Monotonicity of the difference between median and mean of Gamma distributions and of a related Ramanujan sequence

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Abstract

For $n \geq 0$, let λ_n be the median of the $\Gamma(n+1,1)$ distribution. We prove that the sequence $\{\alpha_n = \lambda_n - n\}$ decreases from $\log 2$ to $2/3$ as n increases from 0 to ∞. The difference, $1-\alpha_n$, between the mean and the median thus increases from $1-\log 2$ to $1/3$. This result also proves the following conjecture by Chen & Rubin about the Poisson distributions: Let $Y_\mu \sim$ Poisson(μ), and λ_n be the largest μ such that $P(Y_\mu \leq n) = 1/2$, then $\lambda_n - n$ is decreasing in n.

The sequence $\{\alpha_n\}$ is related to a sequence $\{\theta_n\}$, introduced by Ramanujan, which is known to be decreasing and of the form $\theta_n = \frac{1}{3} + \frac{4}{135(n+k_n)}$, where $\frac{2}{21} < k_n \leq \frac{8}{45}$. We also show that the sequence $\{k_n\}$ is decreasing.

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1 Introduction

Let $Y_\mu \sim$ Poisson(μ), and λ_n be the largest μ such that $P(Y_\mu \le n) = 1/2$. Using the well-known relation between the Poisson and Gamma distributions, we get

$$\frac{1}{2} = P(Y_{\lambda_n} \le n) = P(X_{n+1} > \lambda_n),$$

where $X_{n+1} \sim \Gamma(n+1, 1)$, so that λ_n is the median of the $\Gamma(n+1,1)$ distribution.

Chen and Rubin [1] prove, in our notation, that

$$n + \frac{2}{3} < \lambda_n < n+1, \tag{1}$$

and conjectured that

$$\alpha_n = \lambda_n - n$$

is decreasing in n. By (1),

$$\frac{2}{3} < \alpha_n < 1.$$

This result was sharpened by Choi [2] to

$$\frac{2}{3} < \alpha_n \le \log 2.$$

Choi also gives the following asymptotic expansion for α_n,

$$\alpha_n = \frac{2}{3} + \frac{8}{405\,n} - \frac{64}{5103\,n^2} + \frac{2^7 \cdot 23}{3^9 \cdot 25\,n^3} + O\left(\frac{1}{n^4}\right),$$

which gives

$$\Delta\alpha_n = \alpha_n - \alpha_{n+1} = \frac{8}{405\,n^2} - \frac{1144}{25515\,n^3} + O\left(\frac{1}{n^4}\right),$$

so that $\{\alpha_n\}$ is decreasing for sufficiently large n. In the next section, we will show that the sequence $\{\alpha_n\}$ is in fact decreasing for all $n \ge 0$, with $\alpha_0 = \log 2$ and $\alpha_\infty = \frac{2}{3}$. This proves Conjecture 2 of Chen and Rubin [1].

The analysis of $\{\alpha_n\}$ (or $\{\lambda_n\}$) is closely related to the following problem by Ramanujan [6]: *Show that*

$$\frac{1}{2}e^n = 1 + \frac{n}{1!} + \frac{n^2}{2!} + \cdots + \theta_n \frac{n^n}{n!}, \quad \textit{where } \theta_n \textit{ lies between } \frac{1}{2} \textit{ and } \frac{1}{3}. \tag{2}$$

Ramanujan outlined a solution in [7]. Complete proofs were given by Szegő [9], who also proved that the sequence $\{\theta_n\}$ is decreasing, and Watson [10]. In his first letter to Hardy dated January 16, 1913, see [8], Ramanujan further claims that

$$\theta_n = \frac{1}{3} + \frac{4}{135(n+k_n)}, \quad \textit{where } k_n \textit{ lies between } \frac{8}{45} \textit{ and } \frac{2}{21}. \tag{3}$$

This was proved by Flajolet *et al.* [3]. We will use this result in the next section to prove that the sequence $\{\alpha_n\}$ is decreasing, and in Section 3 we also prove that the sequence $\{k_n\}$ decreases for all $n \ge 0$, from $k_0 = \frac{8}{45}$ to $k_\infty = \frac{2}{21}$.

Remark 1. Ramanujan's claim (3) is given as Exercise 1.2.11.3.13 by Knuth [4]! □

Remark 2. Our interest in the sequence $\{\lambda_n\}$ came from a statistical problem in safety analysis of Swedish nuclear power plants, where we, in order to estimate the mean μ of a Poisson distribution, needed to create un upper 50% confidence limit for μ given the observation n. This confidence limit is $n + \lambda_n$. □

2 Monotonicity of $\{\alpha_n\}$

The values of α_n can easily be computed for small n. For $n \leq 10$ they are given in Table 1.

Table 1: α_n and $\Delta\alpha_n$

n	α_n	$\Delta\alpha_n$
0	0.693147	0.014800
1	0.678347	0.004287
2	0.674060	0.002000
3	0.672061	0.001152
4	0.670909	0.000748
5	0.670161	0.000524
6	0.669637	0.000388
7	0.669249	0.000298
8	0.668951	0.000237
9	0.668715	0.000192
10	0.668522	0.000159

Theorem 3. The sequence $\{\alpha_n\}_0^\infty$ is decreasing in n for all $n \geq 0$.

The proof of the theorem consists of a number of steps. The first step is to establish a relation between α_n and Ramanujan's θ_n.

