Lecture notes
Course “Analysis for PhD students”
Uppsala University, Spring 2018
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Chapter 1

Measures

1.1 Some prerequisites

I will follow closely the textbook “Real analysis: Modern Techniques and Their Applications” by Gerald B. Folland (2nd edition). Throughout these notes, [F] will refer to this book, and (occasionally) a number afterwards will refer to the page.

The following books were also used: “Comprehensive Course in Analysis, Part 1: Real Analysis” by Barry Simon, “Real Analysis and Probability” by R. M. Dudley, “Real and Complex Analysis” by Walter Rudin, “Measure Theory” by Paul R. Halmos.

I will assume the knowledge of:

- Calculus, including Riemann integration;
- Basics of set theory;
- Basics of point-set topology;
- Basics of metric spaces, including compactness.

[F, Chapt 0] should be sufficient.

1.2 Introduction

Intuition. We want to define the notion of measure (“size”) for various subsets of a space X. Ideally, every subset should be assigned a measure, but that turns out to be unrealistic in practice: see the Banach–Tarski paradox below. Conclusion: “sick” sets should not have a measure.

Example 1.1. *Banach–Tarski Paradox*: Let $n \geq 3$ and let B and D be any two balls in $\mathbb{R}^n$. There exists an $m \in \mathbb{N}$ and disjoint sets $\{B_j\}_{j=1}^m$, $\{D_j\}_{j=1}^m$ such that $B = \bigcup_{j=1}^m B_j$, $D = \bigcup_{j=1}^m D_j$, such that for each j, B_j can be moved to D_j using only rotations and translations in $\mathbb{R}^n$.

1.3 Algebras and σ-algebras

Intuition. From the previous section, we decided that we are able to measure only nice enough sets, which we will start calling "measurable". It is natural to demand that complement and unions of two measurable sets is again measurable. This leads us to the notion of algebra of sets. Since we want to do analysis and take limits, we will also want the *countable* union of measurable sets to also be measurable. That is why we will demand that our measurable sets form in fact a σ-algebra.

Definition 1.2. *For X be a non-empty set. A collection $\mathcal{A}$ of subsets of X is called an* ***algebra*** *if*

(i) $\varnothing \in \mathcal{A}$;

(ii) If $E \in \mathcal{A}$, then its complement E^c is also in $\mathcal{A}$;

(iii) If $E_j \in \mathcal{A}$, then $\bigcup_{j=1}^{n} E_j \in \mathcal{A}$ for any (finite) $n \in \mathbb{N}$.

Remark 1.3. From (ii) and (iii) it follows that if $E_j \in \mathcal{A}$ then $\bigcap_{j=1}^{n} E_j$ is also in $\mathcal{A}$ for any (finite) $n \in \mathbb{N}$.

Definition 1.4. *For X be a non-empty set. A collection $\mathcal{M}$ of subsets of X is called an* ***σ-algebra*** *if*

(i) $\varnothing \in \mathcal{M}$;

(ii) If $E \in \mathcal{M}$, then its complement E^c is also in $\mathcal{M}$;

(iii) If $E_j \in \mathcal{M}$, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{M}$.

Remarks 1.5. (a) From (ii) and (iii) it follows that if $E_j \in \mathcal{M}$ then $\bigcap_{j=1}^{\infty} E_j$ is also in $\mathcal{M}$.
(b) If $A, B \in \mathcal{M}$ then $A \setminus B \in \mathcal{M}$. Indeed, $A \setminus B = A \cap B^c$.

Example 1.6. (a) Denote $\mathcal{P}(X) = \{E : E \subseteq X\}$ to be the set of all subsets of X ("the power-set of X"). $\mathcal{P}(X)$ is an example of a σ-algebra on X. Another trivial example of a σ-algebra is $\{\varnothing, X\}$. We will see less trivial examples below.

Proposition 1.7. *(i) Intersection of any family of σ-algebras on X is a σ-algebra.*

(ii) If $\mathcal{F}$ is any collection of subsets of X, then there exists the smallest σ-algebra $\sigma(\mathcal{F})$ on X containing $\mathcal{F}$.

Proof. Part (i) follows directly from the definitions.
Part (ii) follows from (i) since $\sigma(\mathcal{F})$ is clearly the intersection of the family of all possible σ-algebras containing $\mathcal{F}$ (this family is non-empty as $\mathcal{P}(X)$ contains $\mathcal{F}$). $\square$

Definition 1.8. *We call $\sigma(\mathcal{F})$ in Proposition 1.7 the* ***σ-algebra generated by*** *$\mathcal{F}$.*

Example 1.9. Let $X = \prod_{j=1}^{n} X_j$, where $\{X_j\}_{j=1}^{n}$ is a (finite or countable) collection of sets ($n \in \mathbb{N} \cup \infty$), each with a chosen σ-algebra $\mathcal{M}_j$. Then clearly $\mathcal{F} = \left\{\prod_{j=1}^{n} E_j : E_j \in \mathcal{M}_j\right\}$ is not a σ-algebra on X. Instead we define the **product σ-algebra** on X to be the σ-algebra generated by $\mathcal{F}$, denoted by $\otimes_{j=1}^{n} \mathcal{M}_j$.

