Course "Analysis for PhD students", Kozhan

Uppsala University, Spring 2018

HW3

Due on June 04 (electronic submission only; you may request extensions if needed)

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1. (15 points) [Grafakos, Ex. 3.4.2] Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of points with $a_j \in \mathbb{T}^n$. Show that the following three statements are equivalent:

   (i) For any square Q in $\mathbb{T}^n$, we have

   $$\lim_{N\to\infty} \frac{\sharp\{k : 0 \le k \le N-1, a_k \in Q\}}{N} = m(Q),$$

   where $m(Q)$ is the Lebesgue measure of Q (we call $\{a_k\}_{k=0}^{\infty}$ equidistributed on $\mathbb{T}^n$ in this situation).

   (ii) For any $f \in C^\infty(\mathbb{T}^n)$, we have

   $$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(a_j) = \int_{\mathbb{T}^n} f(x)\,dx.$$

   (iii) For every $m \in \mathbb{Z}^n \setminus \{0\}$, we have

   $$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i m\cdot a_k} = 0.$$

   (Hint: in (i)$\Leftrightarrow$(ii), approximate f by step functions; in (iii)$\Rightarrow$(ii), use the Fourier inversion).

2. [Grafakos, Ex. 2.2.9] We know that the Fourier transform of any $f \in L^1(\mathbb{R})$ is a uniformly continuous function decaying to 0 at infinity. This exercise will show that not every such a function is the Fourier transform of some L^1-function.

   (i) (5 points) Prove that for all $0 < \varepsilon < t < \infty$, we have

   $$\left| \int_\varepsilon^t \frac{\sin \xi}{\xi}\, d\xi \right| \le 4.$$

   (ii) (5 points) If $f \in L^1(\mathbb{R})$ is odd, conclude that for all $t > \varepsilon > 0$ we have

   $$\left| \int_\varepsilon^t \frac{\widehat{f}(\xi)}{\xi}\, d\xi \right| \le 4||f||_1.$$

   (iii) (5 points) Let $g(\xi)$ be a continuous odd function that is equal to $\frac{1}{\log \xi}$ for $\xi \ge 2$. Show that there does not exist $f \in L^1(\mathbb{R})$ such that $g = \widehat{f}$.

3. [Folland, Ex.8.7.43] (10 points) Use the Fourier transform to find a solution of the differential equation $u - u'' = f$.

   (Hint: Compute the Fourier transform of $\phi(x) = e^{-|x|}$ on $\mathbb{R}$. Try $u = f * \phi$ as a possible solution and then fix the normalization/scaling. What hypotheses are needed on f?)

4. (i) (10 points) Let $t > 0$. Use the Poisson Summation Formula (no credit if you don't use it) to compute

$$\sum_{n\in\mathbb{Z}} \frac{1}{t^2 + n^2}$$

   (Hint: What is the Fourier transform of $f(x) = e^{-|x|}$?).

   (ii) (5 points) Take $t \to 0$ in (i) to compute $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

5. [Grafakos, Ex. 2.3.4]

   (i) (5 points) Compute the distributional derivative of $\chi_{[a,b]}(x)$ (the characteristic function of the interval $[a, b]$ in $\mathbb{R}$).

   (ii) (5 points) Compute the distributional derivatives $\partial_j \chi_{B(0,1)}$, $j = 1, 2$ (where $B(0, 1) \subset \mathbb{R}^2$ is the disk of radius 1 centered at the origin).

   (iii) (5 points) Compute the (distributional) Fourier transform of $\sin x$ and $\cos x$ (explain why these can be viewed as tempered distributions).

6. (i) (7.5 points) Let

$$\langle u, \phi\rangle = \lim_{\varepsilon\to 0} \int_{\varepsilon\le|x|} \frac{\phi(x)}{x}\, dx, \qquad \phi \in \mathcal{S}(\mathbb{R})$$

   Prove that u is well defined for any $\phi \in \mathcal{S}(\mathbb{R})$ and is a tempered distribution (this exhibits an example of a tempered distribution that is neither a function nor a measure).

   (ii) (7.5 points) Prove that the derivative of the distribution $\log |x| \in \mathcal{S}'(\mathbb{R})$ is the distribution $u \in \mathcal{S}'$ from (i).

7. [Grafakos, Ex. 2.4.3]

   (i) (3 points) Prove that e^x is not in $\mathcal{S}'(\mathbb{R})$.

   (ii) (12 points) Prove that $e^x e^{ie^x}$ is in $\mathcal{S}'(\mathbb{R})$.