Large newforms of the quantized cat map revisited

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Abstract

We study the eigenfunctions of the quantized cat map, desymmetrized by Hecke operators. In the papers [15, 16] it was observed that when the inverse of Planck's constant is a prime exponent $N = p^n$, with $n > 2$, half of these eigenfunctions become large at some points, and half remains small for all points. In this paper we study the large eigenfunctions more carefully. In particular, we answer the question of for which q the L^q norms remain bounded as N goes to infinity. The answer is $q \leqslant 4$.

1 Introduction

The subject of quantum chaos studies the quantum analogs of classical dynamical systems. In particular, much interest concerns chaotic systems and what properties the corresponding quantum system must have or can be expected to have. One of the general heuristic ideas is, that since particles in a chaotic system often move in a space filling manner, the wave function measuring the probability of finding the particle at a specific place must be evenly spread in position space. The wave function is a linear combination of eigenstates corresponding to different energies and the expected behavior is that most eigenstates are nicely spread. We will throughout the paper assume all eigenstates and eigenfunctions to be L^2 normalized, and given this normalization, all eigenstates should be quite similar to each other and to a constant function. To make rigorous statements in the line of this general idea one needs to find some property that measure this. There are several such properties at hand, but maybe the most natural objects to study are the possible limits as the energy goes to infinity (this is often called the semiclassical limit) of subsequences of the induced measures of the eigenstates. For large families of quantum mechanical systems it is known that a sequence of full density, i.e., such that the proportion of elements left out from the sequence tend to zero, of these induced measures converge to the uniform measure [1, 2, 20]. This is known as Schnirelman's theorem and a model having this property is called quantum ergodic. If the sequence of all induced measures tend to the uniform measure the model is called quantum uniquely ergodic. There are also other, more qualitative properties of

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the sequences, which are studied intensively, such including the L^q norms and the Shannon entropy of the eigenstates.

We will study one of the most common "toy models" in quantum chaos, namely the cat map. Given a hyperbolic matrix $A \in SL(2,\mathbb{Z})$ we get a chaotic, time discrete, dynamical system by the mapping taking the point $x \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ to $Ax \in \mathbb{T}^2$ and this is what is known as the cat map. There are several different quantizations of this system [7, 11, 3, 4, 10, 20, 6, 14], but we will follow [12]. In particular this means that our quantized cat map is desymmetrized by Hecke operators, and also that we make the further assumption $A \equiv I \pmod 2$ on the matrix A. This assumption is related to the physical relevance of the model, not to the mathematical truth of our theorems. The quantized cat map is well studied and much is known about the semiclassical limit, which in this model corresponds to letting the integer N, the dimension of the Hilbert space of states, go to infinity. The integer N is often called the inverse of Planck's constant. The most important result about the semiclassical limit is due to Kurlberg and Rudnick and states that the model is uniquely quantum ergodic [12]. It should be noted that the desymmetrization is necessary in the sense that other induced limits exists, if this extra condition is dropped [5].

Also the supremum norms of the eigenstates and the Shannon entropies are asymptotically known as N goes to infinity for our model. The upper bound for the supremum norm of all eigenstates is $O(N^{1/4})$ for "almost all" N (see [15] Theorem 1.1 for the exact formulation) and the lower bound for the Shannon entropy is $1/2 \log N$ [16]. Both these inequalities are sharp, but equality only occurs for special N and very special eigenstates. In short one might say that the number theoretic properties of N starts to come into play. We will restrict to odd N throughout this introduction because it simplifies the expressions below, but the idea is the same also for even N. If $M > 1$ is such that $M^2|N$ then the state space $L^2(\mathbb{Z}_N) = L^2(\mathbb{Z}/N\mathbb{Z}) \cong \mathbb{C}^N$ can be decomposed in two parts, where eigenstates in the first part are called oldforms and eigenstates in the second part are called newforms. The space spanned by the oldforms have dimension N/M^2 and is nothing but blown up images of states corresponding to the Planck's constant N/M^2 living as $N/M-$periodic functions supported on the ideal $M\mathbb{Z}_{N/M} \subset \mathbb{Z}_N$. If $N = M^2$, the unique oldform for M is the function that gives equality in the estimates above.

The newforms are more interesting eigenstates. Here the number theoretic properties of N comes into play again. It is well known that one can use the Chinese remainder theorem to write any eigenstate $\psi_N \in L^2(\mathbb{Z}_N)$ as a tensor product of eigenstates $\psi_{{p_i}^{n_i}} \in L^2(\mathbb{Z}_{{p_i}^{n_i}})$ with $N = \prod_{i=1}^{d} {p_i}^{n_i}$. We therefore formulate the results for $N = p^n$ and the behavior for general N is given directly from the tensor decomposition.

Before we state the known results for the newforms it seems in order to point out that there is a close connection between the newforms and complete character sums over $\mathbb{Z}_N$. Given a specific cat map, the value of a newform can in fact be written as a character sum for "half of all primes" p. These primes are often called split and corresponds to $\left(\frac{D}{p}\right) = 1$, where D is defined in Definition 2.1. Changing point of evaluation corresponds to changing character sum. This was observed by Kurlberg and Rudnick [13] for the case when $N = p$ is a prime, but the construction can be made for $N = p^n$ without any extra effort. More precisely, we have that

$$\psi(b) = \frac{C}{\sqrt{N}} \sum_{y=1}^{N} \chi(y) e\left(\frac{q_b(y)}{p^n}\right),$$

where $|C| = (1 - 1/p)^{-1/2}$, $\chi(y)$ is a multiplicative character on the units of $\mathbb{Z}_N$ and zero for other y and $q_b(y)$ is a second degree polynomial where the linear and constant term is dependent on b. All results stated below applies to this special family of character sums. A general treatment of character sums may be found in Chapter 12 of [9].

For $N = p^n$ with $n = 1$ all eigenstates are uniformly bounded at all points and the same is true for all newforms for $n = 2$. If however $n > 2$, then the set of newforms (defined up to multiplication by scalars) is divided into two parts which are of the same size. In the first part all newforms are uniformly bounded at all points, but in the second part there is a small number of points where the functions are large. At the points where the function is as large as possible, it is of size $p^{[n/3]/2}$, but there are also points where it assumes intermediate values. Since the proportion of points where the function is large goes to zero as p goes to infinity it was proven in [15] that the value distribution is the same for all newforms, when p goes to infinity through the primes. That is, for both large and small newforms, and for any cat map and with any fixed $n \geqslant 2$.