1.4 Borel σ-algebra

Intuition. We want the majority of our every-day sets to be measurable. Typically this includes all of the open sets. This leads to the notion of Borel σ-algebra.

Definition 1.10. *For any topological space X, let τ be the collection of all open sets in X (the topology). We call $\sigma(\tau)$ the **Borel σ-algebra on** X, denoted by $\mathcal{B}(X)$. Elements of $\mathcal{B}(X)$ are called the **Borel sets** of X.*

Examples 1.11. (a) Every open or closed set is a Borel set.

(b) From (ii) and (iii) of Definition 1.4, any countable intersection of open sets is a Borel set. Such sets are called G_δ's (δ stands for intersection ("durchschnitt" in German); G somehow stands for open sets).

(c) Similarly, any countable union of closed sets is a Borel set. Such sets are called F_σ's (σ stands for union ("summe" in German); F somehow stands for closed sets).

(d) In $\mathbb{R}^1$, every half-open interval is an F_σ; is a G_δ; is a Borel set on $\mathbb{R}^1$.

(e) In $\mathbb{R}^1$, the set of all half-open finite intervals generate $\mathcal{B}(\mathbb{R}^1)$. Similarly, $\mathcal{B}(\mathbb{R}^1)$ is generated by all open intervals; by all closed intervals; by all open rays; by all closed rays ([F, Prop 1.2]).

Exercise 1.12. Let $n \in \mathbb{N}$. Show that $\mathcal{B}(\mathbb{R}^n)$ is the same σ-algebra as $\otimes_{j=1}^n \mathcal{B}(\mathbb{R}^1)$ (Note: the equality $\otimes_{j=1}^n \mathcal{B}(X_j) = \mathcal{B}(\prod_{j=1}^n X_j)$ may fail in general).

1.5 Measures

Definition 1.13. *Let X be a set and $\mathcal{M}$ be any σ-algebra on X. Then we call $(X, \mathcal{M})$ a **measurable space**. Sets in $\mathcal{M}$ are called **measurable sets**.*

Example 1.14. Any topological space X with its Borel σ-algebra $\mathcal{B} = \mathcal{B}(X)$ can therefore be regarded as a measurable space $(X, \mathcal{B})$. In what follows we will mostly restrict ourselves to these measurable spaces.

Definition 1.15. *Let X be a set and $\mathcal{M}$ be a σ-algebra on X. A (positive) **measure** on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that*

(i) $\mu(\varnothing) = 0$;

(ii) if $\{E_j\}_{j=1}^\infty$ is a countable collection of disjoint elements of $\mathcal{M}$, then

$$\mu\left(\bigcup_{j=1}^\infty E_j\right) = \sum_{j=1}^\infty \mu(E_j). \tag{1.5.1}$$

*If $\mu(X) < \infty$ (i.e., $\mu(E) < \infty$ for any measurable set E), then we call μ **finite**.*
*If $X = \bigcup_{j=1}^\infty E_j$, where $\mu(E_j) < \infty$ for all j, then we call μ **σ-finite**.*

Remarks 1.16. (a) A **signed measure** is a function $\nu : \mathcal{M} \to [-\infty, \infty)$ or $\nu : \mathcal{M} \to (-\infty, \infty]$ that satisfies (i) and (ii) above, where the convergence of the right-hand side of (1.5.1) is absolute if the left-hand side is finite.

(b) A **complex measure** is a function $\mu : \mathcal{M} \to \mathbb{C}$ that satisfies (i) and (ii) above, where the convergence of the right-hand side of (1.5.1) is absolute. Note that infinite value is not allowed, so a *finite* positive measure is a complex measure, but a *non-finite* positive measure is not a complex measure. Quite annoying.

Definition 1.17. *If μ is a (positive) measure on $(X, \mathcal{M})$, then we call $(X, \mathcal{M}, \mu)$ a **measure space**.*

Examples 1.18. (a) Let $(X, \mathcal{M})$ be any measurable space. Let us fix a point $x_0 \in X$. For any set $E \in \mathcal{M}$, define $\mu(E)$ to be 1 if $x_0 \in E$ and $\mu(E) = 0$ otherwise. Then μ is a measure called *point mass* concentrated at x_0 or the *Dirac measure* at x_0, usually notated by δ_{x_0}.

(b) For any set $E \subseteq X$ define $\mu(E)$ to be the number of points in E (with ∞ allowed). Such μ is a measure on $(X, \mathcal{P}(X))$, and is called the counting measure on X.

(c) More generally, suppose we are given a function $f : X \to [0, \infty]$. Then the formula $\mu(E) = \sum_{x \in E} f(x)$ defines a measure on $(X, \mathcal{P}(X))$. The above two examples are the special cases.