Lemma 4.

$$1 - \theta_n = \int_0^{\alpha_n} e^{-x} \cdot (1 + \frac{x}{n})^n \, dx.$$

Proof. Let, as in Knuth [4],

$$I_1 = \int_n^\infty e^{-t} \cdot t^n \, dt$$

and

$$I_2 = \int_n^{n+\alpha_n} e^{-t} \cdot t^n \, dt.$$

Then, by (2),

$$\frac{I_1}{n!} = P(X_{n+1} > n) = P(Y_n \leq n) = \frac{1}{2} + (1 - \theta_n)\frac{n^n}{n!}e^{-n}$$

and

$$\frac{I_2}{n!} = \frac{1}{2} - P(X_{n+1} < n) = \frac{1}{2} - (1 - \frac{I_1}{n!}) = (1 - \theta_n)\frac{n^n}{n!}e^{-n}.$$

By substituting $t = x + n$ in I_2, we get

$$I_2 = e^{-n} \int_0^{\alpha_n} e^{-x}(n + x)^n \, dx = e^{-n} \cdot n^n \int_0^{\alpha_n} e^{-x}(1 + \frac{x}{n})^n \, dx,$$

which proves the lemma. $\square$

The second step is to constructively estimate the integral in Lemma 4. Let $\gamma_n = 1 - \theta_n$. The sequence $\{\gamma_n\}$ is then increasing for all $n \geq 0$. By (2), we have an explicit expression for θ_n, and hence for γ_n. Later, we will need γ_n for some small values of n, so the first few are given in Table 2.

Table 2: γ_n

n	$\gamma_n = 1 - \theta_n$	decimal
0	$\frac{1}{2}$	0.500000
1	$\frac{4-e}{2}$	0.640859
2	$\frac{10-e^2}{4}$	0.652736
3	$\frac{26-e^3}{9}$	0.657163
4	$\frac{206-3e^4}{64}$	0.659462
5	$\frac{2194-12e^5}{625}$	0.660867

Lemma 5. For $n \geq 3$, with $\gamma_n = 1 - \theta_n$,

$$\begin{aligned} \gamma_n &< \alpha_n - \frac{\alpha_n^3}{6\,n} + \frac{\alpha_n^4}{12\,n^2} + \frac{\alpha_n^5}{40\,n^2} - \frac{0.0022}{n^3}, \\ \gamma_n &> \alpha_n - \frac{\alpha_n^3}{6\,n} + \frac{\alpha_n^4}{12\,n^2} + \frac{\alpha_n^5}{40\,n^2} - \frac{0.0114}{n^3}. \end{aligned}$$

Proof. For $0 < x < 1$,

$$e^{-x}\left(1+\frac{x}{n}\right)^n \begin{cases} = \exp\left(-x + n\log\left(1+\frac{x}{n}\right)\right) = \exp\left(-\frac{x^2}{2\,n} + \frac{x^3}{3\,n^2} - \frac{x^4}{4\,n^3} + \dots\right), \\ < \exp\left(-\left(\frac{x^2}{2\,n} - \frac{x^3}{3\,n^2}\right)\right), \\ > \exp\left(-\left(\frac{x^2}{2\,n} - \frac{x^3}{3\,n^2} + \frac{x^4}{4\,n^3}\right)\right). \end{cases}$$

Now, for $0 < x < 1$,

$$1 - x + \frac{x^2}{2} - \frac{x^3}{6} < e^{-x} < 1 - x + \frac{x^2}{2},$$

so that,

$$\begin{aligned} e^{-x}\left(1+\frac{x}{n}\right)^n &< \exp\left(-\left(\frac{x^2}{2\,n} - \frac{x^3}{3\,n^2}\right)\right) \\ &< 1 - \left(\frac{x^2}{2\,n} - \frac{x^3}{3\,n^2}\right) + \frac{\left(\frac{x^2}{2\,n} - \frac{x^3}{3\,n^2}\right)^2}{2} \\ &= 1 - \frac{x^2}{2\,n} + \frac{x^3}{3\,n^2} + \frac{x^4}{8\,n^2} - \frac{x^5}{6\,n^3} + \frac{x^6}{18\,n^4}. \end{aligned} \tag{4}$$

Integrating (4), using $\alpha_n > \frac{2}{3}$ and $n \geq 3$, gives

$$\begin{aligned}\gamma_n &< \alpha_n - \frac{\alpha_n^3}{6\,n} + \frac{\alpha_n^4}{12\,n^2} + \frac{\alpha_n^5}{40\,n^2} - \frac{\alpha_n^6}{36\,n^3} + \frac{\alpha_n^7}{126\,n^4} \\ &< \alpha_n - \frac{\alpha_n^3}{6\,n} + \frac{\alpha_n^4}{12\,n^2} + \frac{\alpha_n^5}{40\,n^2} - \frac{\alpha_n^6}{n^3} \cdot \left(\frac{1}{36} - \frac{\alpha_n}{126 \cdot 3}\right) \\ &< \alpha_n - \frac{\alpha_n^3}{6\,n} + \frac{\alpha_n^4}{12\,n^2} + \frac{\alpha_n^5}{40\,n^2} - \frac{(\frac{2}{3})^6}{n^3} \cdot \left(\frac{1}{36} - \frac{\frac{2}{3}}{126 \cdot 3}\right) \\ &< \alpha_n - \frac{\alpha_n^3}{6\,n} + \frac{\alpha_n^4}{12\,n^2} + \frac{\alpha_n^5}{40\,n^2} - \frac{0.0022}{n^3},\end{aligned}$$

which proves the first part of the lemma.