In this paper we study the large newforms and try to get more detailed information about the points where these functions are large. One simple way to characterize this is to study their impact on the L^q norm, i.e., study the size of

$$\|\psi\|_q = \left(\frac{1}{N} \sum_{x \in \mathbb{Z}_N} |\psi(x)|^q\right)^{1/q}.$$

More precisely, we ask the following question: *For which $q > 0$ does the L^q norm of the newforms stay bounded as N goes to infinity?* We will call the supremum of all such q the critical exponent of the sequence. Since the measure of the whole space is one, it is obvious that the norm is bounded for all $q \leqslant 2$, but a priori there is no reason why it is bounded away from zero. However, the theorem about the value distribution (Theorem 1.2) in [15] rules out this possibility since it shows that $|\psi(x)| > 1$ for a positive proportion of $x \in \mathbb{Z}_N$. Another simple calculation shows that the L^q norms of oldforms always go to zero or infinity, depending on if q is less than 2, or larger than 2, respectively (as long as p goes to infinity). With these observations in mind it seems that the only non-trivial question that can be raised concerning the boundedness of L^q norms is for the large newforms. Before we state the theorem we have to make a technical restriction on N. The reason is simply that for a finite set of primes p we have no real definition of what a large newform in $L^2(\mathbb{Z}_{p^n})$ is. Thus we need to bound the impact of these primes trivially. We call primes that divides $\mathrm{tr}(A)^2 - 4$ or that divides the element in the lower left corner of A "bad". We also say that 3 is bad. Fix a positive integer m. We say that N is "good" with respect to m if there is no "bad" prime p such that $p^m | N$. In this language it is fair to say that if N is "good", all newforms are small at the "bad" primes. This is the same assumption that was made in Theorem 1.1 in [15], expect that we now include 3 in the "bad" primes.

Theorem 1.1. *The critical exponent for a sequence of large newforms is* 4. *The L^4 norm is bounded if and only if we restrict to the sequence of N where if $p^n \parallel N$ and the newform corresponding to p^n is large, then n is bounded. For $q > 4$ the L^q norm is bounded by $O_m\left(N^{(1-4/q)/6}\right)$.*

Remark. For a general N, a newform is said to be large, if it is large for some newform in its tensor decomposition.

The question of L^q bounds for eigenfunctions of general operators has been studied intensively. For a typical compact manifold the critical exponent for the eigenfunctions of the Laplace-Beltrami operator is 2, there are no $q > 2$ for which the L^q norm is bounded. More interesting is the question of finding a general upper bound for $\|\psi_\lambda\|_q$ as a function of the eigenvalue λ. This was settled by Sogge [19], who proved that

$$\|\psi_\lambda\|_q = O\left(\lambda^{\delta(q)}\right),$$

where $\delta(q) = 1/8 - 1/(4q)$, for $2 \leqslant q \leqslant 6$ and $\delta(q) = 1/4 - 1/q$, for $6 \leqslant q \leqslant \infty$ for compact manifolds without boundary. This estimate is sharp for some manifolds, the most well-known such example is S^2 with the usual metric. In fact, the result of Sogge is more general than it is stated here and may also be generalized even more, see for instance [18].

Although it is not the general behavior, there are manifolds where a non-trivial critical exponent exists. One such example was provided by Zygmund [21], who proved that the eigenfunctions of the Laplacian for the 2−torus has bounded L^4 norm.

In an arithmetic setting the L^q norms has been discussed by Iwaniec and Sarnak [8]. They studied the eigenfunctions of the Laplace-Beltrami operator for so-called arithmetic surfaces, compact and non-compact, where the eigenfunctions also where assumed to be eigenfunctions of the Hecke operators. They proved that the power $1/4$ in the general upper bound for the L^∞ norm stated above can be replaced by $5/24 + \epsilon$ for these eigenfunctions and also that $\|\psi_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$. The proved rate at which the norm blew up was very slow, but they conjectured, that this was in fact not so far from the truth. More precisely, they conjectured that $\|\psi_\lambda\|_\infty$, and consequently all $\|\psi_\lambda\|_q$ with $2 < q \leqslant \infty$, is bounded by $O\left(\lambda^\epsilon\right)$ for all $\epsilon > 0$. For the most famous arithmetic surface, i.e., the modular surface, there has been some progress on this conjecture. Sarnak and Watson has proven, in a still unpublished paper and assuming the Ramanujan conjectures, that the L^4 norm of the Hecke eigenfunctions is in fact bounded by $O\left(\lambda^\epsilon\right)$. The interested reader will find more details about the arithmetic surfaces in [17].

We will derive Theorem 1.1 from an investigation that in fact gives us a much more detailed description. Assume once more that $N = p^n$ and let $k = [n/2]$. Given a fixed large newform ψ of a fixed quantized cat map, we will for each point $b \in \mathbb{Z}_N$, define the level of the point (see Definition 2.5 for the exact definition). If the level is odd, but not maximal, the value of the newform at the point is zero (see Lemma 5.1), but if the level l is even, $|\psi(b)|$ is of the size $p^{l/4}$ (see Theorem 1.2, Proposition 1.3 and Theorem 1.4 for exact statements). A small newform will only have points of level 0. Let b have level $l = 2s$ for some $l < k$. If $b \equiv b' \pmod{p^{n-k+s}}$ it is easy to see that b' also has level l. The values $\psi(b)$ and $\psi(b')$ are related through the following theorem:

Theorem 1.2. *Let $b \in \mathbb{Z}_{p^n}$ have level $2s$ for some $s < k/2$, where $k = [n/2]$ and assume that $\psi(b) \neq 0$. Then there exists an integer x_0 such that $(p^s x_0)^2 \equiv -C + Db^2 \pmod{p^k}$ and*

$$\psi(b + tp^{n-k+s}) = \frac{p^{s/2}}{\sqrt{1 - \left(\frac{D}{p}\right)\frac{1}{p}}} \left(\alpha_\psi(b) e\left(\frac{x_0 t}{p^{k-2s}}\right) + \beta_\psi(b) e\left(\frac{-x_0 t}{p^{k-2s}}\right)\right)$$

where $|\alpha_\psi(b)| = |\beta_\psi(b)| = 1$ and C and D are given by Definition 2.3 and Definition 2.1, respectively.

Using the theorem it is straight forward to prove value distribution results for the newform at each level individually. We will not formulate this in a theorem, but only observe that all levels behave exactly the same, but on different scales. In other words, they all behave as level 0 and the value distribution of this level is given by Theorem 1.2 in [15].

