorem 1.1, Taylor expanding $-\ln(1-1/p)$ and then using Möbius inversion, we reach the equivalent form

$$M = \lim_{x\to\infty} \left[\sum_{p\le x} \frac{1}{p} - \ln(\ln x)\right] = \gamma + \ln A + \sum_{k=2}^{\infty} \frac{\mu(k)}{k} \ln \zeta_P(k).$$

For rational primes Mertens calculated the limit to be $M \approx 0.2614972128$ in his article [10] from 1874. However, this limit is often called Meissel's constant, because as early as 1866, Ernst Meissel announced that

$$\sum_{p} \frac{1}{p(\ln p)^{\alpha}} \approx \frac{1}{\alpha} + 0.2614972128$$

for "very small" α. The similarity of the numerical constants is of course no coincidence, and the fact that they really coincide was proven by Schinzel [11]. In the next theorem we generalize Schinzel's theorem to Beurling primes.

Theorem 2.5. *If M is defined as above, then*

$$\lim_{\alpha\to 0^+} \left[\sum_{p} \frac{1}{p(\ln p)^{\alpha}} - \frac{1}{\alpha}\right] = M.$$

Proof. Let $\mathrm{Li}(x)$ be the function defined for $x > 1$ by

$$\mathrm{Li}(x) = \lim_{\epsilon\to 0^+} \int_0^{1-\epsilon} \frac{dt}{\ln t} + \int_{1+\epsilon}^{x} \frac{dt}{\ln t}.$$

Put $\pi(x) = \mathrm{Li}(x) + R(x)$ and use Lemma 2.1 to get

$$M = \lim_{x\to\infty}\left[\sum_{p\le x}\frac{1}{p} - \ln(\ln x)\right] = \frac{\mathrm{Li}(p_1)}{p_1} - \ln(\ln p_1) + \int_{p_1}^{\infty}\frac{R(t)}{t^2}\,dt.$$

Using Lemma 2.1 again we have that

$$\sum_p \frac{1}{p(\ln p)^\alpha} = \int_{p_1}^{\infty}\frac{\pi(t)}{t^2(\ln t)^\alpha}\,dt + \alpha\int_{p_1}^{\infty}\frac{\pi(t)}{t^2(\ln t)^{\alpha+1}}\,dt = I(\alpha) + \alpha I(\alpha+1).$$

Since both $\int_{p_1}^{\infty}\mathrm{Li}(t)t^{-2}(\ln t)^{-\alpha}\,dt$ and $\int_{p_1}^{\infty}R(t)/t^2\,dt$ are convergent for $\alpha > 0$, we see that $I(\alpha)$ is convergent for $\alpha > 0$. Defining $J(\alpha) = \int_{p_1}^{\infty}\mathrm{Li}(t)t^{-2}(\ln t)^{-\alpha}\,dt$ and using integration by parts we get that

$$J(\alpha+1) = \frac{1}{\alpha}\frac{1}{(\ln p_1)^\alpha}\frac{\mathrm{Li}(p_1)}{p_1} + \frac{1}{\alpha^2}\frac{1}{(\ln p_1)^\alpha} - \frac{1}{\alpha}J(\alpha).$$

This means that

$$\begin{aligned}
\lim_{\alpha\to 0^+}\left[\sum_p\frac{1}{p(\ln p)^\alpha} - \frac{1}{\alpha}\right] &= \lim_{\alpha\to 0^+}\left[I(\alpha) - \frac{1}{\alpha}\right] \\
&= \lim_{\alpha\to 0^+}\left[J(\alpha) - \frac{1}{\alpha}\right] + M - \frac{\mathrm{Li}(p_1)}{p_1} + \ln(\ln p_1) \\
&= \lim_{\alpha\to 0^+}\Bigg[-\alpha J(\alpha+1) + \frac{1}{(\ln p_1)^\alpha}\frac{\mathrm{Li}(p_1)}{p_1} \\
&\quad + \frac{1}{\alpha}\left(\frac{1}{(\ln p_1)^\alpha} - 1\right)\Bigg] + M - \frac{\mathrm{Li}(p_1)}{p_1} + \ln(\ln p_1) = M.
\end{aligned}$$

□

3 Beurling's conjecture

Finally, we shall study a problem posed by Beurling [3]. The problem can be stated as follows:

Which estimation functions $E(x)$ are such that $|N(x) - [x]| < E(x)$ implies that P is in fact the rational primes?

