Chapter 3

$L^p$ spaces

3.1 Banach and Hilbert spaces

Definition 3.1. Let $X$ be a vector space (over $\mathbb{C}$).

(i) We call a function $\| \cdot \| : X \to [0, \infty)$ a norm if it satisfies

(a) (triangle inequality) $\| x + y \| \leq \| x \| + \| y \| ;$
(b) $\| \lambda x \| = |\lambda| \cdot \| x \|$ for any $\lambda \in \mathbb{C};$
(c) $\| x \| = 0$ iff $x = 0.$

Note: if (a) and (b) hold but (c) is not imposed, then we call $\| \cdot \|$ a seminorm.

(ii) $X$ together with a norm $\| \cdot \|$ is called a normed space.

(iii) $X$ is called a Banach space if it’s a normed space that is complete with respect to the norm: that is, if $\{ x_j \}_{j=1}^\infty$ is a Cauchy sequence ($\| x_n - x_m \| \to 0$ as $n, m \to \infty$) then $\| x_j - x \| \to 0$ for some element $x \in X.$

(iv) We call a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ an inner product if it satisfies

(a) (conjugate symmetry) $\langle x, y \rangle = \langle y, x \rangle;$
(b) (linearity in the first argument) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, x \rangle$ for any $\alpha, \beta \in \mathbb{C};$
(c) $\langle x, x \rangle \geq 0$ for all $x$ and $\langle x, x \rangle = 0$ iff $x = 0.$

Note: it is not hard to show that $\| x \| := \sqrt{\langle x, x \rangle}$ is then a norm.

(v) $X$ together with an inner product $\langle \cdot, \cdot \rangle$ is called an inner product space (or a pre-Hilbert space).

(vi) $X$ is called a Hilbert space if it’s an inner product space that is complete with respect to the norm $\| x \| := \sqrt{\langle x, x \rangle}.$
3.2 $L^p$ spaces: definition

**Definition 3.2.** Let $(X, \mathcal{M}, \mu)$ be a measure space. In the following two definition we identify two functions if they are equal to each other $\mu$-a.e..

(i) For $1 \leq p < \infty$, we define the $L^p(\mu)$ space to be the normed space of (equivalence classes of) measurable functions on $X$ such that

$$||f||_p := \left( \int_X |f|^p \, d\mu \right)^{1/p} < \infty$$

(ii) Define the $L^\infty(\mu)$ space to be the normed space of (equivalence classes of) measurable functions on $X$ whose essential supremum $||f||_\infty$ (or $\text{ess sup} |f(x)|$) is finite:

$$||f||_\infty := \inf \{ a \geq 0 : \mu(\{ x : |f(x)| > a \}) = 0 \} < \infty$$

**Remarks 3.3.** (a) That $|| \cdot ||_p$ is in fact a norm (that is, it satisfies the triangle inequality) follows from the Minkowski’s inequality, see Section 3.3.

(b) $|| \cdot ||_p$ for $p < 1$ fails the triangle inequality, so $L^p$ isn’t a normed space for such $p$.

(c) In particular, $|f(x)| \leq ||f||_\infty$ for $\mu$-a.e. $x$, and $||f||_\infty$ is the smallest constant with such property.

(d) If $X$ is $\mathbb{N}$, and $\mu$ is a counting measure, then it is easy to see that each function in $L^p(\mu)$, $1 \leq p \leq \infty$, can be identified with the sequence $\{f_j\}_{j=1}^\infty$ (or $\{f_j\}_{j \in \mathbb{Z}}$, respectively) satisfying $\sum |f_j|^p < \infty$. This special case of $L^p(\mu)$ is then denoted $l^p(\mathbb{N})$. If instead of $\mathbb{N}$, we have any other set $A$ with the counting measure $\mu$, then we also use the notation $l^p(A)$ for $L^p(\mu)$.

(e) $l^\infty(\mathbb{N})$ is then just the space of all bounded sequences.

3.3 A bunch of inequalities

**Definition 3.4.** A function $\phi : (a, b) \to \mathbb{R}$ is called convex if

$$\phi(1 - \lambda)x + \lambda x \leq (1 - \lambda)\phi(x) + \lambda \phi(y)$$

holds for any $x, y \in (a, b)$ and any $\lambda \in [0, 1]$.

**Remarks 3.5.** (a) $a = -\infty$ and/or $b = +\infty$ are allowed.

(b) The condition (3.3.2) can be reparsed to

$$\frac{\phi(t) - \phi(x)}{t - x} \leq \frac{\phi(y) - \phi(t)}{y - t}$$

(3.3.1)

for all $a < x < t < y < b$. This can be easily understood geometrically.

