Pairs of graphs with connective constants and critical probabilities in the same order

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Abstract

We give examples of pairs of planar, quasi-transitive graphs with connective constants and critical probabilities in the same order.

1 Introduction

The connective constant for self-avoiding walks and the critical probabilities for bond and site percolation are three measures of connectivity of graphs. It is likely that for a large class of graphs, these three measures should agree, in the following sense. If G and H are two graphs, [2, 8, 12],

$$\mu(G) \leq \mu(H) \Leftrightarrow p_c^b(G) \geq p_c^b(G) \Leftrightarrow p_c^s(G) \geq p_c^s(H), \tag{1}$$

where μ, p_c^b, and p_c^s denotes the connective constant, bond percolation critical probability and site percolation probability, respectively.

In this paper we show that there exist pairs of planar, quasi-transitive graphs, such that (1) does not hold. More precisely, we find graphs G_1, G_2, H_1 and H_2 such that

$$\mu(G_1) < \mu(H_1), \text{ but } p_c^b(G_1) < p_c^b(H_1), \text{ and}$$

$$\mu(G_2) < \mu(H_2), \text{ but } p_c^s(G_2) < p_c^s(H_2).$$

Remark 1.1. In [11], Wierman gives examples of a pairs of planar, quasi-transitive graphs with bond and site critical probabilities in the opposite order.

2 Definitions

2.1 Connective constants

A walk of length n on a graph G is a alternating sequence of vertices and edges, $\{v_0, e_1, v_1, e_1, \ldots e_n, v_n\}$, such that the edge e_i connects the vertices v_{i-1} and v_i. The walk is self-avoiding if all vertices are distinct. Let f_n denote the number of distinct self-avoiding walks of length n, startig at the origin. The connective constant of the graph G, $\mu(G)$, is defined as

$$\mu(G) = \lim_{n\to\infty} f_n^{1/n}.$$

The connective constant is unknown for all 2-dimensional graphs, with the possible exception of the hexagonal and $(3, 12^2)$ lattices. For the hexagonal lattice, Nienhuis, [7], derived the value $\sqrt{2+\sqrt{2}}$, through non-rigorous methods. If true, this implies, [3, 4], that $\mu((3, 12^2)) \approx 1.711041$.

2.2 Critical probabilities

In the bond (respectively, site) percolation model, each edge (vertex) is randomly declared open, with probability p, or closed, with probability $q = 1 - p$, independently of each other.

For both models, let C_0 denote the set of vertices that can be reached from the origin, by walks on open edges (for the bond model) or open vertices (for the site model).

The bond or site critical probability for a graph G is defined as

$$p_c(G) = \inf_p \{P(|C_0| = \infty) > 0\}.$$

For the bond (respectively, site) model, the critical probability will be denoted $p_c^b(G)$ (respectively, $p_c^s(G)$).

For the bond model the critical probabilities are known for the square, triangular, hexagonal, bowtie and dual bowtie lattices, [5, 9, 10]. For the site model, they are known for triangulated graphs, and the $(3, 6, 3, 6)$ (Kagomé) and $(3, 12^2)$ (extended Kagomé) lattices, [6]. Besides derivatives of these graphs, and trees, the critical probabilities are unknown for all other non-trivial graphs.

3 Bond percolation example

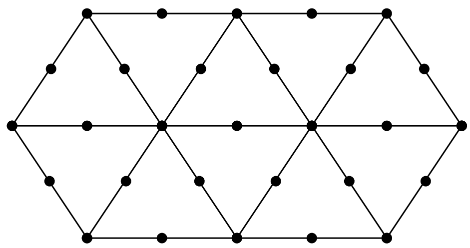

Figure 1: The subdivided triangular lattice.

Let F be the subdivided triangular lattice, the graph obtained if each edge in the triangular lattice $\mathbb{T}$ is substituted for two edges in series (see Figure 1).