This question is of course still a bit vague, for instance, do we want the estimate to hold for all x, or just for x large enough? Theorem 1.3 however shows that $E(x) = c\ln x$ do not have this property for any $c > 0$, even if we demand it for all $x \ge 1$. This means that if Conjecture 1.2 is true, the question of which x the estimate should hold for, is not essential.

Proof of Theorem 1.3. Given $c > 0$ we choose two rational primes p_i and p_j such that the number $q = p_ip_j/(p_i + p_j - 1)$ satisfies that $\ln q > 8/c$. We want to prove that the system composed of the rational primes without p_i and p_j, but with q instead, satisfies $|N(x) - [x]| < c \ln x$. Let $N_1(x)$ be the number of positive integers relatively prime to $p_i p_j$, this is of course

$$N_1(x) = [x] - \left[\frac{x}{p_i}\right] - \left[\frac{x}{p_j}\right] + \left[\frac{x}{p_i p_j}\right],$$

which can be seen using the principle of inclusion-exclusion. Summing over the different powers of q we get

$$N(x) = \sum_{a=0}^{\infty} N_1\left(\frac{x}{q^a}\right).$$

This means that

$$N(x) - [x] = f\left(\frac{x}{q}\right) - f\left(\frac{x}{p_i}\right) - f\left(\frac{x}{p_j}\right) + f\left(\frac{x}{p_i p_j}\right),$$

where $f(x) = \sum_{a=0}^{\infty} \left[\frac{x}{q^a}\right]$. We can estimate $f(x)$ by

$$f(x) = \sum_{a=0}^{\infty} \left[\frac{x}{q^a}\right] < \sum_{a=0}^{\infty} \frac{x}{q^a} = \frac{xq}{q-1}$$

and

$$\begin{aligned} f(x) &= \sum_{a=0}^{\infty} \left[\frac{x}{q^a}\right] = \sum_{a=0}^{\left[\frac{\ln x}{\ln q}\right]} \left[\frac{x}{q^a}\right] > \sum_{a=0}^{\left[\frac{\ln x}{\ln q}\right]} \left(\frac{x}{q^a} - 1\right) \\ &> \frac{xq}{q-1} - \frac{\ln x}{\ln q} - 1 - \frac{q}{q-1} = \frac{xq}{q-1} - \frac{\ln x}{\ln q} - 2 - \frac{1}{q-1} \\ &\geq \frac{xq}{q-1} - \frac{\ln x}{\ln q} - 4. \end{aligned}$$

Using these estimates, we get

$$\begin{aligned} N(x) - [x] &= f\left(\frac{x}{q}\right) - f\left(\frac{x}{p_i}\right) - f\left(\frac{x}{p_j}\right) + f\left(\frac{x}{p_i p_j}\right) \\ &> \frac{q}{q-1}\left(\frac{x}{q} - \frac{x}{p_i} - \frac{x}{p_j} + \frac{x}{p_i p_j}\right) - \frac{\ln\left(\frac{x}{q}\right)}{\ln q} - 4 - \frac{\ln\left(\frac{x}{p_i p_j}\right)}{\ln q} - 4 \\ &= -\frac{2}{\ln q} \ln x - 7 + \frac{\ln(p_i p_j)}{\ln q} > -\frac{2}{\ln q} \ln x - 5, \end{aligned}$$

and in the same way, $N(x) - [x] < \frac{2}{\ln q} \ln x + 6$. This means that $|N(x) - [x]| < \frac{2}{\ln q} \ln x + 6$. Since $p_i, p_j > q$, we have that $N(x) - [x] = 0$ for $x < q$ and this gives us

$$|N(x) - [x]| < \frac{2}{\ln q} \ln x + \frac{6}{\ln q} \ln x < c \ln x.$$

$\square$

The most natural object to study in order to try to estimate the difference $R(t) = N(t) - [t]$ is the difference between $\zeta_P(s)$ and $\zeta(s)$, where $\zeta(s)$ is the Riemann zeta function. Using Lemma 2.1 we see that

$$\zeta_P(s) - \zeta(s) = s \int_1^\infty \frac{R(t)}{t^{s+1}} dt \tag{2}$$

for $\mathrm{Re}(s) > 1$. We are interested in the case when $R(t)$ is small and in this case the right hand side of (2) is well defined for larger half planes. If we, for instance, assume that $R(t) = O((\ln t)^n)$ for some n, then the right hand side gives an analytic continuation to $\mathrm{Re}(s) > 0$. However, it is not possible to get estimates that would allow us to pass even further to the left. This can be seen through the following proposition:

Proposition 3.1. *Let Q be a system of Beurling primes. We have that $|N(t) - [t]| = o(1)$ if and only if $Q = P$, where P is the rational primes.*

Proof. Let $R(t) = N(t) - [t]$. Since $R(t)$ is integer valued the only possibility to get $R(t) = o(1)$ is if $R(t) = 0$ for all $t \geq t_0$. This implies that the value of all Beurling integers must be rational integers, since if some Beurling integer is not an integer, then there exists arbitrarily large Beurling integers, which are non-integers and that is a contradiction. In the same way we see that no integer can appear as a Beurling integer twice. Using these observations we see that $R(t) = o(1)$ imply $Q = P$ and the other implication is obvious. $\square$

Proposition 3.1 shows that one needs something more than just an estimate on $R(t)$ if one wants to make a meromorphic continuation of $\zeta_P(s)$ to something larger than $\mathrm{Re}(s) > 0$. Thus the line $\mathrm{Re}(s) = 0$ can be thought of as some kind of natural boundary for the zeta functions. The general idea is of course that the further to the left we can push the analytic continuation, the better asymptotics we will get and nice boundary behavior will also lead to good asymptotics. Our main belief is that it is not possible to extend the function $(\zeta_P(s) - \zeta(s))/s$ analytically beyond the line $\mathrm{Re}(s) = 0$ and that the behavior at $\mathrm{Re}(s) = 0$ is never better than an infinite number of simple poles and that this gives the property in Conjecture 1.2.

Let us look at Beurling systems Q constructed as

$$Q = (P \backslash \{p_1, p_2, ..., p_m\}) \bigcup \{q_1, q_2, ..., q_n\}, \tag{3}$$

where P is the set of rational primes and q_j are real numbers larger than 1. It was a system of this kind that we used to show Theorem 1.3 and since we believe this theorem to be sharp, we believe that these systems give the best possible estimates. Similar systems, but where the "added primes" are integers, have been studied by Lagarias in [8]. His main result states that all Beurling prime systems, such that all Beurling integers are integers themselves and such that the system have the so called Delone property, are systems of this type, i.e., the Beurling primes are given by (3), where P is the set of rational primes and q_j are integers. The Delone property is the property that all gaps between two consecutive Beurling integers are bounded and bounded away from zero. In the case of integers, the last property is of course the same as the property that two different Beurling integers always have different values (unique factorization as Lagarias calls it). We should observe that if two different Beurling integers have the same value α, then there are at least $n+1$ different Beurling integers having the value α^n, hence the remainder term is of at least logarithmic growth. This shows that a counterexample to Conjecture 1.2 must consist of Beurling integers which all have different values. One can develop this idea further to show that a counterexample to Conjecture 1.2 must have really bad Diophantine approximation properties, but to show the conjecture in such a way seems very hard. There is an open problem (given by Lagarias) of classifying all Beurling systems with the Delone property. To prove that there are no Beurling systems with Delone property such that some Beurling integer is a non-integer (and thereby answering Lagarias question) seems like a first step towards the kind of Diophantine approximation properties that are needed.

Let us try to illustrate the connection between the boundary behavior of ζ_P and the size of $R(t)$. We use the simple case of when we throw away a finite number of primes and add a finite number of Beurling primes. First of all we may observe that $\zeta_Q(s) = \psi(s)\zeta(s)$, where $\zeta(s)$ is the Riemann zeta function and

$$\psi(s) = \frac{\prod_{i=1}^m \left(1 - \frac{1}{p_i^s}\right)}{\prod_{i=1}^n \left(1 - \frac{1}{q_i^s}\right)}.$$

We see that $\psi(s)$ has poles on $\mathrm{Re}(s) = 0$ with a few exceptions. If there exists an injection j such that for all q_i there is a corresponding $p_{j(i)}$ such that $q_i^{k_i} = p_{j(i)}$ for some positive integer k_i, then $\psi(s)$ has no poles, but otherwise we have an infinite sequence of simple poles on the line $\mathrm{Re}(s) = 0$. Let us try to understand this injection criterion a little better. First of all it is easy to realize that if such an injection exists then $N(x) = Ax + O(1)$. Let us prove this: We let Z be all integers which are products of the primes

$$(P \backslash \{p_1, p_2, ..., p_m\}) \bigcup \{p_{j(1)}, p_{j(2)}, ..., p_{j(n)}\}.$$

Z is nothing but all integers relatively prime to some product of thrown away primes. It is easy to see that the number of elements in Z less than x is $A_1 x + O(1)$. Let B denote the finite set

$$B = \left\{ \prod_{i=1}^n q_i^{l_i}; 0 \le l_i < k_i \right\}.$$

Our full system of Beurling integers is BZ, and this clearly has $Ax + O(1)$ elements less than x. In Lemma 3.2 we will show that $A \neq 1$ if the injection exists.