**Theorem 3.6 (Jensen’s Inequality).** Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$. Suppose $\phi$ is convex on $(a, b)$ and let $f \in L^1(\mu)$ with $f(x) \in (a, b)$ for all $x \in X$. Then

$$\phi \left( \int_X f \, d\mu \right) \leq \int_X \phi(f(x)) \, d\mu$$

(3.3.2)

**Proof.** Let $I = \int f \, d\mu$, $a < I < b$. Let $\beta = \sup_{a < x < t} \frac{\phi(I) - \phi(x)}{I - x}$. Then, see (3.3.1), $\beta \leq \frac{\phi(y) - \phi(I)}{y - I}$ for any $I < y < b$. Therefore $\phi(y) \geq \phi(I) + \beta(y - I)$ both for $I < y < b$ as well as $a < y \leq I$ (geometrically this is easy to believe too). Since $f(x) \in (a, b)$, we get

$$\phi(f(x)) \geq \phi(I) + \beta(f(x) - I).$$

Then integrating with respect to $\mu$, we get $\int \phi \circ f \, d\mu \geq \phi(I) + 0$. \hfill $\Box$

30
Definition 3.7. $p, q \in [1, \infty]$ are called conjugate exponents if

$$\frac{1}{p} + \frac{1}{q} = 1.$$ 

Examples 3.8. $p = q = 2$ and $p = 1, q = \infty$ are the most important special cases.

Theorem 3.9 (Young’s Inequality). Suppose $p$ and $q$ are conjugate exponents, $1 < p < \infty$. Then for all $x, y \geq 0$:

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$ 

Proof. Jensen’s inequality with $\phi(x) = e^x$, $X = \{x_1, x_2\}$, and $\mu(\{x_1\}) = 1/p, \mu(\{x_2\}) = 1/q, f(x_1) = p\log x_1, f(x_2) = q\log y$, gives us $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$. \hfill \qed

Theorem 3.10 (Hölder Inequality). Suppose $p$ and $q$ are conjugate exponents, $1 \leq p, q \leq \infty$. If $f$ and $g$ are measurable, then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

(3.3.3)

Remark 3.11. For $p = q = 2$ this is the Schwarz inequality (also, Cauchy–Bunyakovsky in some countries).

Proof. When one of $p$ or $q$ is equal to $\infty$, the result is obvious. So assume $1 < p < \infty$.

The result is also trivial if one of the norms are 0 or $\infty$. Note that scalar multiplication preserves the inequality so we may normalize: $F := |f|/\|f\|_p$ and $G := |g|/\|g\|_q$.

Apply Young’s inequality with $F(x)$ and $G(x)$ instead of $x, y$:

$$F(x)G(x) \leq \frac{1}{p}F(x)^p + \frac{1}{q}G(x)^q$$

which holds for every $x$. Integrating, we get

$$\int FGd\mu \leq \frac{1}{p} + \frac{1}{q} = 1.$$ 

\hfill \qed

Theorem 3.12 (Generalized Hölder’s Inequality). Let $1 \leq p, q, r \leq \infty$ with

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$ 

Then

$$\|fg\|_r \leq \|f\|_p \|g\|_q$$

(3.3.4)

Remark 3.13. This can be generalized even further, see [F, Ex.6.3.31].

Proof. Again, we can assume none of $p, q, r$ are $\infty$. Then let

$$\tilde{f} = |f|^r, \quad \tilde{g} = |g|^r,$$

and $\tilde{p} = \frac{r}{p}, \tilde{q} = \frac{r}{q}$. Then we get $\frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} = 1$, and (3.3.4) becomes reduced to (3.3.3). \hfill \qed

Theorem 3.14 (Minkowski’s Inequality). Let $1 \leq p \leq \infty$. Then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

for any $f, g \in L^p(\mu)$.
Proof. Inequality for \( p = 1 \) and \( p = \infty \) follows from the usual triangle inequality for \( \mathbb{C} \).

For \( 1 < p < \infty \), note that
\[
|f + g|^p \leq |f|^p|f + g|^{p-1} + |g|^p|f + g|^{p-1}.
\]

Then Hölder inequality gives
\[
\int |f| |f + g|^{p-1} \leq \left( \int |f|^p \right)^{1/p} \left( \int (|f + g|^{(p-1)q}) \right)^{1/q},
\]
\[
\int |g| |f + g|^{p-1} \leq \left( \int |g|^p \right)^{1/p} \left( \int (|f + g|^{(p-1)q}) \right)^{1/q},
\]

which, together with \((p-1)q = p\), imply
\[
\int |f + g|^p \leq (\|f\|_p + \|g\|_p) \left( \int |f + g|^p \right)^{1/q}.
\]

Dividing both sides by \( \left( \int |f + g|^p \right)^{1/q} \) (assuming it is non-zero) and using \( 1 - \frac{1}{q} = \frac{1}{p} \), we get, the desired inequality.

\[\square\]

3.4 Completeness

Theorem 3.15. (i) For any \( 1 \leq p \leq \infty \), and any positive measure \( \mu \), \( L^p(\mu) \) is a Banach space.