Further,

$$\begin{aligned}e^{-x}\left(1+\frac{x}{n}\right)^n &> \exp\left(-\left(\frac{x^2}{2\,n} - \frac{x^3}{3\,n^2} + \frac{x^4}{4\,n^3}\right)\right) \\ &> 1 - \left(\frac{x^2}{2\,n} - \frac{x^3}{3\,n^2} + \frac{x^4}{4\,n^3}\right) + \frac{\left(\frac{x^2}{2\,n} - \frac{x^3}{3\,n^2} + \frac{x^4}{4\,n^3}\right)^2}{2} - \frac{\left(\frac{x^2}{2\,n} - \frac{x^3}{3\,n^2} + \frac{x^4}{4\,n^3}\right)^3}{6}.\end{aligned} \tag{5}$$

Integrating (5) gives

$$\begin{aligned}\gamma_n &> \alpha_n - \frac{\alpha_n^3}{6\,n} + \frac{\alpha_n^4}{12\,n^2} + \frac{\alpha_n^5}{40\,n^2} \\ &\quad - \frac{\alpha_n^5}{20\,n^3} - \frac{\alpha_n^6}{36\,n^3} + \frac{\alpha_n^7}{7}\left(\frac{17}{72\,n^4} - \frac{1}{48\,n^3}\right) + \frac{\alpha_n^8}{8}\left(\frac{1}{24\,n^4} - \frac{1}{12\,n^5}\right) \\ &\quad + \frac{\alpha_n^9}{9}\left(\frac{1}{32\,n^6} - \frac{17}{288\,n^5}\right) + \frac{31\alpha_n^{10}}{6480\,n^6} - \frac{17\alpha_n^{11}}{6336\,n^7} + \frac{\alpha_n^{12}}{1152\,n^8} - \frac{\alpha_n^{13}}{4992\,n^9} \\ &= \alpha_n - \frac{\alpha_n^3}{6\,n} + \frac{\alpha_n^4}{12\,n^2} + \frac{\alpha_n^5}{40\,n^2} - \frac{\alpha_n^5}{n^3} \cdot g(\alpha_n, n),\end{aligned}$$

where

$$\begin{aligned}g(\alpha_n, n) &= \frac{1}{20} + \frac{\alpha_n}{36} + \frac{\alpha_n^2}{336} - \frac{13\,\alpha_n^2}{504\,n} - \frac{\alpha_n^3}{192\,n} + \frac{\alpha_n^3}{96\,n^2} + \frac{17\alpha_n^4}{2592\,n^2} - \frac{\alpha_n^4}{288\,n^3} - \frac{31\alpha_n^5}{6480\,n^3} \\ &\quad + \frac{17\alpha_n^6}{6336\,n^4} - \frac{\alpha_n^7}{1152\,n^5} + \frac{\alpha_n^8}{4992\,n^6} \\ &= \frac{1}{20} + \frac{\alpha_n}{36} + \frac{\alpha_n^2}{336} + \frac{\alpha_n^3}{96\,n^2}\left(1 - \frac{n}{2}\right) + \frac{\alpha_n^2}{72\,n^2}\left(\frac{17\alpha_n^2}{36} - \frac{13n}{7}\right) \\ &\quad + \frac{\alpha_n^4}{288\,n^4}\left(\frac{17\alpha_n^2}{22} - n\right) + \frac{\alpha_n^7}{384\,n^5}\left(\frac{\alpha_n}{13} - \frac{n}{3}\right) - \frac{31\alpha_n^5}{6480\,n^3} \\ &< \frac{1}{20} + \frac{\alpha_n}{36} + \frac{\alpha_n^2}{336},\end{aligned}$$

for $n \geq 2$.

Finally, using $\alpha_n \leq \log 2$, we get

$$\alpha_n^5 \cdot g(\alpha_n, n) < \alpha_n^5 \cdot \left(\frac{1}{20} + \frac{\alpha_n}{36} + \frac{\alpha_n^2}{336}\right) \leq (\log 2)^5 \cdot \left(\frac{1}{20} + \frac{\log 2}{36} + \frac{(\log 2)^2}{336}\right) < 0.0114,$$

which proves the second part of the lemma. □

The third step is to invert Lemma 5, that is to give upper and lower bounds for α_n expressed in γ_n.

Lemma 6. For $n \geq 3$,

$$\alpha_n < \gamma_n + \frac{\gamma_n^3}{6\,n} - \frac{\gamma_n^4}{12\,n^2} + \frac{7\gamma_n^5}{120\,n^2} + \frac{0.0068}{n^3},$$
$$\alpha_n > \gamma_n + \frac{\gamma_n^3}{6\,n} - \frac{\gamma_n^4}{12\,n^2} + \frac{7\gamma_n^5}{120\,n^2} - \frac{0.0079}{n^3}.$$

Proof. Lemma 5, with $C_1 = 0.0022$ and $C_2 = 0.0114$, gives

$$\alpha_n < \gamma_n + \frac{\alpha_n^3}{6\,n} - \frac{\alpha_n^4}{12\,n^2} - \frac{\alpha_n^5}{40\,n^2} + \frac{C_2}{n^3}, \tag{6}$$
$$\alpha_n > \gamma_n + \frac{\alpha_n^3}{6\,n} - \frac{\alpha_n^4}{12\,n^2} - \frac{\alpha_n^5}{40\,n^2} + \frac{C_1}{n^3}. \tag{7}$$

Here,

$$\alpha_n < \gamma_n + \frac{\alpha_n^3}{6\,n} - \frac{1}{n^2}\cdot\left(\frac{(\frac{2}{3})^4}{12} + \frac{(\frac{2}{3})^5}{40} - \frac{C_2}{3}\right) = \gamma_n + \frac{\alpha_n^3}{6\,n} - \frac{C_3}{n^2}, \tag{8}$$
$$\alpha_n > \gamma_n + \frac{\alpha_n^3}{6\,n} - \frac{1}{n^2}\cdot\left(\frac{(\log 2)^4}{12} + \frac{(\log 2)^5}{40}\right) = \gamma_n + \frac{\alpha_n^3}{6\,n} - \frac{C_4}{n^2}, \tag{9}$$

where $C_3 > 0$, and

$$\alpha_n < \gamma_n + \frac{(\log 2)^3}{6\,n} = \gamma_n + \frac{C_5}{n}, \tag{10}$$
$$\alpha_n > \gamma_n + \frac{(\frac{2}{3})^3}{6\,n} - \frac{C_4}{3\,n} = \gamma_n + \frac{C_6}{n}. \tag{11}$$