(d) Our "common" Lebesgue measure dx on $\mathbb{R}^1$ will take some work to carefully construct: we do this in Sections 1.8–1.11.

(e) Any measure μ on $(X, \mathcal{B}(X))$ is called a Borel measure.

1.6 Basic properties of measures

Theorem 1.19. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then:*

(i) (Monotonicity) If $E, F \in \mathcal{M}$ and $E \subseteq F$ then $\mu(E) \leq \mu(F)$;

(ii) (Subadditivity) For any (not necessarily disjoint) elements $A_j \in \mathcal{M}$,

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu(A_j). \tag{1.6.1}$$

(iii) (Continuity from below) If $E_j \in \mathcal{M}$ for each j and $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$, then $\mu(\bigcup_{j=1}^{\infty} E_j) = \lim_{j\to\infty} \mu(E_j)$;

(iv) (Continuity from above) If $E_j \in \mathcal{M}$ for each j, $E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots$, and $\mu(E_1) < \infty$, then $\mu(\bigcap_{j=1}^{\infty} E_j) = \lim_{j\to\infty} \mu(E_j)$.

Remarks 1.20. (a) The condition $\mu(E_1) < \infty$ in (iv) is required: e.g., take counting measure on $\mathbb{R}$ and $E_j = \{j, j+1, j+2, \ldots\}$.

(b) If $\mu(E) = 0$ and $F \subseteq E$ then $\mu(F) = 0$ by (i), but only if $F \in \mathcal{M}$. This motivates the definition below.

Proof. More details: [F26].

(i) is immediate from $F = E \cup (F \setminus E)$ and additive property.

(ii) is simple by removing overlaps: $F_1 = E_1$, $F_k = E_k \setminus \cup_{j=1}^{k-1} E_j$.

(iii) follows from $\mu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j \setminus E_{j-1}) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(E_j \setminus E_{j-1}) = \lim_{n\to\infty} E_n$.

(iv) follows by taking complements: define $F_j = E_1 \setminus E_j$, apply part (iii) to get $\mu(\cup F_j) = \lim \mu(F_j) = \mu(E_1) - \lim \mu(E_j)$, then rewrite $\mu(\cup F_j) = \mu(E_1 \setminus \cap E_j) = \mu(E_1) - \mu(\cap E_j)$. $\square$

1.7 Completion

Definition 1.21. *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

(i) If $E \in \mathcal{M}$ and $\mu(E) = 0$ then we call E a ***null set*** *(or μ-null set).*

(ii) *If some property of points $x \in X$ holds except on a null-set, then we say that the property holds* ***almost everywhere*** *(or μ-almost everywhere). We abbreviate it as "a.e.".*

(iii) *Measure space $(X, \mathcal{M}, \mu)$ is called* ***complete*** *if every subset of every μ-null set is measurable. It is very common to just say that μ is complete.*

(iv) *Given a measure μ on $(X, \mathcal{M})$, let $\mathcal{J}$ be the set of all possible subsets of μ-null sets. Define $\overline{\mathcal{M}} := \{E \cup F : E \in \mathcal{M}, F \in \mathcal{J}\}$. Then we define the* ***completion of*** *μ to be the measure $\overline{\mu}$ on $(X, \overline{\mathcal{M}})$ defined by $\overline{\mu}(E \cup F) = \mu(E)$ for $E \in \mathcal{M}, F \in \mathcal{J}$.*

Remarks 1.22. (a) It is easy to see that μ is complete if and only if $\overline{\mu} = \mu$.
(b) It is clear that $\overline{\overline{\mu}} = \overline{\mu}$ for any measure μ.

1.8 Outer measures

Intuition. Before we go any further into the properties of measures, let us now try to carefully construct our usual length and area measures – what we will call the Lebesgue measures on $\mathbb{R}^1$ and $\mathbb{R}^2$, respectively. Our construction will actually be quite general and allow to construct a wide selection of other measures.

The idea is as follows. We first define "measure" of some elementary sets (think of length of intervals on $\mathbb{R}^1$ or area of rectangles on $\mathbb{R}^2$, etc.), and then we hope that this will be extended to measures on the whole σ-algebra, consistently with the property (ii) of Definition 1.15. Now how do we extend this? Through the notion of the outer measure.

For the intuition, imagine we want to define $\mu\left((a, b)\right) = b - a$ and hope to extend this to the whole Borel σ-algebra $\mathcal{B}(\mathbb{R})$. The "outer measure" of any set E will be $\mu^*(A) := \inf \left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j)) : \bigcup_{j=1}^{\infty} (a_j, b_j) \text{ cover } A \right\}$. Similarly we can approximate E from inside by elementary sets and taking the supremum of their total length. If these two "outer" and "inner" numbers coincide, we will call E to be "a nice set", and we will simply define this common value to be the measure $\mu(E)$ of E. Then we just cross our fingers in hope that all such nice sets form a σ-algebra and that μ will satisfy the properties of Definition 1.15.