(ii) \( L^2(\mu) \) is a Hilbert space with the inner product
\[
\langle f, g \rangle := \int_X f(x)\overline{g(x)} \, d\mu(x).
\]

Proof. By Minkowski inequality, \( \| \cdot \|_p \) is a norm, so we just need to check completeness.

Let \( 1 \leq p < \infty \) first. Suppose \( \|f_n - f_m\|_p \to 0 \) as \( n, m \to \infty \). The idea of constructing the limiting function \( f(x) \) is to show that the series on the right-hand side of (3.4.1) converges if we choose \( n_j \) large enough (so that each term in the series is small).

Indeed, proceeding inductively we get \( \|f_{n_{j+1}} - f_{n_j}\|_p < 2^{-j} \) for some indices \( n_1 < n_2 < \ldots \)

Define \( g_k = \sum_{j=1}^{k} |f_{n_{j+1}} - f_{n_j}| \) and \( g(x) = \lim_{k \to \infty} g_k(x) \) (exists for all \( x \) by monotonicity). By Minkowski \( \|g_k\|_p < 1 \) for every \( k \). Since \( g_k \leq g_{k+1} \), we can use the Lebesgue Monotone Convergence theorem to conclude that \( \|g\|_p = \lim \|g_k\|_p \leq 1 \). Since \( g^p \in L^1(\mu) \), this means that \( g(x) < \infty \) for a.e. \( x \). By the definition \( g = \sum_{j=1}^{\infty} |f_{n_{j+1}} - f_{n_j}| \), i.e., the following series also converges (absolutely) for a.e. \( x \):
\[
f(x) := f_{n_1}(x) + \sum_{j=1}^{\infty} (f_{n_{j+1}} - f_{n_j}) \tag{3.4.1}
\]

(define \( f(x) = 0 \) on the null-set where the convergence fails). Note that this can also be rewritten as \( f(x) = \lim_{j \to \infty} f_{n_j}(x) \) a.e.. Note that \( |f| \leq |f_{n_1}| + g \) is in \( L^p \) since both \( f_{n_1} \) and \( g \) are in \( L^p \). We need to show that \( \|f - f_n\|_p \to 0 \).

Choose \( \varepsilon > 0 \) and find \( N \) such that \( \|f_n - f_m\|_p < \varepsilon \) for all \( n, m \geq N \). Then for \( m \geq N \), by Fatou’s lemma
\[
\|f - f_m\|_p^p = \int \liminf_{j \to \infty} |f_{n_j}(x) - f_m(x)|^p \, d\mu \leq \liminf_{j \to \infty} \int |f_{n_j}(x) - f_m(x)|^p \, d\mu = \liminf_{j \to \infty} \|f_{n_j} - f_m\|_p^p \leq \varepsilon^p.
\]
This shows that $\|f - f_m\|_p \to 0$ as $m \to \infty$.

Finally, consider the $p = \infty$ case. Let $\{f_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $L^\infty(\mu)$: $\|f_n - f_m\|_\infty \to 0$ as $n, m \to \infty$. Note that for $\mu$-a.e. $x$ (union of countably many $\mu$-null sets is a $\mu$-null set), $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty$ for all $m, n$. So $f(x) := \lim f_n(x)$ exists $\mu$-a.e., and we define $f = 0$ for other $x$.

Let $\varepsilon_m := \sup_{n \geq m} \|f_n - f_m\|_\infty$. Since $\varepsilon_m \to 0$ by the Cauchy property, we have $\varepsilon_N \leq 1$ for some large enough $N$. Then for a.e. $x$, $|f(x) - f_N(x)| = \lim_{j \to \infty} |f_j(x) - f_N(x)| \leq \lim_{j \to \infty} \|f_j - f_N\|_\infty \leq \varepsilon_N < 1$. So $f - f_N \in L^\infty(\mu)$, which implies $f = f+ (f_N - f) \in L^\infty(\mu)$. The last inequality also shows that $\|f - f_N\|_\infty \to 0$ as $N \to \infty$.

\[ \square \]

### 3.5 Inclusions for $L^p$ and $\ell^p$ spaces

**Intuition.** We want to understand the relationship between $L^p$ spaces for varying $p$. The idea is that $t^2 \geq t$ (lower exponent is better for convergence) if $t \geq 1$, and $t^2 \leq t$ (higher exponent is better for convergence) if $0 \leq t \leq 1$. We make this rigorous in Theorem 3.16.