If j is not injective, then we will have two different Beurling integers with the same value, and by the argument above this gives that $N(x) - Ax$ grows like $C \ln x$ for a sequence of x. The more interesting case is if we add a new Beurling prime q, which is not the k:th root of one of the p_i. Also in this case $R(x) = N(x) - Ax$ will be of the order $C \ln x$ for an infinite sequence of x. To see this we use the inclusion-exclusion principle to write

$$N(x) = \sum_{d | \prod_{i=1}^m p_i} \sum_{k \ge 0} \mu(d) \left[\frac{x}{q^k d} \right],$$

where we have used the multi-index notation $q^k = \prod_{i=1}^n q_i^{k_i}$ and $\mu(d)$ denotes the Möbius function. Since

$$A = \sum_{d|\prod_{i=1}^m p_i} \sum_{k\geq 0} \mu(d)\frac{1}{q^k d} = \sum_{q^k d<x} \mu(d)\frac{1}{q^k d} + O\left(\frac{1}{x}\right),$$

this implies that

$$R(x) = \sum_{q^k d<x} \mu(d)\left\{\frac{x}{q^k d}\right\} + O(1), \tag{4}$$

where $\{y\}$ denotes the fractional part of y. If $q^k = p_i$, the identity $q^k \times d = 1 \times (p_i d)$ gives large cancellation in the sum above, but otherwise we can choose x so that the cancellation is small. Since the sum contains $C(\ln x)^n$ terms asymptotically, we will get rather large $R(x)$ for some x. In other words, in general we have that

$$N(x) - Ax = R(x) = O\left((\ln x)^n\right), \tag{5}$$

but if there are no identities of the form $q^k = p_i$, then for some x, this is a good approximation. Let us make this even simpler, just to illustrate what happens. Let us look at two different Beurling prime systems, in both systems we have thrown away the prime 2, but in the first case we add the prime $\sqrt{2}$ and in the second case we add the prime 4. In the first case we have $\zeta_Q(s) = \zeta(s)(1+2^{-s/2})$, a zeta function without poles on $\mathrm{Re}(s) = 0$ and the Beurling integers are $\mathbb{Z}\cup\sqrt{2}\mathbb{Z}$, which gives $N(x) = [x] + [x/\sqrt{2}]$. In the second case $\zeta_Q(s) = \zeta(s)(1 + 2^{-s})^{-1}$, which have poles for $s = i\pi n/\ln 2$, where n is an odd integer. The Beurling integers are the ordinary integers such that the number of times they are divisible by 2 is even. Equation (4) shows that we have

$$N(x) = \frac{2}{3}x + \sum_{k=0}^{[\log_2 x]} (-1)^k \left\{\frac{x}{2^k}\right\} + O(1).$$

Choosing x written in binary notation as 1010...10 gives $|N(x) - 2/3x| \approx 1/6\log_2 x$.

Let us prove that the zeta functions of systems discussed above have poles on $\mathrm{Re}(s) = 0$ as long as $N(x) = x + o(x)$:

Lemma 3.2. *Let $p_1, p_2, ..., p_m$ be rational primes and let $q_1, q_2, ..., q_n$ be ordinary real numbers larger than one. Define $\psi(s)$ to be*

$$\psi(s) = \frac{\prod_{i=1}^m \left(1 - \frac{1}{p_i^s}\right)}{\prod_{i=1}^n \left(1 - \frac{1}{q_i^s}\right)}$$

and assume that $\psi(1) = 1$ and that $\psi(s) \not\equiv 1$. Then $\psi(s)$ has a pole ρ with $\mathrm{Re}(\rho) = 0$.