In order to derive Theorem 1.1 we need to bound the values for all levels l, not only for the levels where $l < k$. In general, we prove the following proposition:

Proposition 1.3. *If b has level $2s$ then*

$$|\psi(b)| \leqslant \frac{2}{\sqrt{1 - \left(\frac{D}{p}\right)\frac{1}{p}}} p^{s/2}.$$

It is easy to calculate the number of points that have a fixed level $l > 0$ (see Lemma 2.1) and Theorem 1.1 follows from the fact that the value of ψ at a point of even level l is of size $p^{l/4}$. In fact, one can calculate the asymptotic relations of each level individually. Fix a normalized newform ψ and let ψ_s be the restriction of ψ to the points of level $2s$, for some $s \leqslant n/3$. In other words, we let

$$\psi_s(x) = \begin{cases} \psi(x) & \text{if } x \text{ has level } 2s \\ 0 & \text{else} \end{cases}.$$

We now have the following theorem:

Theorem 1.4. *If ψ_s denotes the restriction of ψ to the points of level $2s$, for some $s \leqslant n/3$, then $p^{-s(1-4/q)/2}\|\psi_s\|_q$ is uniformly bounded away from both zero and infinity as N goes to infinity.*

That the sequence is bounded follows directly from Proposition 1.3. On the other hand Theorem 1.4 shows that for a positive proportion of points of level $2s$, Proposition 1.3 is not off by more than a constant. Note also that the value of the newform at a point of maximal level does not have to be 0 even if this level is odd. One can formulate an analogous statement for the restriction to maximal level, but we will not do this.

2 Basic assumptions and definitions

The main purpose of this chapter is to introduce the necessary notation and ideas from [12, 15, 16]. We will let $A \in SL(2, \mathbb{Z})$ be the hyperbolic matrix that determines the cat map and assume that $N = p^n$, where $p > 3$ is a prime and

$n \geqslant 2$. We will also use the notation $k = [n/2]$. In order for our quantization to be consistent, we assume that A is congruent to the identity modulo 2. We also assume that the element in the lower left corner of A is invertible modulo N. Having made these assumptions it turns out (or will turn out) that the results are more or less independent of A. One of the reasons for this is that the Hecke operators we will study only depends on the following parameter:

Definition 2.1. Let $D \equiv (\mathrm{tr}(A)^2 - 4)/(4c^2) \pmod{N}$, where c is the element in the lower left corner of A.

Remark. The parameter D says if A is possible to diagonalize modulo N. If D is a non-zero square modulo p, we can diagonalize A. If D is not a square, we can not diagonalize A. Observe also that a matrix is diagonalized simultaneously with A if and only if it commutes with A modulo N.

Definition 2.2. Given $D \in \mathbb{Z}_N$ we let

$$H_D = \left\{ \begin{pmatrix} a & bD \\ b & a \end{pmatrix} ; a, b \in \mathbb{Z}_N \ , \ a^2 - Db^2 = 1 \right\}.$$

Our state space is $L^2(\mathbb{Z}_N) = L^2(\mathbb{Z}/N\mathbb{Z})$ with the inner product

$$\langle \phi, \psi \rangle = \frac{1}{N} \sum_{x \in \mathbb{Z}_N} \phi(x) \overline{\psi(x)}.$$

A quantization, like the quantized cat map, assigns to every smooth real valued function of the classical phase space (this is usually called an observable) an Hermitian operator acting on the state space. It also prescribes how the system evolves in time through a unitary operator called the quantum propagator. We will denote the quantum propagator for the quantized cat map by $U_N(A)$. The most important properties of $U_N(A)$ is that it is well defined for $A \in SL(2, \mathbb{Z}_N)$, that the function mapping A to $U_N(A)$ is a representation of $SL(2, \mathbb{Z}_N)$, and that it has the so-called exact Egorov property (see [12] page 48). The representation is known as the Weil representation and it can be defined by explicit formulas for the generators of the group, see [12] Section 4.3, although there are more natural ways to introduce this representation in more abstract settings. The element r occurring in the formulas is nothing but the inverse of 2 modulo N and we will also use this notation.

The Hecke operators corresponding to A and N are the commutative group of operators $U_N(g)$, such that $g = xI + yA$, $x, y \in \mathbb{Z}$ and $\det(g) \equiv 1 \pmod{N}$. However one may, without loss of generality, assume that the Hecke operators are given by $\{U_N(h); h \in H_D\}$ (see [16] page 1060) and we will do this. In H_D there is a neighborhood of the identity which forms a cyclic subgroup of order p^k, namely $\{h \in H_D; h \equiv I \pmod{p^{n-k}}\}$. The joint eigenfunctions of the Weil representation restricted to this subgroup form subspaces of $L^2(\mathbb{Z}_N)$ which were called V_C with the following convention in [15, 16]:

Definition 2.3. For $C \in \mathbb{Z}_{p^k}$ we say that $\psi \in V_C$ if

$$U_N \begin{pmatrix} 1 & 2p^{n-k}D \\ 2p^{n-k} & 1 \end{pmatrix} \psi = e\left(\frac{C}{p^k}\right) \psi. \tag{1}$$

Different Hecke eigenfunctions in V_C are very similar, but we also introduced the notation $\widetilde{C}$ which was more detailed than C and eigenfunctions having the same $\widetilde{C}$, where even more similar.

Definition 2.4. Let $\alpha = p^{\lceil k/2 \rceil}$. Given a Hecke eigenfunction ψ we define $\widetilde{C} \in \mathbb{Z}$ to be the integer in the intervall $[1, p^{n-\lceil k/2 \rceil}]$ such that

$$U_N \begin{pmatrix} 1+2D\alpha^2 & (2\alpha + D\alpha^3)\, D \\ 2\alpha + D\alpha^3 & 1+2D\alpha^2 \end{pmatrix} \psi = e\left(\frac{\widetilde{C}}{p^{n-\lceil k/2 \rceil}}\right)\psi. \tag{2}$$

Let B denote the matrix appearing in the left hand side of (2) and B' denote the matrix appearing in the left hand side of (1). An easy calculation shows that $B^m \equiv B' \pmod N$ for $m = p^{n-k}/\alpha$ and this implies that $C \equiv \widetilde{C} \pmod{p^k}$.

One of the main observations in [15, 16] was that the value of a Hecke eigenfunction ψ at the point b can be written as an exponential sum over the solutions to the equation $x^2 \equiv -C + Db^2 \pmod{p^k}$. How many solutions this equation has is more or less determined by how many times p divides $-C + Db^2$. A natural guess, which turns out to be more or less correct, is that the number of terms in the exponential sum determines the size of the sum. However, for the b such that $-C + Db^2 \equiv 0 \pmod{p^k}$ there are different sizes of cancellations for different b, so in this case we need to be careful with the definition of the level.