Definition 1.23. *Let X be a set and $\mathcal{P}(X)$ be its power-set (the set of all subsets). An* ***outer measure*** *on X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that*

(i) $\mu^*(\varnothing) = 0$*;*

(ii) $\mu^*(A) \le \mu^*(B)$ *if* $A \subseteq B$*;*

(iii) $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \le \sum_{j=1}^{\infty} \mu^*(A_j)$.

Remark 1.24. "Outer measure" is not a measure, typically!

Proposition 1.25. *Let $\mathcal{S}$ be a family of subsets of X such that $\varnothing \in \mathcal{S}$ and X is a countable union of sets in $\mathcal{S}$. Let $\mu_0 : \mathcal{S} \to [0, \infty]$ be any function satisfying $\mu_0(\varnothing) = 0$. Then for any set $A \in \mathcal{P}(X)$,*

$$\mu^*(A) := \inf \left\{ \sum_{j=1}^{\infty} \mu_0(E_j) : E_j \in \mathcal{S}, A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}$$

defines an outer measure on X.

Proof. [F29], [S681]

$\mu^* \geq 0$ and $\mu^*(\varnothing) = 0$ are obvious.

Property (ii) follows easily since any cover of B is also a cover of A.

Property (iii): fix $\varepsilon > 0$, and for each j, choose a cover $\{S_{j,k}\}_{k=1}^{\infty}$ of A_j such that $\sum_{k=1}^{\infty} \mu_0(S_{j,k}) - \frac{\varepsilon}{2^j} \leq \mu^*(A_j)$. Then $\bigcup_{j=1}^{\infty} A_j$ is covered by $\bigcup_{j,k} S_{j,k}$ with $\sum_{j,k} \mu_0(S_{j,k}) \leq \varepsilon + \sum_{j=1}^{\infty} \mu^*(A_j)$. This implies (iii) since ε was arbitrary. ☐

Examples 1.26. (a) Taking $X = \mathbb{R}^1$ and $\mu_0\left((a,b]\right) = b - a$, and $\mathcal{S}$ to be the set of all finite half-intervals, we obtain an outer measure μ^* called the Lebesgue outer measure.

(b) More generally, we can take $\mu((a,b]) = F(b) - F(a)$ for some non-decreasing, right-continuous function F. This leads to the outer measure μ_F^* which is called the Lebesgue–Stieltjes outer measure. More on this, see Section 1.11 below.

1.9 μ^*-measurable sets

Intuition. μ^*-measurable sets are basically those "nice" sets for which the inner and outer measures coincide. This condition can be restated as (1.9.1) so that we don't in fact need to define the inner measure.

Definition 1.27. *Given a set X and an outer measure μ^*, we say that a set $A \subseteq X$ is μ^***-measurable** if*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \qquad \text{for all } E \subseteq X. \tag{1.9.1}$$

Remarks 1.28. (a) If E is an elementary set containing A, then the outer measure of A is $\mu^*(A) = \mu^*(E \cap A)$, and the inner measure of A is $\mu^*(E) - \mu^*(E \cap A^c)$, so (1.9.1) is indeed the right condition.

(b) Inequality $\leq$ in (1.9.1) is automatic by subadditivity (iii) of Definition 1.23, so A is μ^*-measurable if and only if the inequality $\geq$ holds in (1.9.1).

1.10 Carathéodory theorem: construction of a measure from an outer measure

Theorem 1.29 (Carathéodory)**.** *Let μ^* be an outer measure on X, and $\mathcal{M}$ be the collection of all μ^*-measurable sets. Then $\mathcal{M}$ is a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a (complete) measure.*

Proof. [F29–30]

(i): $\varnothing \in \mathcal{M}$ is trivial: $\mu^*(E) = \mu^*(\varnothing) + \mu^*(E)$.

(ii): $A \in \mathcal{M}$ implies $A^c \in \mathcal{M}$ is immediate from the definition since (1.9.1) is symmetric in A and A^c.

(iii), algebra: applying measurability condition (1.9.1) to A and then B we get $\mu^*(E) = \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c)$ for any set E. Now using $A \cup B = (A^c \cap B) \cup (A \cap B^c) \cup (A \cap B)$ and subadditivity of μ^*, we get $\mu^*(E \cap (A \cup B)) \leq \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B)$. Combining them, we get $\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap A^c \cap B^c)$ which means $A \cup B \in \mathcal{M}$ by definition of μ^*-measurability.

(iii), σ-algebra: note that it suffices to take *disjoint* unions. So let $A_j \in \mathcal{M}$ be disjoint. Define $B_n = \bigcup_{j=1}^{n} A_j$ and $B = \bigcup_{j=1}^{\infty} A_j$. We want to show that for any E, we have $\mu^*(E) \geq \mu^*(E \cap B) + \mu^*(E \cap B^c)$.

