Course "Analysis for PhD students", Kozhan

Uppsala University, Spring 2018

HW2

Due on May 14 (you may request extensions if needed)

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1. [Folland, ex. 6.1.7] (15 points) Suppose $f \in L^p(X) \cap L^\infty(X)$ for some $1 \leq p < \infty$. Show that
$$||f||_\infty = \lim_{q\to\infty} ||f||_q.$$

2. [Folland, Ex. 6.1.14] Let $g \in L^\infty(X)$, $1 \leq p \leq \infty$, and let T be the linear operator on $L^p(X)$ given by $T(f) = fg$.
   (i) (3 point) Show that T is bounded with $||T|| \leq ||g||_\infty$, where $||T||$ is the operator norm of T.
   (ii) (12 points) Suppose μ is σ-finite on X. Show that $||T|| = ||g||_\infty$.

3. [Rudin, Ex. 3.11] (15 points) Let $\mu(X) = 1$ and let $f, g : X \to (0, \infty)$ be two measurable functions satisfying $f(x)g(x) \geq 1$. Prove that $||f||_1 \, ||g||_1 \geq 1$.

4. [Folland, Ex. 6.3.33]
   (i) (10 points) Let $1 < p < \infty$ and define $(Tf)(x) = x^{-1/p} \int_0^x f(t)\, dt$. Let q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Show that if $f \in L^q((0, \infty))$ (with respect to the Lebesgue measure), then Tf belongs to $C_0((0, \infty))$.
   (ii) (5 points) What can go wrong if $p = \infty$ in the above statement? What can go wrong if $p = 1$ in the above statement?

5. Recall that a trigonometric polynomial on $\mathbb{T}^1$ was defined to be any function $\mathbb{T}^1 \to \mathbb{C}$ of the form $p(t) = \sum_{m=-n}^{n} c_m e^{2\pi imt}$ with $n \in \mathbb{N}$ and $c_j \in \mathbb{C}$. In this case we say that $\deg p \leq n$.
   (i) (2 points) How many zeros does $\frac{1}{2}(e^{2\pi int} + e^{-2\pi int})$ have on $\mathbb{T}$? How many zeros does $\frac{1}{2i}(e^{2\pi int} - e^{-2\pi int})$ have on $\mathbb{T}$?
   (ii) (8 points) If $\deg p = n$, show that p can have no more than $2n$ zeros of $\mathbb{T}$ (Hint: make a change of variables to reduce trigonometric polynomials to the usual polynomials of a complex variable).

6. (15 points) Let $f \in L^1(\mathbb{T})$ and $g \in L^\infty(\mathbb{T})$. Show that
$$\lim_{n\to\infty} \int_{\mathbb{T}} f(t)g(nt)\, dt = \widehat{f}(0)\widehat{g}(0).$$
   (Hint: approximate f in L^1 by trigonometric polynomials).

7. [Simon, Ex. 3.5.12]

In class we discussed only the convergence of (square) partial sums and their Cesàro means. Another commonly used approach is to use Abel sums. Given $f \in C(\mathbb{T})$ and any $0 \le r < 1$, let

$$(A_r f)(x) := \sum_{m=-\infty}^{\infty} r^{|m|} \widehat{f}(m) e^{2\pi i m x},$$

where $\widehat{f}(m)$ are the Fourier coefficients of f.

(i) (6 points) Prove that
$$(A_r f)(t) = (P_r * f)(t),$$
where
$$P_r(t) = \frac{1 - r^2}{1 + r^2 - 2r\cos 2\pi t}$$
known as the Poisson kernel.

(ii) (6 points) Prove that $\{P_a(t)\}_{r \to 1}$ forms an approximate identity on $\mathbb{T}$.

(iii) (3 points) Prove that $A_r f \to f$ uniformly as $r \uparrow 1$ for any $f \in C(\mathbb{T})$ and $A_r f \to f$ in $|| \cdot ||_p$-norm as $r \uparrow 1$ for any $f \in L^p(\mathbb{T})$ $(1 \le p < \infty)$.