This gives, using (8) and recalling that $\alpha_n > \frac{2}{3}$ and $\gamma_n \geq \gamma_3$ for $n \geq 3$,

$$\begin{aligned}
\alpha_n^3 &< \alpha_n^2\left(\gamma_n + \frac{\alpha_n^3}{6\,n} - \frac{C_3}{n^2}\right) < \alpha_n\gamma_n\left(\gamma_n + \frac{\alpha_n^3}{6\,n} - \frac{C_3}{n^2}\right) + \frac{\alpha_n^5}{6\,n} - \frac{C_3\alpha_n^2}{n^2} \\
&< \gamma_n^2\left(\gamma_n + \frac{\alpha_n^3}{6\,n} - \frac{C_3}{n^2}\right) + \frac{\alpha_n^3(\gamma_n\alpha_n + \alpha_n^2)}{6\,n} - \frac{C_3(\gamma_n\alpha_n + \alpha_n^2)}{n^2} \\
&= \gamma_n^3 + \frac{\alpha_n^3(\gamma_n^2 + \gamma_n\alpha_n + \alpha_n^2)}{6\,n} - \frac{C_3(\gamma_n^2 + \gamma_n\alpha_n + \alpha_n^2)}{n^2} \\
&< \gamma_n^3 + \frac{\alpha_n^3(\gamma_n^2 + \gamma_n\alpha_n + \alpha_n^2)}{6\,n} - \frac{C_3(\gamma_3^2 + \gamma_3\cdot\frac{2}{3} + (\frac{2}{3})^2)}{n^2} \\
&< \gamma_n^3 + \frac{\alpha_n^3(\gamma_n^2 + \gamma_n\alpha_n + \alpha_n^2)}{6\,n} - \frac{C_7}{n^2},
\end{aligned} \tag{12}$$

and, in the same way, using (9) and recalling that $\alpha_n \leq \log 2$ and $\gamma_n < \frac{2}{3}$,

$$\begin{aligned}
\alpha_n^3 &> \gamma_n^3 + \frac{\alpha_n^3(\gamma_n^2 + \gamma_n\alpha_n + \alpha_n^2)}{6\,n} - \frac{C_4\left((\frac{2}{3})^2 + \frac{2}{3}\log 2 + (\log 2)^2\right)}{n^2} \\
&> \gamma_n^3 + \frac{\alpha_n^3(\gamma_n^2 + \gamma_n\alpha_n + \alpha_n^2)}{6\,n} - \frac{C_8}{n^2}.
\end{aligned} \tag{13}$$

Using (10) and (11), we get

$$
\begin{align}
\alpha_n^3 &< \alpha_n^2 \left(\gamma_n + \frac{C_5}{n}\right) < \alpha_n \gamma_n \left(\gamma_n + \frac{C_5}{n}\right) + \frac{C_5 \alpha_n^2}{n} \notag \\
&< \gamma_n^2 \left(\gamma_n + \frac{C_5}{n}\right) + \frac{C_5(\gamma_n \alpha_n + \alpha_n^2)}{n} < \gamma_n^3 + \frac{C_5\left(\gamma_n^2 + \gamma_n \alpha_n + \alpha_n^2\right)}{n} \notag \\
&< \gamma_n^3 + \frac{C_5\left((\frac{2}{3})^2 + \frac{2}{3}\log 2 + (\log 2)^2\right)}{n} = \gamma_n^3 + \frac{C_9}{n}, \tag{14} \\
\alpha_n^3 &> \gamma_n^3 + \frac{C_6\left(\gamma_3^2 + \gamma_3 \cdot \frac{2}{3} + (\frac{2}{3})^2\right)}{n} = \gamma_n^3 + \frac{C_{10}}{n}, \tag{15}
\end{align}
$$

and, with the same method,

$$
\begin{align}
\alpha_n^4 &< \gamma_n^4 + \frac{C_5\left(\gamma_n^3 + \gamma_n^2 \alpha_n + \gamma_n \alpha_n^2 + \alpha_n^3\right)}{n} \notag \\
&< \gamma_n^4 + \frac{C_5\left((\frac{2}{3})^3 + (\frac{2}{3})^2 \log 2 + \frac{2}{3}(\log 2)^2 + (\log 2)^3\right)}{n} = \gamma_n^4 + \frac{C_{11}}{n}, \tag{16} \\
\alpha_n^4 &> \gamma_n^4 + \frac{C_6\left(\gamma_3^3 + \gamma_3^2 \cdot \frac{2}{3} + \gamma_3 (\frac{2}{3})^2 + (\frac{2}{3})^3\right)}{n} = \gamma_n^4 + \frac{C_{12}}{n}, \tag{17}
\end{align}
$$

and

$$
\begin{align}
\alpha_n^5 &< \gamma_n^5 + \frac{C_5\left(\gamma_n^4 + \gamma_n^3 \alpha_n + \gamma_n^2 \alpha_n^2 + \gamma_n \alpha_n^3 + \alpha_n^4\right)}{n} \notag \\
&< \gamma_n^5 + \frac{C_5\left((\frac{2}{3})^4 + (\frac{2}{3})^3 \log 2 + (\frac{2}{3})^2 (\log 2)^2 + \frac{2}{3}(\log 2)^3 + (\log 2)^4\right)}{n} = \gamma_n^5 + \frac{C_{13}}{n}, \tag{18} \\
\alpha_n^5 &> \gamma_n^5 + \frac{C_6\left(\gamma_3^4 + \gamma_3^3 \cdot \frac{2}{3} + \gamma_3^2 (\frac{2}{3})^2 + \gamma_3 (\frac{2}{3})^3 + (\frac{2}{3})^4\right)}{n} = \gamma_n^5 + \frac{C_{14}}{n}. \tag{19}
\end{align}
$$