Definition 2.5. Fix a Hecke eigenfunction $\psi \in V_C$ and let $b \in \mathbb{Z}_N$. For even n we define the level l at b to be the largest integer $l \leqslant (2n+2)/3$ such that $p^l | -\widetilde{C} + Db^2 - p^{k+(k-2[k/2])}3^{-1}rCD$ and for odd n we define the level l at b to be the largest integer $l \leqslant (2n+2)/3$ such that $p^l | -\widetilde{C} + Db^2 - p^{k+(k-2[k/2])}3^{-1}CD$. If $l = [(2n+2)/3]$ we say that b has maximal level.

Remark. The levels $l < k$ can be defined by the simpler formula $p^l \parallel -C + Db^2$. Also note that if C/D is a quadratic non-residue modulo p, then all b have level 0. One can show that the level is independent of the representative of $\widetilde{C}$ modulo $p^{n-\lceil k/2 \rceil}$.

We will mostly discuss even levels and for those we will use the notation $l = 2s$. This is the same s as appears in [15, 16]. The following notation (already used in Theorem 1.2) was also used in these papers:

Definition 2.6. For a point b of level $2s < k$ we let x_0 be some integer such that $(p^s x_0)^2 \equiv -C + Db^2 \pmod{p^k}$ if such an integer exists.

Remark. Note that $p \nmid x_0$.

Definition 2.7. If C and D are both invertible modulo N and C/D is a quadratic residue, then a function $\psi \in V_C$ which is a joint eigenfunction for all $U_N(h)$ with $h \in H_D$ is called a large newform.

Remark. That C and D are invertible and C/D is a square modulo N is equivalent to the same statement modulo p.

Since the aim of the paper is to study the large newforms we will from now on assume that $p \nmid C, D$ and that C/D is a square modulo p.

Lemma 2.1. *Let* $(2n-1)/3 \geqslant l > 0$. *The number of* $b \in \mathbb{Z}_N$ *of level* l *is*

$$\frac{2N}{p^l}\left(1-\frac{1}{p}\right).$$

The number of b *of level* 0 *is* $N(1-2/p)$ *and the number of* b *of maximal level* $l_{\max} = [(2n+2)/3]$ *is* $2N/p^{l_{\max}}$.

Proof. Let us first assume that $(2n-1)/3 \geqslant l > 0$. The condition that b have level l can be written on the form $b^2 \equiv E \pmod{p^l}$, but $b^2 \not\equiv E \pmod{p^{l+1}}$, where E is easily derived from Definition 2.5. We observe that $E \equiv C/D \pmod{p}$ and therefore $b^2 \equiv E \pmod{p^l}$ has 2 solutions modulo p^l. This shows the total number of points in $\mathbb{Z}_N$ of level l to be $2N/p^l - 2N/p^{l+1}$. The number of b of level 0 is $N - 2N/p$ and the number of b of level $l_{\max}$ is $2N/p^{l_{\max}}$ by the same arguments. □

3 Exponential sums

As we have mentioned above, the value of a Hecke eigenfunction may be calculated as the value of an exponential sum. The calculations focus on finding the absolute value of the following object in the special case when q is of degree three:

Definition 3.1. Let m be a nonnegative integer. For $q \in \mathbb{Z}_{p^m}[x]$ we define

$$S(q,m) = \sum_{z=1}^{p^m} e\left(\frac{q(z)}{p^m}\right).$$

In [16] the following three lemmata (Lemma 6.1, Lemma 6.2 and Lemma 6.3) were developed:

Lemma 3.1. *Let* $q(z) = a_3z^3 + a_2z^2 + a_1z + a_0$ *and assume that* $p|a_3$ *but* $p \nmid a_2$. *Then* $|S(q,m)| = p^{m/2}$.

Lemma 3.2. *Let* $q(z) = a_3z^3 + a_1z + a_0$ *and assume that* $p \nmid a_3$ *and that* $p^2 \nmid a_1$. *Then* $|S(q,m)| \leqslant 2p^{m/2}$.

Lemma 3.3. *Let* $q(z) = a_3z^3 + p^2a_1z + a_0$ *and assume that* $p \nmid a_3$. *For* $m \geqslant 3$ *we have that* $|S(q,m)| = p^2|S(q_1, m-3)|$, *where* $q_1(z) = a_3z^3 + a_1z$.

We will also need a special case of Lemma 6.4 from the same paper. For the readers convenience we only formulate the part of the statement we need:

Lemma 3.4. *Let* $q(z) = a_3z^3 + a_1z + a_0$ *and assume that* $p \nmid a_3$ *and that* $m \leqslant 2$. *Then* $|S(q,m)| \leqslant 2p^{m/2}$.

As a final recollection of the work in [16], we observe that in the proof of Lemma 3.2, the special case $d = 1$ was shown for the following lemma:

Lemma 3.5. *Let* $q(z) = a_3z^3 + a_1z$, *assume that* $p \nmid a_3$ *and that* $p^d \parallel a_1$. *If* d *is odd and* $(3d+1)/2 \leqslant m$ *we have that* $S(q,m) = 0$.

Proof. By the remark above we may assume that $d \geqslant 3$. We use Lemma 3.3 $(d-1)/2$ times (observe that d is small enough for this) and get $|S(q,m)| = p^{d-1}|S(q_1, m-3(d-1)/2)|$, where q_1 is a polynomial of the same form as q and with $d' = 1$. The lemma now follows from the fact that $S(q_1, m-3(d-1)/2)$ fulfills the lemma. Observe in particular that $m - 3(d-1)/2 \geqslant 2 = (3d'+1)/2$. $\square$

Lemma 3.6. *Let $q(z) = a_3 z^3 + a_1 z$, assume that $p \nmid a_3$ and that $p^d \parallel a_1$. If d is even $|S(q,m)| \leqslant 2p^{(2m+d)/4}$.*

Proof. We use Lemma 3.3 $c = \min(d/2, [m/3])$ times and get $|S(q,m)| = p^{2c}|S(q_1, m-3c)|$, where q_1 is a polynomial of the same form as q and either $d = 0$ or $m - 3c \leqslant 2$. According to Lemma 3.2 and Lemma 3.4 we get $|S(q,m)| \leqslant p^{2c} 2p^{(m-3c)/2} = 2p^{(m+c)/2} \leqslant 2p^{(2m+d)/4}$. $\square$

Remark. Note that it is obvious from the proof that we always have the estimate $|S(q,m)| \leqslant 2p^{(m+[m/3])/2}$.