By μ^*-measurability of A_n, we get that for any E, $\mu^*(E \cap B_n) = \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1})$, so by induction $\mu^*(E \cap B_n) = \sum_{j=1}^n \mu^*(E \cap A_j)$. Since $B_n \in \mathcal{M}$ (finite union), we get

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \geq \sum_{j=1}^{n} \mu^*(E \cap A_j) + \mu^*(E \cap B^c). \tag{1.10.1}$$

Take $n \to \infty$ and apply subadditivity: we get $\mu^*(E) \geq \mu^*(\bigcup_{j=1}^\infty (E \cap A_j)) + \mu^*(E \cap B^c) = \mu^*(E \cap B) + \mu^*(E \cap B^c)$, which is what we need for μ^*-measurability of B.

This shows that $\mathcal{M}$ is a σ-algebra.

Now let us show that μ^* restricted to $\mathcal{M}$ is a measure.

$\mu^*(\varnothing) = 0$ is given since μ^* is an outer measure.

To show countable additivity for disjoint unions, take $E = B$ and $n \to \infty$ in (1.10.1): we get $\mu^*(B) \geq \sum_{j=1}^\infty \mu^*(A_j)$. The reverse inequality is immediate from subadditivity of outer measures, so we get the required equality.

Finally, showing that μ^* on $\mathcal{M}$ is complete is easy: if $\mu^*(A) = 0$ for some set A, then also $\mu^*(E \cap A) = 0$ by monotonicity, so μ^*-measurability of A is equivalent to $\mu^*(E) \geq \mu^*(E \cap A^c)$ which holds by monotonicity again. □

1.11 Lebesgue–Stieltjes measures on $\mathbb{R}$: construction

Intuition. Given a finite Borel measure μ on $\mathbb{R}$, we can form a function $F : \mathbb{R} \to [0, \infty)$ given by $F(x) = \mu((-\infty, x])$. Such a function is non-decreasing (by (i) of Theorem 1.19) and right-continuous (by (iv) of Theorem 1.19). Notice then that $\mu((a, b]) = F(b) - F(a)$. Now in the converse direction, we use the Carathéodory theorem to construct a measure μ_F corresponding to any non-decreasing, right-continuous function F. In particular $F(x) = x$ corresponds to the Lebesgue measure on $\mathbb{R}$ ("length").

Theorem 1.30. *Given a non-decreasing, right-continuous function $F : \mathbb{R} \to \mathbb{R}$, there is a unique Borel measure μ_F on $\mathbb{R}$ such that*

$$\mu_F((a, b]) = F(b) - F(a).$$

Any Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets is of the form μ_F for some non-decreasing, right-continuous function F.

Remarks 1.31. (a) If $F(x) = x$, we call $\overline{\mu}_F$ (the completion of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_F)$) the **Lebesgue measure** on $\mathbb{R}$, denoted by m.

(b) For a general F, we call $\overline{\mu}_F$ (the completion of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_F)$) the **Lebesgue–Stieltjes measure** associated to F.

Exercise 1.32. What will be different in the statement of the theorem if we start off with a left-continuous function F instead?

Proof. [F31–35]

Let $\mathcal{S}$ be the set of all finite half-intervals $(a, b]$. We can apply proposition 1.25 with $\mu_0((a, b]) = F(b) - F(a)$ to obtain the outer measure μ_0^* (the Lebesgue–Stieltjes outer measure). Then we apply the Carathéodory Theorem 1.29 to obtain a complete measure μ on the σ-algebra of μ_0^*-measurable sets. We are left with showing three things: 1) that μ^* and μ_0 agree on half-intervals; 2) that any Borel set in $\mathbb{R}$ is μ_0^*-measurable; 3) uniqueness.

3): suppose μ_1 and μ_2 are two measures that agree on all intervals $(a, b]$. Note that the collection of sets where μ_1 and μ_2 agree forms a σ-algebra. Therefore μ_1 and μ_2 agree on the minimal σ-algebra generated by intervals $(a, b]$, that is they agree on all of the Borel sets.

We show (1) and (2) in steps below:

Step 1: μ_0 is countably additive on $\mathcal{S}$.

Proof: finite additiveness (for disjoint unions) is obvious from telescoping summation. Finite sub-additiveness (for non-disjoint unions) is also clear too by subdividing intervals into disjoint or coinciding intervals. Now to prove countable additivity, let $I = (a, b]$ be equal to the countable disjoint union of $I_j = (a_j, b_j]$. By finite additivity, for any n, $\mu_0(I) \geq \sum_{j=1}^{n} \mu_0(I_j)$, which gives $\mu_0(I) \geq \sum_{j=1}^{\infty} \mu_0(I_j)$. To prove the converse, fix $\varepsilon > 0$, and for every $n \geq 0$, use the right-continuity of F to get $\delta_n > 0$ such that $F(b_n + \delta_n) < F(b_n) + \frac{\varepsilon}{2^n}$, and similarly $\delta > 0$ such that $F(a+\delta) < F(a) + \varepsilon$. Now the compact interval $[a+\delta, b]$ is covered by open intervals $(a_n, b_n + \delta_n)$ ($n \in \mathbb{N}$), we can choose a finite subcover (with indices $\Lambda_n \subset \mathbb{N}$). By finite subadditivity, we get

$$\mu_0(I) - \varepsilon \leq F(b) - F(a+\delta) \leq \sum_{\Lambda_n} (F(b_n + \delta_n) - F(a_n)) \leq \sum_{n=1}^{\infty} (F(b_n) - F(a_n) + \frac{\varepsilon}{2^n}).$$