Combining (12) with (14), (16) and (18), gives

$$
\begin{align}
\alpha_n^3 &< \gamma_n^3 + \frac{\gamma_n^2}{6\,n}\left(\gamma_n^3 + \frac{C_9}{n}\right) + \frac{\gamma_n}{6\,n}\left(\gamma_n^4 + \frac{C_{11}}{n}\right) + \frac{1}{6\,n}\left(\gamma_n^5 + \frac{C_{13}}{n}\right) - \frac{C_7}{n^2} \notag \\
&< \gamma_n^3 + \frac{\gamma_n^5}{2\,n} + \frac{1}{6\,n^2}\left(C_9 \left(\frac{2}{3}\right)^2 + C_{11} \cdot \frac{2}{3} + C_{13} - 6 \cdot C_7\right) = \gamma_n^3 + \frac{\gamma_n^5}{2\,n} + \frac{C_{15}}{n^2}, \tag{20}
\end{align}
$$

and, using (13) with (15), (17) and (19),

$$
\begin{align}
\alpha_n^3 &> \gamma_n^3 + \frac{\gamma_n^2}{6\,n}\left(\gamma_n^3 + \frac{C_{10}}{n}\right) + \frac{\gamma_n}{6\,n}\left(\gamma_n^4 + \frac{C_{12}}{n}\right) + \frac{1}{6\,n}\left(\gamma_n^5 + \frac{C_{14}}{n}\right) - \frac{C_8}{n^2} \notag \\
&> \gamma_n^3 + \frac{\gamma_n^5}{2\,n} + \frac{1}{6\,n^2}\left(C_{10} \cdot \gamma_3^2 + C_{12} \cdot \gamma_3 + C_{14} - 6 \cdot C_8\right) = \gamma_n^3 + \frac{\gamma_n^5}{2\,n} + \frac{C_{16}}{n^2}. \tag{21}
\end{align}
$$

Further, inserting (16-21) into (6) and (7), we get

$$
\begin{aligned}
\alpha_n &< \gamma_n + \frac{\gamma_n^3 + \frac{\gamma_n^5}{2\,n} + \frac{C_{15}}{n^2}}{6\,n} - \frac{\gamma_n^4 + \frac{C_{12}}{n}}{12\,n^2} - \frac{\gamma_n^5 + \frac{C_{14}}{n}}{40\,n^2} + \frac{C_2}{n^3} \\
&< \gamma_n + \frac{\gamma_n^3}{6\,n} - \frac{\gamma_n^4}{12\,n^2} + \frac{7\gamma_n^5}{120\,n^2} + \frac{1}{n^3}\left(\frac{C_{15}}{6} - \frac{C_{12}}{12} - \frac{C_{14}}{40} + C_2\right),
\end{aligned}
$$

and

$$\begin{aligned}\alpha_n &> \gamma_n + \frac{\gamma_n^3 + \frac{\gamma_n^5}{2\,n} + \frac{C_{16}}{n^2}}{6\,n} - \frac{\gamma_n^4 + \frac{C_{11}}{n}}{12\,n^2} - \frac{\gamma_n^5 + \frac{C_{13}}{n}}{40\,n^2} + \frac{C_1}{n^3} \\ &> \gamma_n + \frac{\gamma_n^3}{6\,n} - \frac{\gamma_n^4}{12\,n^2} + \frac{7\gamma_n^5}{120\,n^2} + \frac{1}{n^3}\left(\frac{C_{16}}{6} - \frac{C_{11}}{12} - \frac{C_{13}}{40} + C_1\right).\end{aligned}$$

Finally, computing

$$\frac{C_{15}}{6} - \frac{C_{12}}{12} - \frac{C_{14}}{40} + C_2 < 0.0068$$

and

$$\frac{C_{16}}{6} - \frac{C_{11}}{12} - \frac{C_{13}}{40} + C_1 > -0.0079$$

finishes the proof of the lemma. □

In order to estimate $\Delta\alpha_n = \alpha_n - \alpha_{n+1}$, using Lemma 6, we first need to estimate $\Delta\gamma_n = \gamma_n - \gamma_{n+1}$.

Lemma 7.

$$0 > \Delta\gamma_n = \gamma_n - \gamma_{n+1} > -\frac{1364}{42525} \cdot \frac{1}{n(n+1)}.$$

Proof. $\gamma_n - \gamma_{n+1} = (1-\theta_n) - (1-\theta_{n+1}) = \theta_{n+1} - \theta_n < 0$, and, by (3),

$$\begin{aligned}\theta_{n+1} - \theta_n &= \frac{1}{3} + \frac{4}{135(n+1+k_{n+1})} - \frac{1}{3} - \frac{4}{135(n+k_n)} \\ &= \frac{4}{135}\left(\frac{1}{n+1+k_{n+1}} - \frac{1}{n+k_n}\right) \\ &> \frac{4}{135}\left(\frac{1}{n+1+\frac{8}{45}} - \frac{1}{n+\frac{2}{21}}\right) = -\frac{4}{135} \cdot \frac{341}{315} \cdot \frac{1}{(n+\frac{53}{45})(n+\frac{2}{21})} \\ &> -\frac{1364}{42525} \cdot \frac{1}{n(n+1)}.\end{aligned}$$