Lemma 3.7. *Let $S_m(t) = S(a_3 z^3 + tz, m)$ where $p \nmid a_3$. There exists a constant $C > 0$ such that if $0 \leqslant d \leqslant 2m/3$ is even, then a positive proportion of all $t \in \mathbb{Z}_{p^{m-d}}$ fulfill $|S_m(p^d t)| > Cp^{(2m+d)/4}$.*

Proof. Define the inner product in $L^2(\mathbb{Z}_{p^m})$ to be

$$\langle \phi, \psi \rangle = \frac{1}{p^m} \sum_{z \in \mathbb{Z}_{p^m}} \phi(z)\overline{\psi(z)}.$$

Observe that $\left\{ e\left(\frac{-tz}{p^m}\right) \right\}$ is an orthonormal basis for $L^2(\mathbb{Z}_{p^m})$. This leads to

$$1 = \left\| e\left(\frac{a_3 z^3}{p^m}\right) \right\|_2^2 = \sum_{t \in \mathbb{Z}_{p^m}} \left| \left\langle e\left(\frac{a_3 z^3}{p^m}\right), e\left(\frac{-tz}{p^m}\right) \right\rangle \right|^2 = \sum_{t \in \mathbb{Z}_{p^m}} \left| \frac{1}{p^m} S_m(t) \right|^2$$
$$= \frac{\|S_m\|_2^2}{p^m}.$$

In other words, the sum of $|S_m(t)|^2$ is p^{2m}. According to Lemma 3.3 we have that $|S_m(p^2 t)| = p^2 |S_{m-3}(t)|$ for $m \geqslant 3$ and according to Lemma 3.5 $S_m(pt) = 0$ for $p \nmid t$ and $m \geqslant 2$. This shows that the sum over all $|S_m(t)|^2$ with $p \nmid t$ is $p^{2m} - p^{2m-1} \geqslant \frac{4}{5}p^{2m}$. On the other hand $|S_m(t)|^2 \leqslant 4p^m$ for these t by Lemma 3.2. Thus, for more than $p^m/7$ of all $t \in \mathbb{Z}_{p^m}$ we must have the estimate $|S_m(t)|^2 \geqslant p^m/4$. For $m \leqslant 2$ Lemma 3.4 shows the estimate $|S_m(t)|^2 \leqslant 4p^m$ for all t and we get our estimate by comparing this with the sum of all $|S_m(t)|^2$. This finishes the argument for $d = 0$.

Since $d \leqslant 2m/3$ we may use Lemma 3.3 $d/2$ times and get $|S_m(p^d t)| = p^d |S_{m-3d/2}(t)| > p^d C p^{(m-3d/2)/2} = Cp^{(2m+d)/4}$ for a positive proportion of $t \in \mathbb{Z}_{p^{m-d}}$. $\square$

4 Evaluating newforms

In [15, 16] methods for evaluating newforms were developed. The main ingredient in the proofs of the main theorems in this paper, is the use of those methods. This chapter serves to recapitulate the main ideas of this evaluation procedure.

We let δ_x denote the function defined on $\mathbb{Z}_N$ which is 1 at x and 0 else. The following functions play an important role in the evaluation of the newforms:

Definition 4.1. Given $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{Z}_N^2$ we let $\zeta_x : \mathbb{Z}_N \to \mathbb{C}$ be defined by

$$\zeta_x = \sum_{t \in \mathbb{Z}_{p^k}} e\left(\frac{x_1 t}{p^{n-k}}\right) \delta_{x_2 + p^k t}.$$

Remark. In [15] these functions were called $\zeta_{0,x}$, but the extra index is not necessary in our presentation and we will therefore be omitted.

The reason that we study these functions is that they transform in a simple manner when we apply $U_N(B)$ to them. For our purposes the following statement, which combines Lemma 5.1 in [16] and parts of Lemma 3.1 in [15], will suffice:

Lemma 4.1. *Let $B \in SL(2, \mathbb{Z}_N)$ and if n is odd we also assume that $B \equiv I \pmod{p}$. If $x' = Bx$ then*

$$U_N(B)\zeta_x = e\left(\frac{r(x_1' x_2' - x_1 x_2)}{N}\right) \zeta_{x'}.$$

From this lemma one may observe that ζ_x is an eigenfunction of $U_N(h)$ as long as $h \equiv I \pmod{p^{n-k}}$, or in other words, each function ζ_x belongs to some space V_C. In fact, $\zeta_x \in V_C$ if and only if $x_1^2 - Dx_2^2 \equiv -C \pmod{p^k}$. Moreover, we can calculate the coefficients for a given Hecke eigenfunction ψ in some basis of ζ_x functions.

Lemma 4.2. *Let $\psi \in V_C$ be a normalized Hecke eigenfunction and assume that p does not divide C or D. If n is odd we also assume that $p \nmid x_2$. Then*

$$|\langle \psi, \zeta_x \rangle| = \begin{cases} \frac{1}{\sqrt{N\left(1 - \left(\frac{D}{p}\right)\frac{1}{p}\right)}} & \text{if } x_1^2 - Dx_2^2 \equiv -C \pmod{p^k} \\ 0 & \text{if } x_1^2 - Dx_2^2 \not\equiv -C \pmod{p^k} \end{cases}.$$

Remark. For odd n this is a part of the statement of Lemma 3.3 in [15] and for even n it follows from Lemma 4.1 since all elements x such that $x_1^2 - Dx_2^2 \equiv -C \pmod{p^k}$ forms an orbit of the group H_D.

Fixing a point $b \in \mathbb{Z}_N$, we may now observe that for most of the functions ζ_x, the value at b is zero. In fact, Lemma 4.2 shows that $\psi(b)$ is a sum over all solutions to the equation $x^2 \equiv -C + Db^2 \pmod{p^k}$. Note that this is a very short sum in this context. In particular, if $x^2 \equiv -C + Db^2 \pmod{p^k}$ has no solutions, then $\psi(b) = 0$.