**Theorem 3.16.** For any $1 \leq p < q \leq r \leq \infty$, $L^q \subseteq L^p + L^r$, that is any function in $L^q(\mu)$ is the sum of a function in $L^p(\mu)$ and a function in $L^r(\mu)$.

**Proof.** Let us split $f \in L^q(\mu)$ into two parts – where $|f| > 1$ and where $|f| \leq 1$: $f = g + h$ with $g = f \chi_{|x|>1}$ and $h = f \chi_{|x| \leq 1}$. Since $f \in L^q$, we also have $g, h \in L^q$. Now, $|g|^p \leq |g|^q$, so $g \in L^p$, and $|h|^r \leq |h|^q$, so $h \in L^r$ (if $r = \infty$, then $|h| \leq 1$ clearly implies $|h|^\infty \leq 1$).

**Theorem 3.17.** For any $1 \leq p < q < r \leq \infty$, $L^p \cap L^r \subseteq L^q$.

**Proof.** One can follow the same idea as before: $f = g + h$ with $g = f \chi_{|x|>1}$ and $h = f \chi_{|x| \leq 1}$. Since $f \in L^p \cap L^r$, we also have $g, h \in L^p \cap L^r$. Now, as before $g \in L^r$ implies $g \in L^q$ (as in the previous proof, since $|g| \geq 1$, we can pass to the lower exponent), and $h \in L^p$ implies $h \in L^q$ (since $|h| \leq 1$, we can pass to the higher exponent). This means $f \in L^q$.

Alternatively, one can prove the inequality

\[ \|f\|_q \leq \|f\|_p^{\lambda} \|f\|_p^{1-\lambda}, \]

where $\lambda \in (0, 1)$ is defined from $\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}$. This is a direct corollary of generalized Hölder’s inequality if we take $|f|^\lambda$ and $|f|^{1-\lambda}$ instead of $f$ and $g$, and $\frac{p}{q}$, $\frac{r}{q}$, $q$ instead of $p, q, r$, respectively, in (3.3.4). \[ \square \]

**Theorem 3.18.** If $\mu(X) < \infty$ and $1 \leq p < q \leq \infty$, then $L^p(\mu) \supseteq L^q(\mu)$.

**Remark 3.19.** The inclusion fails if $\mu(X) = \infty$ as a simple counterexample $f(x) \equiv 1$ on $(\mathbb{R}, B(\mathbb{R}), m)$ shows.

**Proof.** Note that $\mu(X) < \infty$ means that function 1 is in any $L^p$. So we only need to worry about functions $f$ on the set $\{x : |f| > 1\}$ and not on $\{x : |f| \leq 1\}$.

Indeed, let $f \in L^q$, and let as before $f = g + h$. Then $h$ is in every $L^r$ ($1 \leq r \leq \infty$), while $g \in L^p$ (we can go to lower exponent). Therefore $f \in L^p$.

Alternatively, one can prove that

\[ \|f\|_p \leq \|f\|_q \mu(X)^{\frac{1}{p} - \frac{1}{q}}. \]

which follows from Hölder’s inequality with functions $|f|^p$ and 1 and exponents $\frac{p}{p}$ and $\frac{q}{q-p}$:

\[ \|f\|_p = \int |f|^p \cdot 1 \, d\mu \leq \|f\|_q |\mu(X)|_{q/p}^{1-1} |\mu(X)|_{(q-p)/q} = \|f\|_q \mu(X)^{(q-p)/q}. \]

\[ \square \]
Theorem 3.20. For any $1 \leq p < q \leq \infty$, we have $\ell^p(\mathbb{N}) \subset \ell^q(\mathbb{N})$.

Remark 3.21. One can take any set $A$ instead of $\mathbb{N}$.

Proof. If $f \equiv \{f_j\}_{j=1}^{\infty} \in \ell^p$ then $\sum_{j=1}^{\infty} |f_j| < \infty$, so $f_j \to 0$, so eventually $|f_j| < 1$.

Decompose as above $f = g + h$ where $g = f \chi_{\{|f|>1\}}$ and $h = f \chi_{\{|f|\leq1\}}$. Then $g$ is supported on finitely many points, so $g \in \ell^r$ for any $r$. While for $h$: $h \in \ell^p$ implies that $h \in \ell^q$ (we can go to higher exponent since $|h| \leq 1$).

Alternatively, one can prove that for sequences we have

$$||f||_q \leq ||f||_p$$

which follows by applying (3.5.1) with $r = \infty$ and combining it with $||f||_\infty \leq ||f||_p$. \qed

3.6 Dense subspaces of $L^p$ spaces

Intuition. Given a function in $L^p(\mu)$ space, it is natural to ask how well we can approximate it by a simpler class of functions, such as simple or continuous functions. We explore these questions here.