□

Proof. **(Theorem 3)** Let $C_1 = 0.0068$ and $C_2 = 0.0079$ denote the constants of Lemma 6 and $C_\gamma = \frac{1364}{42525}$ denote the constant of Lemma 7. Then, by Lemma 6,

$$\begin{aligned}\alpha_n - \alpha_{n+1} &> \gamma_n + \frac{\gamma_n^3}{6\,n} - \frac{\gamma_n^4}{12\,n^2} + \frac{7\gamma_n^5}{120\,n^2} - \frac{C_2}{n^3} \\ &\quad - \left(\gamma_{n+1} + \frac{\gamma_{n+1}^3}{6(n+1)} - \frac{\gamma_{n+1}^4}{12(n+1)^2} + \frac{7\gamma_{n+1}^5}{120(n+1)^2} + \frac{C_1}{(n+1)^3}\right) \\ &= \gamma_n - \gamma_{n+1} + \frac{\gamma_n^3 - \gamma_{n+1}^3}{6\,n} + \frac{\gamma_{n+1}^3}{6\,n(n+1)} + \frac{7}{120} \cdot \frac{\gamma_n^5 - \gamma_{n+1}^5}{n^2} \\ &\quad + \frac{7}{120} \cdot \gamma_{n+1}^5 \cdot \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) - \frac{\gamma_n^4 - \gamma_{n+1}^4}{12\,n^2} \\ &\quad - \frac{\gamma_{n+1}^4}{12} \cdot \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) - \frac{C_2}{n^3} - \frac{C_1}{(n+1)^3}. \qquad (22)\end{aligned}$$

As $\Delta\gamma_n = \gamma_n - \gamma_{n+1} < 0$, we get, for $n \geq 3$, using $\gamma_3 \leq \gamma_n < \gamma_{n+1} < \frac{2}{3}$,

$$\begin{aligned}
&\gamma_n^3 - \gamma_{n+1}^3 = \Delta\gamma_n \cdot (\gamma_n^2 + \gamma_n\gamma_{n+1} + \gamma_{n+1}^2) > \Delta\gamma_n \cdot 3 \cdot \left(\frac{2}{3}\right)^2 = \frac{4}{3} \cdot \Delta\gamma_n, \\
&\gamma_n^4 - \gamma_{n+1}^4 = \Delta\gamma_n \cdot (\gamma_n^3 + \gamma_n^2\gamma_{n+1} + \gamma_n\gamma_{n+1}^2 + \gamma_{n+1}^3) < \Delta\gamma_n \cdot 4 \cdot \gamma_3^3, \\
&\gamma_n^5 - \gamma_{n+1}^5 = \Delta\gamma_n \cdot (\gamma_n^4 + \gamma_n^3\gamma_{n+1} + \gamma_n^2\gamma_{n+1}^2 + \gamma_n\gamma_{n+1}^3 + \gamma_{n+1}^4) > \Delta\gamma_n \cdot 5 \cdot \left(\frac{2}{3}\right)^4 = \frac{80}{81}\Delta\gamma_n.
\end{aligned}$$

Further, for $n \geq 3$,

$$\gamma_4 \leq \gamma_{n+1} < \frac{2}{3}, \quad \frac{3}{4} \leq \frac{n}{n+1} < 1,$$

$$\frac{2}{n(n+1)^2} < \frac{1}{n^2} - \frac{1}{(n+1)^2} < \frac{2}{n^2(n+1)}.$$

Inserting these estimates into (22), and using Lemma 7, we get

$$\begin{aligned}
\alpha_n - \alpha_{n+1} &> \Delta\gamma_n + \frac{\frac{4}{3}\Delta\gamma_n}{6\,n} + \frac{\gamma_4^3}{6\,n(n+1)} + \frac{7}{120} \cdot \frac{\frac{80}{81}\Delta\gamma_n}{n^2} + \frac{7}{120} \cdot \gamma_4^5 \cdot \frac{2}{n(n+1)^2} \\
&\quad - \frac{4\gamma_3^3\Delta\gamma_n}{12\,n^2} - \frac{\left(\frac{2}{3}\right)^4}{12} \cdot \frac{2}{n^2(n+1)} - \frac{C_2}{n^3} - \frac{C_1}{(n+1)^3} \\
&> \frac{1}{n(n+1)} \cdot \left(\frac{\gamma_4^3}{6} - C_\gamma\right) + \frac{1}{n^2(n+1)} \cdot \left(-\frac{2}{9} \cdot C_\gamma - \frac{8}{243}\right) \\
&\quad + \frac{1}{n(n+1)^2} \cdot \frac{7}{60} \cdot \gamma_4^5 - \frac{C_2}{n^3} - \frac{C_1}{(n+1)^3} + \frac{C_\gamma}{n^3(n+1)} \left(\frac{\gamma_3^3}{3} - \frac{14}{243}\right) \\
&> \frac{1}{n(n+1)} \cdot \left(\frac{\gamma_4^3}{6} - C_\gamma\right) - \frac{1}{n^3}\left(\frac{2}{9} \cdot C_\gamma + \frac{8}{243} - \left(\frac{3}{4}\right)^2 \cdot \frac{7}{60} \cdot \gamma_4^5 + C_1 + C_2\right) \\
&\quad + \frac{C_\gamma}{n^3(n+1)} \left(\frac{\gamma_3^3}{3} - \frac{14}{243}\right) \\
&> \frac{0.0157}{n(n+1)} - \frac{0.0466}{n^3} + \frac{0.0369}{n^3(n+1)} > 0 \quad \text{if } n > 3.17,
\end{aligned}$$

so that $\{\alpha_n\}$ is decreasing for $n > 3$. Checking in Table 1 that $\{\alpha_n\}$ is decreasing also for $n \leq 3$ finishes the proof. □

3 Monotonicity of $\{k_n\}$

Theorem 8. The Ramanujan sequence $\{k_n\}$ of (3) is decreasing for all $n \geq 0$.

To prove this theorem we will use the technique of Flajolet *et al.* [3] in their proof of (3), but we need to improve some of their estimates.