If $x^2 \equiv -C + Db^2 \pmod{p^k}$ has some solutions (but not the maximal number) Theorem 5.4 in [16] and Theorem 3.5 in [15] tells us how to evaluate $\psi(b)$. These theorems can be summarized by the following theorem:

Theorem 4.3. *Let $\psi \in V_C$ be a normalized Hecke eigenfunction and assume that p does not divide C or D. Let $b \in \mathbb{Z}_N$ and assume that the equation $x^2 \equiv -C + Db^2 \pmod{p^k}$ has the solutions $x \equiv \pm x_0 p^s + p^{k-s}\mathbb{Z}_{p^s} \pmod{p^k}$ for some x_0 and s such that $p \nmid x_0$ and $0 \leqslant s < k/2$.*

If n is even then

$$\psi(b) = \frac{1}{\sqrt{1 - \left(\frac{D}{p}\right)\frac{1}{p}}} \left(\alpha_\psi(b) \sum_{z=1}^{p^s} e\left(\frac{q_+(z)}{p^s}\right) + \beta_\psi(b) \sum_{z=1}^{p^s} e\left(\frac{q_-(z)}{p^s}\right) \right), \tag{3}$$

and if n is odd and $p \nmid b$ then

$$\psi(b) = \frac{1}{\sqrt{p - \left(\frac{D}{p}\right)}} \left(\alpha_\psi(b) \sum_{z=1}^{p^{s+1}} e\left(\frac{q_+(z)}{p^{s+1}}\right) + \beta_\psi(b) \sum_{z=1}^{p^{s+1}} e\left(\frac{q_-(z)}{p^{s+1}}\right) \right). \tag{4}$$

Here $q_\pm(z) = r\left(\Theta_\psi(b)z \pm x_0 Dbz^2 + p^{k-2s}3^{-1}D^2b^2z^3\right)$, $|\alpha_\psi(b)| = |\beta_\psi(b)| = 1$ and the function $\Theta_\psi(b)$ is given by

$$\Theta_\psi(b)p^k \equiv -x_0^2 p^{2s} - \tilde{C} + Db^2 - p^{2(k-s)}3^{-1}rD^2b^2 \pmod{p^{n-k+s}}.$$

The careful reader may have observed that the case when n is odd and $p|b$ was left out from the theorem. However, that case will always be covered by Theorem 3.4 in [15], which states that:

Theorem 4.4. *Assume that n is odd and let $\psi \in V_C$ be a normalized Hecke eigenfunction. If p does not divide C or D and b fulfills that $-C + Db^2 \equiv x_0^2 \pmod{p^k}$ for some $p \nmid x_0$, then*

$$\psi(b) = \frac{1}{\sqrt{1 - \left(\frac{D}{p}\right)\frac{1}{p}}} (\alpha_\psi(b) + \beta_\psi(b)).$$

α_ψ and β_ψ are functions satisfying $|\alpha_\psi(b)| = |\beta_\psi(b)| = 1$, $\alpha_\psi(b + p^{k+1}t) = e\left(\frac{x_0 t}{p^k}\right)\alpha_\psi(b)$ and $\beta_\psi(b + p^{k+1}t) = e\left(\frac{-x_0 t}{p^k}\right)\beta_\psi(b)$.

Finally, we turn to the case when the equation $x^2 \equiv -C + Db^2 \pmod{p^k}$ has the maximal number of solutions, i.e., when $-C + Db^2 \equiv 0 \pmod{p^k}$. In this case Theorem 5.5 in [16] and Theorem 3.6 in [15] can be summarized in the following way:

Theorem 4.5. *Let $\psi \in V_C$ be a normalized Hecke eigenfunction and assume that p does not divide C or D. Let $b \in \mathbb{Z}_N$ and assume that $-C + Db^2 \equiv 0 \pmod{p^k}$. If n is even then*

$$\psi(b) = \frac{\alpha_\psi(b)}{\sqrt{1 - \left(\frac{D}{p}\right)\frac{1}{p}}} \sum_{z=1}^{p^{[k/2]}} e\left(\frac{q(z)}{p^{[k/2]}}\right), \tag{5}$$

where $q(z) = r\left(\Theta_\psi(b)z + p^{k-2[k/2]}3^{-1}CDz^3\right)$, $|\alpha_\psi(b)| = 1$ and

$$\Theta_\psi(b)p^k \equiv -\tilde{C} + Db^2 - p^{k+(k-2[k/2])}3^{-1}rCD \pmod{p^{[3k/2]}}.$$

If n is odd then

$$\psi(b) = \frac{\alpha_\psi(b)}{\sqrt{p - \left(\frac{D}{p}\right)}} \sum_{z=1}^{p^{[k/2]+1}} e\left(\frac{q(z)}{p^{[k/2]+1}}\right), \tag{6}$$

where $q(z) = \Theta_\psi(b)z - p^{k-2[k/2]}3^{-1}2CDz^3$, $|\alpha_\psi(b)| = 1$ *and*

$$\Theta_\psi(b)p^k \equiv -\widetilde{C} + Db^2 - p^{2[k/2]}3^{-1}CD \pmod{p^{[3k/2]+1}}.$$

5 The proofs of the main theorems

Lemma 5.1. *If the level* $l \leqslant (2n-1)/3$ *of* b *is odd, we have that* $\psi(b) = 0$.

Proof. If the level l is less than k, the equation $x^2 \equiv -C + Db^2 \pmod{p^k}$ has no solutions and so we are done. If $k \leqslant l$, $\psi(b)$ is given by Theorem 4.5. We have to treat the cases n even and n odd separately. Assume first that n is even. In this case $\psi(b)$ is given by (5), recall in particular that

$$\Theta_\psi(b)p^k \equiv -\widetilde{C} + Db^2 - p^{k+(k-2[k/2])}3^{-1}rCD \pmod{p^{[3k/2]}}. \tag{7}$$

The right hand side of (7) is divisible by p exactly l times, thus $p^{l-k} \parallel \Theta_\psi(b)$. The coefficient in front of z^3 is divisible by p exactly $k - 2[k/2]$ times and this number is always less than or equal to $l - k$. Hence, the exponential sum is actually an exponential sum over $\mathbb{Z}_{p^{3[k/2]-k}}$, where the coefficient in front of z^3 is invertible and the linear term is divisible by p exactly $l - 2k + 2[k/2]$ times. This number is odd and the claim now follows from Lemma 3.5 since

$$\frac{3(l - 2k + 2[k/2]) + 1}{2} \leqslant 3[k/2] - k.$$

Let us now assume that n is odd. In this case $\psi(b)$ is given by (6), where this time

$$\Theta_\psi(b)p^k \equiv -\widetilde{C} + Db^2 - p^{k+(k-2[k/2])}3^{-1}CD \pmod{p^{[3k/2]+1}}. \tag{8}$$

The same reduction as above gives a sum over $\mathbb{Z}_{p^{3[k/2]-k+1}}$, where the coefficient in front of z^3 is invertible and the linear coefficient is divisible by p exactly $l - 2k + 2[k/2]$ times. The claim now follows from Lemma 3.5 since

$$\frac{3(l - 2k + 2[k/2]) + 1}{2} \leqslant 3[k/2] - k + 1.$$

□

The idea behind the proof of Theorem 1.2 is that, in some sense, Theorem 1.2 is a special case of Theorem 4.3. Unfortunately, it is not so easy to read off our theorem from Theorem 4.3 and in reality a better description of the proof of Theorem 1.2 is that it is a close analysis of the proofs of Theorem 5.4 in [16] and Theorem 3.5 in [15]. The notation is these proofs are not well equipped for our situation and for that reason we will rewrite the proofs completely.