Theorem 3.22. Let $(X, \mathcal{M}, \mu)$ be a measure space.

Let $S$ be the class of (complex-valued) simple measurable functions $\sum_{j=1}^{n} \alpha_j \chi_{E_j}$, where $n < \infty$, $\alpha_j \in \mathbb{C}$, $\mu(E_j) < \infty$.

Let $\tilde{S}$ be the class of (complex-valued) simple measurable functions $\sum_{j=1}^{\infty} \alpha_j \chi_{E_j}$, where $n < \infty$, $\alpha_j \in \mathbb{C}$, but with $\mu(E_j)$ allowed to be infinite.

(i) $S$ is dense in $L^p(\mu)$ for any $1 \leq p < \infty$.

(ii) $\tilde{S}$ is dense in $L^\infty(\mu)$.

Remark 3.23. In general (i) wouldn’t work for $p = \infty$, as the counterexample $f \equiv 1$ on $L^\infty(\mathbb{R}, m)$ shows.

Proof. (i) Clearly, $S \subset L^p$. Now, given $f \in L^p \cap L^+$, approximate $f$ from below by simple functions $\phi_n$ as usual (see the proof of Proposition 2.8). Then $0 \leq \phi_n \leq f$, $\phi_n \not\to f$. Note that $\phi_n \leq f$, so $\phi_n \in L^p$, so $\mu(E_j) < \infty$ for any $\phi_n$. Since $|f - \phi_n|^p \leq |f|^p$, we can use Dominated Convergence Theorem to conclude that $\lim ||f - \phi_n||_p = \lim(\int |f - \phi_n|^p d\mu)^{1/p} = 0$, in other words, $f$ is in the closure of $S$. For complex $f$, we approximate $\text{Re} f$ and $\text{Im} f$ separately.

(ii) For $f \in L^\infty(\mu) \cap L^+$, first we choose a representative of the equivalence class of $f$ that is bounded. Then we use again the approximation $\{\phi_n\}$ from the proof of Proposition 2.8. Clearly, $\phi_n \in \tilde{S}$ and $||\phi_n - f||_\infty \leq \frac{1}{n}$ for $n$ large enough. \qed

Theorem 3.24. Let $(X, \mathcal{M}, \mu)$ be a measure space with $X$ locally compact and Hausdorff. Suppose $\mu$ is regular, Borel, $\sigma$-finite. Let $1 \leq p < \infty$.

Then $C_c(X)$ is dense in $L^p(\mu)$.

Remark 3.25. It is clear that for $p = \infty$ this fails in general. For example, if $X = \mathbb{R}^n$, $\mu = m^n$, then the completion of $C_c(\mathbb{R}^n)$ in the $|| \cdot ||_\infty$-norm is not $L^\infty$ but $C_0(\mathbb{R}^n)$, the space of all continuous functions on $\mathbb{R}^n$ which vanish at $\infty$, that is, those $f$ for which $\lim_{|x| \to \infty} f(x) = 0$. This can be generalized to more general setting than $X = \mathbb{R}^n$. 

34
Proof. By the previous theorem, we just need to be able to \( \| \cdot \|_p \)-approximate functions \( \chi_E \) with \( \mu(E) < \infty \) by \( C_c(X) \) functions.

Given \( \varepsilon > 0 \), by regularity and \( \sigma \)-finiteness of \( \mu \) (see Theorem 1.35 — we can choose compact rather than just closed by using inner regularity; this works even for \( \sigma \)-finite case as countable intersection of compacts is compact for Hausdorff spaces), we can find a compact set \( K \subseteq E \) and an open set \( U \supseteq E \) such that \( \mu(U \setminus K) < \varepsilon / \varepsilon^1/p \). By Urysohn’s lemma applied to the closed sets \( K \) and \( U_c \), we can find a function \( f \in C_c(X) \) such that \( \chi_K \leq f \leq \chi_U \). Then \( \| \chi_E - f \|_p \leq \mu(U \setminus K)^{1/p} < \varepsilon^{1/p} \).

\[
3.7 \text{ Linear functionals}
\]

Recall that in Section 2.24 we had discussed linear functionals on the space \( C_c(X) \) of continuous compactly supported functions. Linear functionals can of course be defined over arbitrary vector spaces.

**Definition 3.26.** Let \( X \) be a vector space over \( \mathbb{C} \).

(i) We say that a map \( \phi : X \to \mathbb{C} \) is a **linear functional** on \( X \) if \( \phi(x + y) = \phi(x) + \phi(y) \) and \( \phi(\alpha x) = \alpha \phi(x) \) for \( \alpha \in \mathbb{C} \).

