Course "Analysis for PhD students", Kozhan

Uppsala University, Spring 2018

HW1

Due on Apr 2 (you may request extensions if needed)

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1. [Simon, Ex. 4.2.9]

   Let $K$ be the generalized Cantor set corresponding to sequence $\{\alpha_j\}_{j=1}^\infty$ (see [Folland, p.39]): $K_0 = [0,1]$; $K_j$ is obtained from $K_{j-1}$ by removing the (open) middle $\alpha_j$-th from each of the intervals comprising $K_{j-1}$; finally $K = \bigcap_{j=1}^\infty K_j$.

   (i) (5pts) Show that for any choice of $\{\alpha_j\}_{j=1}^\infty$ (with $\alpha_j \in (0,1)$ for each $j$), $K$ is nowhere dense (i.e., its closure has empty interior).

   (ii) (5pts) Show that for any choice of $\{\alpha_j\}_{j=1}^\infty$ (with $\alpha_j \in (0,1)$ for each $j$), $K$ has no isolated points.

   (iii) (5pts) Show that $\sum_{j=1}^\infty \alpha_j < \infty$ iff $m(K) > 0$, where $m$ stands for the Lebesgue measure.

2. [Folland, Ex. 7.1.4]

   Let $X$ be a locally compact Hausdorff space.

   (i) (5pts) Let $f \in C_c(X)$ (compactly supported continuous function $X \to \mathbb{C}$) and $f(x) \geq 0$ for all $x$. Show that $f^{-1}([a,\infty))$ is a compact $G_\delta$ set for any $a > 0$.

   (ii) (5pts) Let $K \subseteq X$ be a compact $G_\delta$ set. Show that there exists $f \in C_c(X)$ such that $K = f^{-1}(\{1\})$ (Suggestion: apply Urysohn's lemma to suitably chosen sets).

   (iii) (5pts) Show that the $\sigma$-algebra $Ba_1$ generated by all compact $G_\delta$ sets coincides with the smallest $\sigma$-algebra $Ba_2$ satisfying the property that every $f \in C_c(X)$ is $(Ba_2, \mathcal{B})$-measurable (Remark: this $\sigma$-algebra is called the Baire $\sigma$-algebra, and its sets are called Baire sets).

3. [Rudin, Ex. 1.9]

   (15pts) Let $(X, \mathcal{M}, \mu)$ be a (positive) measure space and let $f : X \to [0,\infty]$ be measurable, $\int_X f\,d\mu = c \in (0,\infty)$. Let $\alpha > 0$. Prove that

$$\lim_{n\to\infty} \int_X n \log\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right) d\mu = \begin{cases} \infty & \text{if } 0 < \alpha < 1, \\ c & \text{if } \alpha = 1, \\ 0 & \text{if } 1 < \alpha < \infty. \end{cases}$$

   (Hint: if $\alpha \geq 1$, the integrands are dominated by $\alpha f$. If $\alpha < 1$, Fatou's lemma can be applied.)

4. [Folland, Ex 2.3.20–21]

   (i) (5pts) Let $f_n, g_n, f, g \in L^1(\mu)$ for some (positive) measure $\mu$. Suppose $f_n \to f$ and $g_n \to g$ pointwise $\mu$-a.e.; $|f_n| \leq g_n$ for all $x$; $\int g_n\,d\mu \to \int g\,d\mu$. Prove that $\int f_n\,d\mu \to \int f\,d\mu$. (Hint: rework the proof of the Dominated Convergence Theorem).

   (ii) (5pts) Suppose $f_n, f \in L^1(\mu)$ and $f_n \to f$ for $\mu$-a.e. $x$. Show that $||f_n - f||_1 \to 0$ iff $||f_n||_1 \to ||f||_1$.

5. [Rudin, Ex. 8.12]

   (i) (7.5pts) Show that for any $0 < A < +\infty$, $e^{-xt} \sin x \in L^1([0, A] \times [0, +\infty), dx \otimes dt)$ (Note: this fails for $A = +\infty$)

   (ii) (7.5pts) Use Fubini's/Tonelli's theorem to compute $\lim_{A \to \infty} \int_0^A \frac{\sin x}{x} dx$ (use part (i)).

6. [Folland, Ex. 3.2.11]

   Let $\mu$ be a (positive) measure. Let us call a collection of functions $\{f_\alpha\}_{\alpha \in A} \subset L^1(\mu)$ **uniformly integrable** if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\left| \int_E f_\alpha \, d\mu \right| < \varepsilon$ for all $\alpha \in A$ whenever $\mu(E) < \delta$.

   (i) (5pts) Show that any finite subset of $L^1(\mu)$ is uniformly integrable.

   (ii) (10pts) Let $f_n, f \in L^1(\mu)$ and $||f_n - f||_1 \to 0$. Show that $\{f_n\}_{n=1}^\infty$ is uniformly integrable.

7. [Folland, Ex. 3.4.25]

   If $E$ is a Borel set in $\mathbb{R}^n$, the density $D_E(x)$ of $E$ at the point $x$ is defined as

$$D_E(x) = \lim_{r \to 0} \frac{m(E \cap B(r,x))}{m(B(r,x))},$$

   whenever the limit exists (recall that $m$ is the Lebesgue measure, and $B(r, x) := \{y \in \mathbb{R}^n : |x - y| < r\}$).

   (i) (5pts) Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.

   (ii) (10pts) For any $\alpha \in (0, 1)$, find an example of a Borel set $E \subset \mathbb{R}$ and a point $x$ such that $D_E(x) = \alpha$.