Proof of Theorem 1.2. We fix $b \in \mathbb{Z}_N$ and study the values of $\psi(b')$ for $b' = b + tp^{n-k+s}$. Theorem 4.4 proves the theorem for n odd and $s = 0$ and we may therefore assume $p \nmid b$ for odd n. By Lemma 4.2 we know that we can write $\psi = R\sum a_x\zeta_x$, where the sum is taken over all $x \in \mathbb{Z}_{p^{n-k}} \times \mathbb{Z}_{p^k}$ such that $x_1^2 - Dx_2^2 \equiv -C \pmod{p^k}$. Actually, the functions ζ_x are dependent on the exact representative of x_2 we choose, but we will postpone the exact choice of representative for now. If we let

$$R = \frac{1}{\sqrt{p^{n-2k}\left(1-\left(\frac{D}{p}\right)\frac{1}{p}\right)}}$$

the coefficients a_x (where $p \nmid x_2$ if n is odd) will have absolute value 1. Note that

$$\psi(b') = R \sum a_x \zeta_x(b'), \tag{9}$$

where the sum is over all x such that $x_2 \equiv b \pmod{p^k}$ and x_1 solves the equation $x_1^2 \equiv -C + Db^2 \pmod{p^k}$. Since b has level $2s$ and $\psi(b) \neq 0$ we know that there must exist an integer x_0, not divisible by p, such that $(x_0 p^s)^2 \equiv -C + Db^2 \pmod{p^k}$. The solutions to $x_1^2 \equiv -C + Db^2 \equiv (x_0 p^s)^2 \pmod{p^k}$ are given by $x_1 \equiv \pm x_0 p^s \pmod{p^{k-s}}$. Let us now define

$$B(s) = \begin{pmatrix} 1 + rDp^{2(k-s)} & p^{k-s}D \\ p^{k-s} & 1 + rDp^{2(k-s)} \end{pmatrix}.$$

By induction it is easy to show that

$$B(s)^z = \begin{pmatrix} 1 + rDz^2p^{2(k-s)} & \left(p^{k-s}z + 3^{-1}rDp^{3(k-s)}(z^3 - z)\right) D \\ p^{k-s}z + 3^{-1}rDp^{3(k-s)}(z^3 - z) & 1 + rDz^2p^{2(k-s)} \end{pmatrix}.$$

Let $m = n - 2k + s$. We observe that the two orbits

$$\left\{ B(s)^z \begin{pmatrix} x_0 p^s \\ b \end{pmatrix} ; z = 0, 1, ..., p^m - 1 \right\}$$

and

$$\left\{ B(s)^z \begin{pmatrix} -x_0 p^s \\ b \end{pmatrix} ; z = 0, 1, ..., p^m - 1 \right\}$$

corresponds exactly to the x in (9). We may therefore choose these representatives for x in (9). In other words, we introduce $\zeta_{\pm,z} = \zeta_{B(s)^z \left(\begin{smallmatrix} \pm x_0 p^s \\ b \end{smallmatrix}\right)}$ and write

$$\psi(b') = R \left(\sum_{z=0}^{p^m-1} a_{+,z}\zeta_{+,z}(b') + \sum_{z=0}^{p^m-1} a_{-,z}\zeta_{-,z}(b') \right).$$

Since $\zeta_x(b') = e\left(\frac{x_1 t}{p^{k-s}}\right)\zeta_x(b)$ this can be reduced to

$$\psi(b') = R \left(e\left(\frac{x_0 t}{p^{k-2s}}\right) \sum_{z=0}^{p^m-1} a_{+,z}\zeta_{+,z}(b) + e\left(\frac{-x_0 t}{p^{k-2s}}\right) \sum_{z=0}^{p^m-1} a_{-,z}\zeta_{-,z}(b) \right). \tag{10}$$

The task is now to show that

$$a_{\pm,z}\zeta_{\pm,z}(b) = a_{\pm,0}\, e\left(\frac{q_\pm(z)}{p^m}\right), \tag{11}$$

where $q_\pm(z) = r\left(\Theta_\psi(b)z \pm x_0Dbz^2 + p^{k-2s}3^{-1}D^2b^2z^3\right)$ is the polynomial occurring in Theorem 4.3. This follows from Lemma 4.1 together with the definition of ζ_x, remembering that ψ is a Hecke eigenfunction. (11) was shown in the proofs of Theorem 5.4 in [16] and Theorem 3.5 in [15], and we therefore omit the (quite tedious) exact calculations proving this statement.

At this point we have reproved Theorem 4.3 (combine (10) and (11)), also showing that the functions α_ψ and β_ψ appearing in the equations (3) and (4) fulfill

$$\alpha_\psi(b + tp^{n-k+s}) = \alpha_\psi(b)e\left(\frac{x_0 t}{p^{k-2s}}\right) \text{ and } \beta_\psi(b + tp^{n-k+s}) = \beta_\psi(b)e\left(\frac{-x_0 t}{p^{k-2s}}\right).$$

To conclude our proof we must show that the exponential sums in (3) and (4) have absolute value $p^{s/2}$ and $p^{(s+1)/2}$ respectively. If $s = 0$ this is trivial for even n and otherwise we must have $p \nmid b$ also in the even case. The claim now follows immediately from Lemma 3.1. □