Taking $\varepsilon \to \infty$ gives $\mu_0(I) \leq \sum_{n=1}^{\infty} \mu_0(I_n)$.

Step 2: μ_0 and μ^* agree on $\mathcal{S}$.

Proof: Let $I \in \mathcal{S}$. $\mu^*(I) \leq \mu_0(I)$ is obvious from the definition of μ^* since we can just cover I with I. Now suppose $I \subseteq \bigcup_{j=1}^{\infty} I_j$, $I_j \in \mathcal{S}$. Define $B_n = I \cap (I_n \setminus \cup_{j=1}^{n-1} I_j)$. Then B_n's are disjoint, in $\mathcal{S}$, with union equal I, so by countable additivity of μ_0, $\mu_0(I) = \sum_{j=1}^{\infty} \mu_0(B_n) \leq \sum_{j=1}^{\infty} \mu_0(I_n)$. This holds for any cover of I, so $\mu_0(I) \leq \mu^*(I)$. This proves their equality.

Step 3: Every half-interval $A := (a, b]$ is μ^*-measurable.

Choose any $E \subseteq X$ with $\mu^*(E) < \infty$. We need $\geq$ in (1.9.1). Given $\varepsilon > 0$, choose countable $I_j \in \mathcal{S}$ so that $\cup_j I_j$ cover E and $\sum_j \mu_0(I_j) \leq \mu^*(E) + \varepsilon$. Then $\cup_j (I_j \cap A)$ cover $E \cap A$ and $\cup_j (I_j \cap A^c)$ cover $E \cap A^c$. Therefore $\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \sum_j \mu_0(I_j \cap A) + \sum_j \mu_0(I_j \cap A^c) = \sum_j \mu_0(I_j)$ by countable additivity of μ_0 on $\mathcal{S}$. Thus we showed $\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E) + \varepsilon$. Now we take $\varepsilon \to 0$.

Step 4: Since half-intervals generate $\mathcal{B}(\mathbb{R})$, and the set of μ^*-measurable sets is a σ-algebra, we obtain that every Borel set is μ^*-measurable.

Remark: One can show that Borel sets are always μ^*-measurable in Theorem 1.29 as long as μ^* is a so-called "metric outer measure", see, e.g., [F, Prop 11.16]. $\square$

1.12 Regular sets and regular measures

Intuition. Measurable sets are often not easy to understand, and it would be nice to be able to approximate them with the sets that we understand better: for example open or compact sets.

Definition 1.33. *Let X be a topological space, and $(X, \mathcal{M}, \mu)$ be a measure space.*

(i) We say that a set $E \in \mathcal{M}$ is ***outer regular*** *if*

$$\mu(E) = \inf \{\mu(U) : E \supseteq E \text{ and } U \text{ is open and measurable}\}$$

(ii) We say that a set $E \in \mathcal{M}$ is ***inner regular*** *if*

$$\mu(E) = \sup \{\mu(K) : K \subseteq E \text{ and } K \text{ is compact and measurable}\}$$

(iii) We say that the measure μ is ***outer regular*** *if every set in $\mathcal{M}$ is outer regular.*

(iv) *We say that the measure μ is* ***inner regular*** *if every set in $\mathcal{M}$ is inner regular.*

(v) *We say that the measure μ is* ***regular*** *if it is both outer and inner regular.*

Remark 1.34. There are slight variations in terminology in the literature for (inner/outer) regular measures, so keep this mind when reading books!

Theorem 1.35. *Let $(X, \mathcal{M}, \mu)$ be a measure space with μ σ-finite, and $\mathcal{M} \supseteq \mathcal{B}(X)$. If μ is outer regular then:*

(i) *For any $E \in \mathcal{M}$ and $\varepsilon > 0$, there exists a closed set C and an open set U such that $C \subseteq E \subseteq U$ and $\mu(U \setminus C) < \varepsilon$.*

(ii) *For any $E \in \mathcal{M}$, there exists a G_δ set G and an F_σ set F such that $F \subseteq E \subseteq G$ and $\mu(G \setminus F) = 0$.*

Remarks 1.36. (a) Definitions of G_δ and F_σ are in Section 1.4.
(b) This is closely related to [F, 1.19–1.20], but more general, borrowed from [R, 2.17].