Lemma 3.1. *The bond percolation critical probability of F is*

$$p_c^b(F) = \sqrt{p_c^b(\mathbb{T})}.$$

Proof. The percolation models on the two graphs can be coupled as follows. Assuming that a bond percolation configuration with edge probability p have been generated on F, let each edge in $\mathbb{T}$ be open if and only if the corresponding two edges in F are open. This gives a bond percolation configuration with edge probability p^2 on $\mathbb{T}$. Since an infinite open self-avoiding walk on F (starting at a vertex of degree 6) corresponds exactly to an infinite open self-avoiding walk on $\mathbb{T}$, the lemma is proved. □

Lemma 3.2. *The connective constant of F is*

$$\mu(F) = \sqrt{\mu(\mathbb{T})}.$$

Proof. Let f_n denote the number of self-avoiding walks of length n on F, starting from a vertex of degree 6. Let g_n denote the number of self-avoiding walks of length n on $\mathbb{T}$. Then $f_{2n} = g_n$, and

$$\mu(F) = \lim_{n\to\infty} f_{2n}^{1/2n} = \lim_{n\to\infty} \sqrt{g_n}^{1/n} = \sqrt{\mu(G)}.$$

□

Combined with bounds for the connective constants for the triangular lattice $\mathbb{T}$ and the dual bowtie lattice $D(\mathbb{B})$, [2], and the exact values for the bond percolation critical probabilities, $p_c^b(\mathbb{T}) = 2\sin(\pi/18)$ and $p_c^b(D(\mathbb{B})) \approx 0.595482$, the above lemmas give

$$\mu(F) \leq \sqrt{4.25142} < 2.07670 \leq \mu(D(\mathbb{B})), \text{ and}$$

$$p_c^b(F) = \sqrt{2\sin\tfrac{\pi}{18}} < 0.59548 < p_c^b(D(\mathbb{B})).$$

4 Site percolation example

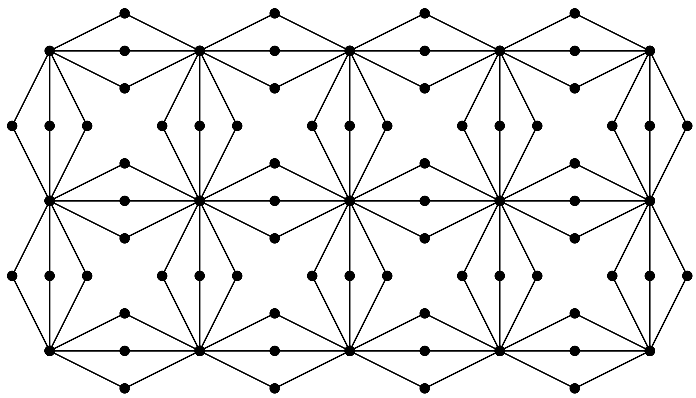

Figure 2: The graph F_3, of the type used in the site percolation example.

Let F_k be the graph obtained by replacing each edge in the square lattice $\mathbb{Z}^2$ with k paths of length 2, see Figure 2. The graphs F_k all are subgraphs of triangulated graphs, so ([1, 6]), $p_c^s(F) > 1/2 = p_c^s(\mathbb{T})$.

Lemma 4.1. *The connective constant of F_k satisfies*

$$\mu(F_k) \geq \sqrt{k\mu(\mathbb{Z}^2)}.$$

Proof. Let f_n denote the number of self-avoiding walks of length n on F_k, starting at a vertex of degree $4k$. Let g_n denote the number of self-avoiding walks of length n on $\mathbb{Z}^2$. Note that to each edge in a self-avoiding walk on $\mathbb{Z}^2$, corresponds at least k walks of length 2 on F, so that $f_{2n} \geq k^n g_n$. Thus

$$\mu(F_k) = \lim_{n\to\infty} f_{2n}^{1/2n} \geq \lim_{n\to\infty} \sqrt{k^n g_n}^{1/n} = \sqrt{k\mu(\mathbb{Z}^2)}.$$

□

Consequently, for large enough k ($k = 7$ suffices), $\mu(F_k) > \mu(\mathbb{T})$.

Remark 4.2. If we allow multiple edges, and consider the k edges between two adjacent vertices as k distinct self-avoiding walks of length n, an analogous but easier example for the site percolation case can be found.

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