(ii) The space of all linear functionals on \( X \) forms a vector space which is called the **algebraic dual space** of \( X \).

**Remark 3.27.** If \( X \) is equipped with a partial order \( \leq \) that is compatible with the vector addition and scalar multiplication (in the natural way you’d expect), then we call a functional positive if \( x \geq 0 \) implies \( \phi(x) \geq 0 \). We encountered this in Section 2.24 in the special case when \( X \) was the (partially ordered) space of continuous compactly supported functions.

**Definition 3.28.** Now let \( X \) be a Banach space with norm \( \| \cdot \| \).

(i) We say that a linear functional \( \phi \) is **bounded** (or **continuous**) if there is \( C > 0 \) such that \( |\phi(x)| \leq C \|x\| \) for all \( x \in X \).

(ii) The space of all bounded linear functionals on \( X \) forms a vector space which is called the **dual space** of \( X \), denoted by \( X^* \).

**Remarks 3.29.** (a) Some authors may call \( X^* \) the **continuous dual space** or **topological dual space**. We will just call it dual.

(b) Clearly, \( X^* \), the dual space of \( X \), is a subspace of the algebraic dual space of \( X \).

(c) If \( X \) is a Banach space, then \( X^* \) is easily seen to be a normed space with the norm defined by

\[
\|\phi\| = \sup\{|\phi(x)| : x \in X, \|x\| \leq 1\} = \sup\left\{ \frac{|\phi(x)|}{\|x\|} : x \in X, x \neq 0 \right\}.
\]

In fact, it is not much work to show that \( X^* \) is complete, i.e., a Banach space.

(d) A rough way to state the Remark 2.76(c) (which is also referred to as a Riesz–Markov representation theorem) is to say that the dual \( C_0(X)^* \) of \( C_0(X) \) (space of continuous functions vanishing at infinity, the completion of \( C_c(X) \)) is the space of all (complex, in particular finite) regular Borel measures on \( X \).
3.8 Duals of $L^p$

**Intuition.** Given $g \in L^q(\mu)$ (1 $\leq q < \infty$), according to Hölder’s inequality, the map $f \mapsto \int fg \, d\mu$ is a bounded linear functional on $L^p(\mu)$. Does every bounded linear functional on $L^p$ arise in this way? The answer is yes for $1 \leq p < \infty$ (at least if $\mu$ is $\sigma$-finite), but not for $p = \infty$.

**Theorem 3.30.** Suppose $1 \leq p < \infty$ and $\mu$ is a $\sigma$-finite (positive) measure. Then for any bounded linear functional $\phi$ on $L^p(\mu)$ there is a unique $g \in L^q(\mu)$ (where $\frac{1}{p} + \frac{1}{q} = 1$) such that

$$
\phi(f) = \int fg \, d\mu, \quad \text{for all } f \in L^p(\mu).
$$

(3.8.1)

Moreover, $\|\phi\| = \|g\|_q$.

**Remarks 3.31.** (a) Morally, one can say that $(L^p)^* = L^q$ if $1 \leq p < \infty$ and $\mu$ is $\sigma$-finite.

(b) The statement of the theorem is still correct without the $\sigma$-finiteness assumption provided that $1 < p < \infty$.

(c) In particular, $(L^p)^* = L^p$ for $1 < p < \infty$. Spaces satisfying such condition are called reflexive.

(d) The statement for $p = \infty$ fails. Indeed, $(L^\infty)^*$ is much bigger than $L^1$.

**Proof.** Uniqueness: if $g$ and $\tilde{g}$ both satisfy (3.8.1) then we can take $f = \chi_E$ for any measurable $E$ with $\mu(E) < \infty$, giving $\int_E (g - \tilde{g}) \, d\mu = 0$. This implies $g - \tilde{g} = 0$ a.e. using $\sigma$-finiteness of $\mu$.

We will suppose that $\mu(X) < \infty$ and leave the extension to the $\sigma$-finite case as an exercise ([IF90]). Note that (3.8.1) for $f = \chi_E$ takes the form $\phi(\chi_E) = \int g \, d\mu$, which looks like a (complex) $\mu$-absolutely continuous measure. This motivates us to define

$$
\nu(E) = \phi(\chi_E), \quad \text{for any } E \in \mathcal{M}.
$$

We want to show that $\nu(E) = \int_E \chi_E \, g \, d\mu = \int_E g \, d\mu$ for some $g \in L^q$. To do this, we will show: (1) $\nu$ is a (complex) measure; (2) $\nu$ is $\mu$-a.c.; (3) equality (3.8.1) holds with $g := \frac{d\nu}{d\mu} \in L^1$; (4) $g$ is in $L^q(\mu)$ and $\|g\|_q = \|\phi\|$. 