Proof. If $n = 0$, then $\psi(1) = 1$ shows that $m = 0$, which gives $\psi(s) \equiv 1$. Thus we may assume that $n > 0$. Furthermore, we assume that $\psi(s)$ has no pole ρ with $\mathrm{Re}(\rho) = 0$. We can without loss of generality assume that $p_i \neq q_j$ for $i = 1, ..., m$ and $j = 1, ..., n$. $\psi(s)$ will have a pole at $s = 2\pi i/\ln q_j$ unless one of the p_i fulfill

$p_i = q_j^k$, where $k \geq 2$. If $p_i = q_j^{k_1} = q_l^{k_2}$ then $(1 - p_i^{-s})(1 - q_j^{-s})^{-1}(1 - q_l^{-s})^{-1}$ will have a pole at $s = 2\pi i k_1 k_2 / \ln p_i$ and therefore we may order p_i in a way so that $p_i = q_i^{k_i}$ for $i = 1, ..., n$ and $k_i \geq 2$. From this it is obvious that we must have $m \geq n$. The condition $\psi(1) = 1$ can now be rewritten as

$$\begin{aligned}\prod_{i=n+1}^{m}\left(1-\frac{1}{p_i}\right)^{-1} &= \prod_{i=1}^{n}\frac{1-\frac{1}{q_i^{k_i}}}{1-\frac{1}{q_i}} = \prod_{i=1}^{n}\left(1+\frac{1}{q_i}+...+\frac{1}{q_i^{k_i-1}}\right)\\ &= \sum_{a_1=0}^{k_1-1}\sum_{a_2=0}^{k_2-1}\cdots\sum_{a_n=0}^{k_n-1}\frac{1}{\prod_{i=1}^{n} q_i^{a_i}}\\ &= \frac{1}{\prod_{i=1}^{n} q_i^{k_i-1}}\sum_{a_1=0}^{k_1-1}\sum_{a_2=0}^{k_2-1}\cdots\sum_{a_n=0}^{k_n-1}\prod_{i=1}^{n} q_i^{a_i}.\end{aligned}$$

We observe that $\prod_{i=1}^{n} q_i^{a_i}$ is the natural basis for the field extension $\mathbb{Q}(q_1, q_2 ..., q_n)$ and in particular the right hand side must be an irrational number. But the left hand side is obviously rational and therefore we have a contradiction and we draw the conclusion of the theorem. □

We have tried to illustrate that a close analysis of the behavior of the zeta functions on the line $\mathrm{Re}(s) = 0$ might be the key to proving Conjecture 1.2.

Proposition 3.3. *There are no Beurling prime number systems such that ζ_P has a pole of order n at $s = 0$ and such that $|N(x) - [x]| = o((\ln x)^n)$.*

Proof. Assume that $|N(x) - [x]| = o((\ln x)^n)$. For $\mathrm{Re}(s) > 0$ we have that

$$\zeta_P(s) = \zeta(s) + s\int_1^\infty \frac{N(t) - [t]}{t^{s+1}}\, dt,$$

and the estimate above and the fact that $\zeta(s)$ does not have a pole at $s = 0$ immediately gives

$$\zeta_P(s) = o\left(s\int_1^\infty \frac{(\ln t)^n}{t^{s+1}}\, dt\right) = o(s^{-n})$$

as $s \to 0^+$. This clearly shows that ζ_P can not have a pole of order n at $s = 0$. □

An analogous calculation shows that if the zeta function of a Beurling system has a pole of order n at $s \in i\mathbb{R}$, then it does not have the property that $|N(x) - [x]| = o((\ln x)^{n-1})$. Unfortunately we can no longer rule out that $|N(x) - [x]| = o((\ln x)^n)$, which is what we want to rule out. To get a better theorem we will have to assume that the number of poles is infinite. However, in some cases Proposition 3.3 is enough, and gives us the estimates we have argued for above more directly:

Corollary 3.4. *Let $Q = (P\backslash\{p_1, p_2, ..., p_m\})\bigcup\{q_1, q_2, ..., q_n\}$, where P is the rational primes, $m < n$ and $q_j > 1$ for $j = 1, 2, ..., n$. Then we have*

$$\limsup_{x\to\infty}\frac{|N(x) - [x]|}{\ln x} > 0.$$

Proof. We have that $\zeta_Q(s) = \psi(s)\zeta(s)$, where $\zeta(s)$ is the Riemann zeta function and

$$\psi(s) = \frac{\prod_{i=1}^{m}\left(1 - \frac{1}{p_i^s}\right)}{\prod_{i=1}^{n}\left(1 - \frac{1}{q_i^s}\right)}.$$

Since $\psi(s)$ has a pole at $s = 0$ and $\zeta(0) \neq 0$ we are done. $\square$

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