Proof. (i) If $\mu(E) < \infty$, then use outer regularity gives an open set U such that $U \supseteq E$ with $\mu(U \setminus E) < \varepsilon/2$. If $\mu(E) = \infty$, then by σ-finiteness, $X = \bigcup_{j=1}^\infty X_j$ with $\mu(X_j) < \infty$, so $\mu(E \cap X_j) < \infty$, and by outer regularity we can find $U_j \supseteq E \cap X_j$ with $\mu(U_j \setminus (E \cap X_j)) < \varepsilon/2^{j+1}$. Then let $U = \bigcup_{j=1}^\infty U_j$: we get $U \setminus E = \bigcup U_j \setminus E \subseteq \bigcup U_j \setminus (E \cap X_j)$, so that $\mu(U \setminus E) < \varepsilon/2$.

Now applying this to E^c instead of E, we get a closed $C \subseteq E$ with $\mu(E \setminus C) < \varepsilon/2$, which completes the proof of (i).

(ii) Apply (i) with $\varepsilon = 1/n$ to get closed set C_n and open sets U_n with $C_n \subseteq E \subseteq U_n$ and $\mu(U_n \setminus C_n) < 1/n$. Then take $F = \bigcup C_j$ (an F_σ set) and $G = \bigcap U_j$ (a G_δ set). Then $F \subseteq E \subseteq G$ and $\mu(G \setminus F) \le \mu(U_n \setminus C_n) < 1/n$ for all n. □

1.13 Lebesgue–Stieltjes measures on $\mathbb{R}$: regularity

Theorem 1.37. *For any F (see Theorem 1.30), the Lebesgue–Stieltjes measure $\overline{\mu}_F$ is regular.*

Remark 1.38. One can extend this to $\mathbb{R}^n$: any Borel measure on $\mathbb{R}^n$ that is finite on compacts is necessarily regular.

Proof. [F36]

Let us use μ instead of $\overline{\mu}_F$.

Outer regularity: $\mu(E) \le \inf$ is obvious from inclusion $E \subset U$. For the other direction, given $\varepsilon > 0$, by definition of μ^*, we can choose $\bigcup_{j=1}^\infty (a_j, b_j] \supseteq E$ with $\sum \mu((a_j, b_j]) \le \mu(E) + \varepsilon$. Then choose $\delta_j > 0$ with $F(b_j + \delta_j) - F(\delta_j) < \frac{\varepsilon}{2^j}$ (right-continuity). Then $E \subseteq \bigcup (a_j, b_j + \delta_j)$ with $\sum \mu((a_j, b_j + \delta_k)) \le \sum \mu((a_j, b_j]) + \varepsilon \le \mu(E) + 2\varepsilon$.

Inner regularity: Suppose E is bounded. Need to show $\mu(E) \le \sup$. If E is closed, then it's obvious. If not, then by (i), choose an open $U \supseteq \overline{E} \setminus E$ such that $\mu(U) \le \mu(\overline{E} \setminus E) + \varepsilon$ (note: $\overline{E} \setminus E$ is Borel so μ-measurable). Then we take $K = \overline{E} \setminus U = E \setminus U$ (draw a pic), compact and $\subseteq E$, get $\mu(K) = \mu(E) - \mu(E \cap U) = \mu(E) - [\mu(U) - \mu(U \setminus E)] \ge \mu(E) - \mu(U) + \mu(\overline{E} \setminus E) \ge \mu(E) - \varepsilon$. Finally, if E is unbounded, divide E into countable union of disjoint bounded sets, e.g., $E_j = E \cap (j, j+1]$, and then approximate each of them by a compact K_j, then take finite union. □

Remark 1.39. Combining with the results of the previous section, we obtain that any Borel set (or more generally, any Lebesgue set) is an F_σ-set plus a set of μ-measure zero (or a G_δ set minus a set of μ-measure zero).

1.14 Lebesgue measure on $\mathbb{R}$: translations and dilations

Theorem 1.40. *Let m be the Lebesgue measure on $\mathbb{R}$ and $\mathcal{L}$ be the σ-algebra of the Lebesgue measurable sets.*

(i) If $E \in \mathcal{L}$ then for any $s \in \mathbb{R}$, $\{x + s : x \in E\} =: E + s$ is also in $\mathcal{L}$ and $m(E + s) = m(E)$.

(ii) If $E \in \mathcal{L}$ then for any $r \in \mathbb{R}$, $\{rx : x \in E\} =: rE$ is also in $\mathcal{L}$ and $m(rE) = |r|\, m(E)$.

Remarks 1.41. (a) More generally, the Lebesgue measure m^n on $\mathbb{R}^n$ (defined, e.g., through volumes of $\mathbb{R}^n$-parallelepipeds and then extended via Carathéodory theorem; alternatively, it can be defined through the product of one-dimensional measures – to be discussed later). This measure also behaves well under translations, dilations, as well as rotations or more generally under any linear transformations, see [F70–76].