Proof of Proposition 1.3. This follows immediately from Theorem 1.2 for $s < k/2$. For $s \geqslant k/2$ we have to analyze Theorem 4.5. If we let $q(z)$ denote the polynomial in this theorem we have to show that $|S(q, [k/2])| \leqslant 2p^{s/2}$ for even n and that $|S(q, [k/2]+1)| \leqslant 2p^{(s+1)/2}$ for odd n. The level is defined in such a way that p^{2s-k} divides the coefficient in the linear term of the polynomial. Assume that k is even. For such k the polynomial fulfills the assumptions of Lemma 3.6, thus for even n we get $|S(q, [k/2])| = |S(q, k/2)| \leqslant 2p^{(2k/2+2s-k)/4} = 2p^{s/2}$ and similarly for odd n we get $|S(q, [k/2] + 1)| = |S(q, k/2 + 1)| \leqslant 2p^{(2(k/2+1)+2s-k)/4} = 2p^{(s+1)/2}$. If on the other hand we assume k to be odd, we know that both the linear coefficient (since this was divisible by p^{2s-k}) and the third degree coefficient is divisible by p. Thus we may cancel one p in the numerators and denominators to arrive at p shorter exponential sums of length $p^{[k/2]-1}$ and $p^{[k/2]}$ respectively. We now see that the new polynomial q_1 fulfills the assumption of Lemma 3.6 and the linear coefficient is divisible by p at least $2s-k-1$ times. If n is even we get $|S(q, [k/2])| = p|S(q_1, (k-3)/2)| \leqslant 2p^{1+(2(k-3)/2+2s-k-1)/4} = 2p^{s/2}$ and if n is odd we get $|S(q, [k/2] + 1)| = p|S(q_1, (k-1)/2)| \leqslant 2p^{1+(2(k-1)/2+2s-k-1)/4} = 2p^{(s+1)/2}$. □

Proof of Theorem 1.4. Let ψ_s be the restriction of ψ to the points of level $2s$, for some s such that $s \leqslant n/3$. The theorem is obvious for $s = 0$. Let us now assume that the level is $0 < l = 2s < k$ and take a point b' of level l. We study the values of $-C + Db^2$, for $b = b' + tp^{2s}$ as t runs though $\mathbb{Z}_p$. Since $2Db'$ is invertible we see that $-C + D(b' + tp^{2s})^2$ runs through all values modulo p^{2s+1} that are congruent to zero modulo p^{2s}. Of these numbers, $(p-1)/2$ are squares, $(p-1)/2$ are non-squares and the last number is zero, saying that the corresponding point b has higher level than b'. For the points b corresponding to non-squares $\psi(b) = 0$ and for b corresponding to squares we can use Theorem 1.2 (actually even if it were to happen that $\psi(b) = 0$). In other words, for exactly half the points of level $0 < l = 2s < k$ we may use Theorem 1.2 and for the other half $\psi(b) = 0$. Since

$$\alpha_\psi(b)e\left(\frac{x_0 t}{p^{k-2s}}\right) + \beta_\psi(b)e\left(\frac{-x_0 t}{p^{k-2s}}\right)$$

is bounded and also bounded away from zero for most t, it is easy to see that $p^{-s(q-4)/2}\|\psi_s\|_q^q$ is bounded and also bounded away from zero, by combining this observation with Lemma 2.1. This shows the theorem for levels $l < k$.

Now assume $l = 2s \geqslant k$. We can no longer use Theorem 1.2, but Proposition 1.3 works just as well for the upper bound. To prove the lower bound we once more have to study the expressions from Theorem 4.5. Let us therefore review the proof of Lemma 5.1: In our new case, l is even, which makes $l - 2k + 2[k/2]$ even. We want to use Lemma 3.7 and in order to do this we note that

$$3/2(l - 2k + 2[k/2]) \leqslant 3[k/2] - k$$

for even n and

$$3/2(l - 2k + 2[k/2]) \leqslant 3[k/2] - k + 1$$

for odd n. Fix a point b' and study $b = b' + tp^k$ for $t \in \mathbb{Z}_{p^{n-k}}$. The polynomial $q(z)$ is of the form $q(z) = a_3 z^3 + a_1 z$ where a_3 is fixed (but dependent on the parity of k and n) and a_1 changes with b. Observe that $\Theta(b' + tp^k) - \Theta(b') = 2Db't$ (see (7) and (8)) and since $p \nmid 2Db'$, this shows that a_1 takes all values of $\mathbb{Z}_{p^{[k/2]}}$ and $\mathbb{Z}_{p^{[k/2]+1}}$ respectively, the same number of times. The theorem now follows from Lemma 3.7. $\square$

Proof of Theorem 1.1. We first note that since $\|\psi_1 \otimes \psi_2\|_q = \|\psi_1\|_q \|\psi_2\|_q$ it is enough to study the case where $N = p^n$ and $p > 3$ is such that D and the lower left element in A is invertible modulo p. If $p|C$ we know that D is a square and by Proposition 4.4 in [15] we see that the newforms are uniformly bounded at all points. If $p \nmid C$ and C/D is not a square, we get the same conclusion from Theorem 4.2 in [15]. The only thing left to study is the large newforms, i.e., where C/D is a non-zero square modulo p.

Let ψ_s be the restriction of the large newform ψ to the points of level $2s$ for some $s < n/3$ and let $\psi_{\max}$ be the restriction to the points of maximal level. Obviously, $\|\psi\|_q^q$ is the sum of all $\|\psi_s\|_q^q$. By Theorem 1.4 we know that $\|\psi_s\|_q^q \approx p^{s(q-4)/2}$, where $\|\psi_s\|_q^q \approx p^{s(q-4)/2}$ should be understood in the sense that the $p^{-s(q-4)/2}\|\psi_s\|_q^q$ is uniformly bounded away from both zero and infinity as N goes to infinity. According to Theorem 4.3 in [15] $|\psi(b)| = O\left(p^{[n/3]/2}\right)$ for all points, including those with maximal level. This, together with Lemma 2.1, gives us

$$\|\psi_{\max}\|_q^q = O\left(\frac{p^{q[n/3]/2}}{p^{[(2n+3)/3]}}\right) = O\left(p^{nq/6 - 2n/3}\right) = O\left(p^{n(q-4)/6}\right).$$

These estimates show that $\|\psi\|_q^q$ is bounded for $q < 4$. Moreover, if n is bounded, so is the number of different levels, and therefore $\|\psi\|_4^4$ is bounded. On the other hand, if $q = 4$ and n goes to infinity, the number of different levels, and therefore the sum over all $\|\psi_s\|_q^q$ goes to infinity. Since $n \geqslant 3$ the maximal level is at least 2. In other words, ψ_1 is non-trivial (for $n = 3$ this was denoted $\psi_{\max}$ above). Theorem 1.4 shows that $\|\psi_1\|_q^q$ explodes for $q > 4$ and this causes $\|\psi\|_q$ to explode. Finally, it is easy to see that for $q > 4$, $\|\psi\|_q^q$, the sum of all $\|\psi_s\|_q^q$, is dominated by $O\left(p^{n(q-4)/6}\right)$ (corresponding to the term $\|\psi_{\max}\|_q^q$) and this ends the proof. $\square$

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