First, we need an asymptotic expansion for θ_n. Marsaglia [5] provides a method which gives an arbitrary number of terms in the expansion, the first being

$$\theta_n = \frac{1}{3} + \frac{4}{135\,n} - \frac{8}{2835\,n^2} - \frac{16}{8505\,n^3} + \frac{8992}{12629925\,n^4} + O\left(\frac{1}{n^5}\right). \tag{23}$$

Solving for k_n in (3) gives

$$k_n = \frac{4}{135(\theta_n - \frac{1}{3})} - n, \tag{24}$$

which, after inserting (23), gives the expansion

$$k_n = \frac{2}{21} + \frac{32}{441\,n} - \frac{50752}{4584195\,n^2} + O\left(\frac{1}{n^3}\right),$$

which shows that $\{k_n\}$ is decreasing for sufficiently large n, as the difference

$$\Delta k_n = k_n - k_{n+1} = \frac{32}{441\,n^2} + O\left(\frac{1}{n^3}\right) \tag{25}$$

obviously is positive for $n > n_0$, for some sufficiently large n_0.

In order to specify n_0, we need constructive bounds in (23) of the type

$$\begin{aligned}
\theta_n &< \frac{1}{3} + \frac{4}{135\,n} - \frac{8}{2835\,n^2} - \frac{16}{8505\,n^3} + \frac{8992}{12629925\,n^4} + \frac{C_1}{n^5},\\
\theta_n &> \frac{1}{3} + \frac{4}{135\,n} - \frac{8}{2835\,n^2} - \frac{16}{8505\,n^3} + \frac{8992}{12629925\,n^4} + \frac{C_2}{n^5},
\end{aligned}$$

which give corresponding bounds for k_n:

$$\begin{aligned}
k_n &< \frac{2}{21} + \frac{32}{441\,n} - \frac{50752}{4584195\,n^2} + \frac{D_1}{n^3},\\
k_n &> \frac{2}{21} + \frac{32}{441\,n} - \frac{50752}{4584195\,n^2} + \frac{D_2}{n^3}.
\end{aligned}$$

Then, $\Delta k_n = k_n - k_{n+1} > A_1/n^2 - A_2/n^3 > 0$ if $n > n_0 = A_2/A_1$. Checking that $\Delta k_n > 0$ for $n \leq n_0$ can then be done numerically, provided that n_0 is not too large.

Flajolet *et al.* [3] give constructive bounds for the quantity

$$D_n = 2 \cdot \theta_n,$$

introduced by Knuth [4] as an example of asymptotic expansions, namely

$$D_{10}(n) - \Delta_{10}(n) \leq D(n) \leq D_{10}(n) + \Delta_{10}(n), \tag{26}$$

where

$$\begin{aligned}
D_{10}(n) = \sum_{k=0}^{9} \frac{d_k}{n^k} &= \frac{2}{3} + \frac{8}{135\,n} - \frac{16}{2835\,n^2} - \frac{32}{8505\,n^3} + \frac{17984}{12629925\,n^4} + \frac{13159709}{9699782400\,n^5}\\
&\quad - \frac{977069}{1039262400\,n^6} - \frac{36669961}{28291032000\,n^7} + \frac{117191}{56582064\,n^8} - \frac{479}{561330\,n^9},
\end{aligned} \tag{27}$$

and the remainder $\Delta_{10}(n)$ is estimated by

$$\Delta_{10}(n) < F_1 \cdot n^{3/2} \cdot 2^{-n/2} + \frac{F_2}{n^5}, \tag{28}$$

where

$$F_1 = 13.06 \text{ and } F_2 = 56.59398.$$

Both constants, F_1 and F_2, depend on the coefficients, c_k, in the expansion

$$\log\left(\frac{z^2}{2(1-(1+z)e^{-z})}\right) = \sum_{k=1}^{\infty} c_k \cdot z^k. \tag{29}$$

Remark 9. There is a misprint in [3] in their asymptotic expansion of $D_{10}(n)$ on page 109, where the term

$$\frac{17984}{12629925\, n^4} \text{ is given as } \frac{1794}{12629925\, n^4}.$$

□

The estimate of $\Delta_{10}(n)$ used in [3] is

$$\Delta_{10}(n) < \frac{10^{-7}}{n^3} + \frac{57}{n^5}, \text{ for } n \geq 116, \tag{30}$$

which is insufficient for our needs, as we need an estimate of order

$$\Delta_{10}(n) < \frac{C}{n^5} \quad \text{for } n \geq n_0.$$

This can, however, be obtained by replacing their estimate of the first term in (28) by

$$13.06 \cdot n^{3/2} \cdot 2^{-n/2} < \frac{K_0}{n^5} \quad \text{for } n \geq 116,$$

with $K_0 = 13.06 \cdot 116^{13/2} \cdot 2^{-58} < 0.001189$. The numerator 57 in (30) is actually $F_2 = 56.59398$, so that

$$\Delta_{10}(n) < \frac{56.595169}{n^5}. \tag{31}$$

Unfortunately, performing the analysis outlined above, only shows that $\{k_n\}$ is decreasing for $n > n_1 > 26324$, so we need to improve the bound in (31). We will do this by a more careful estimation of the remainder term, $\Delta_{10}(n)$ of (28).

As both terms on the right hand side of (28) depend on $|c_k|$, of (29), it is natural to try to improve the estimate given in Lemma 4 of [3]:

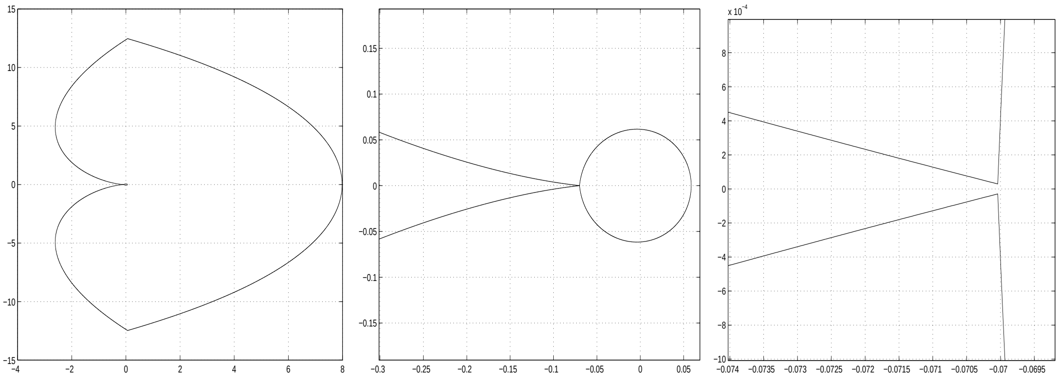
$$|c_k| < \frac{10.96714833}{\pi^k}, \quad \text{for all } k \geq 1.$$

This can be achieved by a slight modification of their proof, and by noting that we only need an estimate for $k > 10$.