(b) It is not hard to see that, up to a normalization, m^n is a unique translation invariant Borel measure on $\mathbb{R}^n$.

Proof. [F37]

Collection of open intervals are translation and dilation invariant, so the same is true for Borel sets. Three measures $m(E)$, $m_s(E) := m(E + s)$, $m_r(E) = m(rE)/|r|$ all agree on intervals, and then by σ-additivity, also on $\mathcal{B}(\mathbb{R})$. Finally their completions must also be the same measure. ☐

1.15 Lebesgue measure on $\mathbb{R}$: measurable and non-measurable sets

Examples 1.42. (a) Any point, and therefore any countable set (e.g., $\mathbb{Q}$), is clearly a Lebesgue measurable set of measure 0.

(b) Does there exist a Lebesgue measurable set that isn't Borel? Yes (many!). One way to see it to compare their cardinality ($2^{\mathfrak{c}}$ and $\mathfrak{c}$, respectively), see [F39].

(c) Does there exist a Lebesgue non-measurable set? The classical example is a *Vitali set*: subdivide $\mathbb{R}$ into disjoint sets of the form $r + \mathbb{Q}$ (put it another way, into elements of the quotient group $\mathbb{R}/\mathbb{Q}$). Choose one point in $[0, 1]$ from each of the disjoint sets (using the axiom of choice). Denote this set V. Enumerate $\mathbb{Q} \cup [-1, 1] = \{q_j\}_{j=1}^{\infty}$, and note that $V_j = V + q_j$ are pairwise disjoint sets satisfying $[0, 1] \subseteq \bigcup_{j=1}^{\infty} V_j \subseteq [-1, 2]$. If V were Lebesgue measurable, then every V_k has equal measure, and we get $1 \le \sum_{j=1}^{\infty} m(V_k) \le 3$ which is a contradiction.

1.16 The Cantor set

The Cantor set is obtained by repeatedly removing one thirds of the interval from $[0, 1]$. Namely, first we remove the set $U_{1,1} = (\frac{1}{3}, \frac{2}{3})$ (one open interval of length $\frac{1}{3}$). Then remove one thirds of the remaining two intervals: $U_{2,1} = (\frac{1}{9}, \frac{2}{9})$ and $U_{2,2} = (\frac{7}{9}, \frac{8}{9})$ (two open intervals of total length $\frac{2}{9}$). We continue in the same fashion: on the n-th step we remove $U_{n,1} = (\frac{1}{3^n}, \frac{2}{3^n}), \ldots, U_{n,2^{n-1}} = (1 - \frac{2}{3^n}, 1 - \frac{1}{3^n})$ (2^{n-1} disjoint open intervals of total length $\frac{2^{n-1}}{3^n}$). Define

$$C_n = [0, 1] \setminus \bigcup_{k=1}^{n} \bigcup_{j=1}^{2^{n-1}} U_{k,j}$$

and the **Cantor set** to be

$$C = [0, 1] \setminus \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{2^{n-1}} U_{k,j} = \bigcap_{n=1}^{\infty} C_n.$$

Exercise 1.43. Show that

(i) C is compact, nowhere dense (i.e., its closure has empty interior), totally disconnected (the only connected subset are single points), perfect (non-empty, closed, with every point being an accumulation point), with no isolated points.

(ii) $m(C) = 0$.

(iii) $\mathrm{card}(C) = \mathfrak{c}$.

Equivalently, the Cantor set can also be defined as the set of all points in $[0, 1]$ whose base-three expansion has digits 0 or 2 only, see [F38].

Let us also define the **Cantor function** $c(x)$ as follows: $c(x) = \frac{1}{2}$ on $U_{1,1}$; $c(x) = \frac{1}{4}$ on $U_{2,1}$, $c(x) = \frac{3}{4}$ on $U_{2,2}$; and so on: $c(x) = \frac{2j-1}{2^k}$ on $U_{k,j}$. This defines $c(x)$ on $[0, 1] \setminus C$ (note that this set is dense everywhere on $[0, 1]$). Then for $x \in C$ we define

$$c(x) = c(\sum_{j=1}^{\infty} \frac{a_j}{3^j}) = \sum_{j=1}^{\infty} \frac{a_j/2}{3^j}$$

(each a_j is 0 or 2 here by the discussion above).

Exercise 1.44. Show that

(i) c defined above is continuous on $[0, 1]$.

(ii) c is C^1 on $[0, 1] \setminus C$ with $c'(x) = 0$ there.

(iii) For $x \in C$, $\lim_{\varepsilon \downarrow 0} \frac{c(x+\varepsilon)-c(x-\varepsilon)}{2\varepsilon} = \infty$.

Remarks 1.45. (a) Note that $c(1) - c(0) = 1$, even though $c'(x) = 0$ Lebesgue-a.e.!

(b) One can form the so-called generalized Cantor sets by removing not always 1/3's of the intervals, but varying amounts. These Cantor sets can have positive Lebesgue measure [F39].