(1) Finite-additivity of $\nu$ follows from linearity of $\phi$ and $\chi_{A \cup B} = \chi_A + \chi_B$ for disjoint sets $A$ and $B$. For $\sigma$-additivity, let $E_\infty = \bigsqcup_{j=1}^\infty A_j$. Define $E_n = \bigsqcup_{j=1}^n A_j$. We need to show $\nu(E_\infty) = \sum_{j=1}^\infty \nu(A_j)$ equals $\lim \nu(E_n)$. We use continuity of $\phi$ to get $|\nu(E_\infty) - \nu(E_n)| = |\phi(\chi_{E_\infty} - \chi_{E_n})| \leq C ||\chi_{E_\infty} - \chi_{E_n}||_p$. Now note that $||\chi_{E_\infty} - \chi_{E_n}||_p = |\mu(E_\infty \setminus E_n)|^{1/p} \to 0$ by continuity of $\mu$. (recall that $p < \infty$).

(2) If $\mu(E) = 0$, then $\chi_E(x) = 0$ ($\mu$-a.e.), so that $||\chi_E||_p = 0$, which implies $\nu(E) = \phi(\chi_E) = 0$ by linearity. Therefore $\nu \ll \mu$.

(3) By (2) and the Radon-Nikodym theorem, $d\nu = g \, d\mu$ for some $g \in L^1(\mu)$. In other words, $\int_E d\nu = \phi(\chi_E) = \int_E g \, d\mu = \int_X \chi_E g \, d\mu$.

By linearity of integral and of $\phi$, we get (3.8.1) for any $f$ that is a simple function.

We can further extend (3.8.1) to $f \in L^\infty$: indeed, by Theorem 3.22, we can find simple functions $s_n \to f$ in $|| \cdot ||_\infty$-norm, which implies $s_n \to f$ in $|| \cdot ||_p$-norm since $\mu(X) < \infty$, and then we can take limits of both sides in (3.8.1) with $s_n$. We will get (3.8.1) with $f \in L^p(\mu)$ later; having $f \in L^\infty(\mu)$ will be sufficient for now.

(4) Suppose first that $1 < p < \infty$ (so that $p \neq 1$, $q \neq \infty$). Then define $f = |g|^{q-1}sgn g$. Note that $|f|^p = |g|^q = fg$, so we expect from (3.8.1)

$$
\int_X |g|^q \, d\mu = \int fg \, d\mu = \phi(f) \leq ||\phi|| \cdot ||f||_p = ||\phi|| \left( \int_X |g|^q \, d\mu \right)^{1/p},
$$

36
but we cannot plug \( f \) into (3.8.1) since we don’t have \( f \in L^\infty \). To fix this, let \( f_n = |g|^q \text{sgn} \chi_{E_n} \) where \( E_n = \{ x : |g(x)| \leq n \} \). Then \( |f_n|^p = |g|^q = fg \) on \( E_n \), \( f_n \in L^\infty \), and we get
\[
\int_{E_n} |g|^q \, d\mu = \int_X f_n g \, d\mu = \phi(f_n) \leq ||\phi||_p ||f_n||_p = ||\phi|| \left( \int_{E_n} |g|^q \, d\mu \right)^{1/p},
\]
which implies \( \left( \int \chi_{E_n} |g|^q \, d\mu \right)^{1/q} \leq ||\phi|| \). Applying Monotone Convergence Theorem, we get \( g \in L^q \) and \( ||g||_q \leq ||\phi|| \). This allows us to extend (3.8.1) to \( f \in L^p(\mu) \) in the exact same way as before: for any \( f \in L^p \), take simple functions \( s_n \to f \) in \( ||\cdot||_p \)-norm, and then take limits of both sides of (3.8.1). Because \( g \in L^q \), this works now. Finally, having (3.8.1) for all \( f \in L^p \) allows us to use Hölder's inequality to conclude \( ||\phi|| \leq ||g||_q \), so we get \( ||\phi|| = ||g||_q \).

Now let \( p = 1, q = \infty \). Take any \( M < ||g||_\infty \), and let \( A = \{ x : |g(x)| > M \} \). Note that \( 0 < \mu(A) < \infty \), and we can take \( f = \chi_A \text{sgn} g \). Since \( f \in L^\infty \), (3.8.1) can be applied to get
\[
M \mu(A) \leq \int_A |f| \, d\mu = \int_X f g \, d\mu = \phi(f) \leq ||\phi|| ||f||_1 = ||\phi|| \mu(A),
\]
so we proved that \( M < ||g||_\infty \) implies \( M \leq ||\phi|| \). This proves that \( ||g||_\infty \leq ||\phi|| \) and in particular \( g \in L^\infty \). This allows to extend (3.8.1) to all \( f \in L^1 \), and then use Hölder’s inequality to conclude \( ||\phi|| \leq ||g||_\infty \), so that \( ||\phi|| = ||g||_q \).

\[ \square \]

### 3.9 Riesz Representation Theorem

**Intuition.** The duality theorem from the previous section states in particular that \((L^2)^* = L^2\), or more precisely, every bounded linear functional on \( L^2(\mu) \) has the form
\[
\phi(f) = \int f g \, d\mu, \quad \text{for all } f \in L^2(\mu)
\]
for some \( g \in L^2 \). The last expression can also be written as \( \phi(f) = \langle f, g \rangle \). This is the special case \( H = L^2(\mu) \) of the Riesz Representation Theorem which holds for an arbitrary Hilbert space \( H \).

**Theorem 3.32** (Riesz Representation Theorem). Let \( H \) be a Hilbert space. For any \( g \in H \), define
\[
\phi_g(f) = \langle f, g \rangle, \quad \text{for any } f \in H.
\]
Then \( \phi_g \in H^* \), and conversely, every bounded linear functional on \( H \) has the form \( \phi_g \) for a unique \( g \in H \).

**Proof.** [F187]

The uniqueness of \( g \): if \( \langle f, g \rangle = \langle f, \tilde{g} \rangle \) for all \( f \in H \), then taking \( f = g - \tilde{g} \), we get \( ||g - \tilde{g}|| = 0 \), i.e., \( g = \tilde{g} \).

Now for existence, if \( \phi \) is a bounded linear functional on \( H \), then either \( \phi \equiv 0 \) (in which case we take \( g = 0 \)), or otherwise, let \( K = \ker \phi = \{ f \in H : \phi(f) = 0 \} \). Note that in order for (3.9.2) to hold, we must have \( g \in K^\perp \). Choose any \( z \in K^\perp \) \((K \text{ is a closed proper subspace in } H, \text{ so } K^\perp \neq \{ 0 \})\). Then for arbitrary \( f \), \( \phi(f)z - \phi(z)f \) is in \( K \), so
\[
0 = \langle \phi(f)z - \phi(z)f, z \rangle = \phi(f)||z||^2 - \phi(z) \langle f, z \rangle.
\]
Rearranging we get
\[
\phi(f) = \left\langle f, \frac{\phi(z)}{||z||^2} \right\rangle,
\]
so we can take \( q = \frac{2||z||}{||z||^2} \).

\[ \square \]
3.10 Linear operators: definition

Definition 3.33. Let $X$ and $Y$ be normed spaces.

(i) We say that a function $T : X \to Y$ is a **bounded linear operator** if $T$ is linear and there exists $C > 0$ such that $\|T(x)\|_Y \leq C \|x\|_X$ for all $x \in X$.

(ii) The **operator norm** $\|T\|$ of a bounded linear operator $T$ is defined to be the smallest such constant $C$, or, in other words:

$$\|T\| = \sup \left\{ \frac{\|T(x)\|_Y}{\|x\|_X} : x \in X, x \neq 0 \right\}.$$ 

(iii) The space of all bounded linear operators from $X$ to $Y$ is denoted by $L(X,Y)$.

Remarks 3.34. 1. In particular, $L(X,\mathbb{C}) = X^*$.

2. It can be shown that if $Y$ is Banach, then $L(X,Y)$ with the operator norm is also a Banach space.

3.11 Riesz–Thorin Interpolation Theorem

Theorem 3.35 (Riesz–Thorin Interpolation Theorem). Suppose $(X,M,\mu)$ and $(Y,N,\nu)$ are two (σ-finite) measure spaces, and let $p_0, p_1, q_0, q_1 \in [1,\infty]$. For each $0 < t < 1$, let $p_t, q_t$ be defined through

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$ 

Let $T : L^{p_0}(\mu) + L^{p_1}(\mu) \to L^{q_0}(\nu) + L^{q_1}(\nu)$ be linear and satisfy

$$\|T(f)\|_{q_0} \leq M_0 \|f\|_{p_0},$$

$$\|T(f)\|_{q_1} \leq M_1 \|f\|_{p_1}.$$ 

Then for any $0 \leq t \leq 1$, $T$ is a bounded linear operator from $L^{p_t}(\mu)$ to $L^{q_t}(\nu)$ and $\|T(f)\|_{q_t} \leq M_0^{1-t} M_1^t \|f\|_{p_t}$.

Proof. This is mostly complex analysis and will be skipped, see [F200–202] if interested. \qed