Lemma 10. For $k > 10$, we have

$$|c_k| < \frac{0.4593}{(\frac{6}{5}\pi)^k}.$$

Figure 1: A plot of $f(z)$ on $\mathcal{A}$ and zoomed close to the origin.



Proof. Recall that c_k are defined, in (29), as the coefficients in the expansion of $\log f(z)$, where

$$f(z) = \frac{z^2}{2(1-(1+z)e^{-z})},$$

and thus, by Cauchy's formula, can be written

$$c_k = \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{A}} \frac{\log f(z)}{z^{k+1}}\, dz,$$

where $\mathcal{A}$ is a contour encircling the origin, and chosen so that $\log f(z)$ is well-defined on it.

In [3], $\mathcal{A}$ is chosen as $\mathcal{D}$, the boundary of the square $|\Re z| \le \pi$, $|\Im z| \le \pi$, that is with side 2π. We will use the slightly larger square $|\Re z| \le 6\pi/5$, $|\Im z| \le 6\pi/5$, with side $12\pi/5$. Figure 1, and the argument principle, shows that there are no poles or zeros of $f(z)$ on $\mathcal{A}$. We will estimate $\log f(z)$ separately on the four sides of the square. Let, for $-1 \le \tau \le 1$,

$$\begin{aligned}
\mathcal{A}_1 &: z = \frac{6}{5}\pi(1+\mathrm{i}\tau),\\
\mathcal{A}_2 &: z = \frac{6}{5}\pi(\tau+\mathrm{i}),\\
\mathcal{A}_3 &: z = \frac{6}{5}\pi(-1+\mathrm{i}\tau),\\
\mathcal{A}_4 &: z = \frac{6}{5}\pi(\tau-\mathrm{i}),
\end{aligned}$$

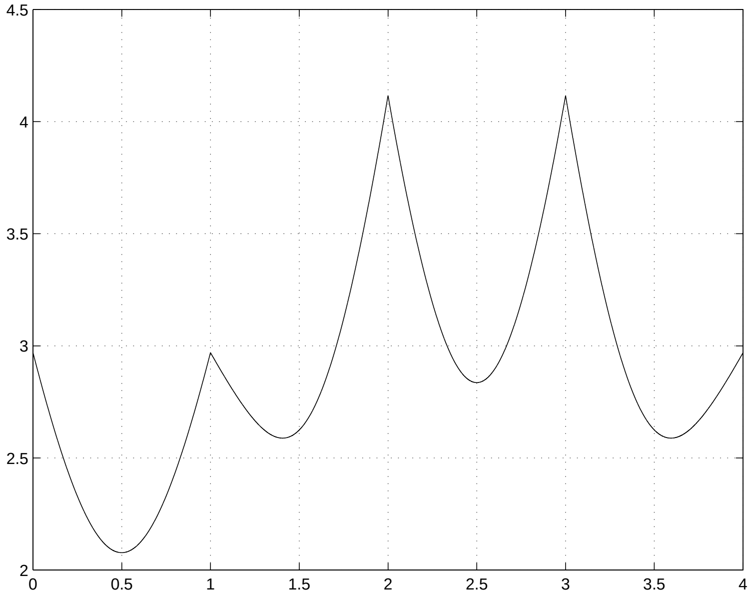
and let

$$a_i = \max|\log f(z)| \text{ on } \mathcal{A}_i.$$

As $|z| \ge \frac{6}{5}\pi$ on $\mathcal{A}$, we get, for $k > 10$,

$$\begin{aligned}
\oint_{\mathcal{A}_1} \frac{|\log f(z)|}{|z|^{k+1}}\, dz &\le \frac{a_1}{(\frac{6}{5}\pi)^{k+1}} \int_{-1}^{1} \frac{d\tau}{|1+\mathrm{i}\tau|^{k+1}} = \frac{2a_1}{(\frac{6}{5}\pi)^{k+1}} \int_{0}^{1} \frac{d\tau}{(1+\tau^2)^{\frac{k+1}{2}}}\\
&< \frac{2a_1}{(\frac{6}{5}\pi)^{k+1}} \int_{0}^{1} \frac{d\tau}{(1+\tau^2)^6} = \frac{2a_1}{(\frac{6}{5}\pi)^{k+1}} \left(\frac{63\pi}{1024} + \frac{61}{320}\right)\\
&= \frac{a_1}{(\frac{6}{5}\pi)^{k+1}} \left(\frac{63\pi}{512} + \frac{61}{160}\right),
\end{aligned}$$

Figure 2: A plot of $|\log f(z)|$ on $\mathcal{A}$, in the order $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$,$\mathcal{A}_4$.



and, in the same way,

$$\oint_{\mathcal{A}_i} \frac{|\log f(z)|}{|z|^{k+1}}\, dz < \frac{a_i}{(\frac{6}{5}\pi)^{k+1}} \left(\frac{63\pi}{512} + \frac{61}{160} \right), \text{ for } k > 10,$$

so that

$$\oint_{\mathcal{A}} \frac{|\log f(z)|}{|z|^{k+1}}\, dz < \frac{a_1 + a_2 + a_3 + a_4}{(\frac{6}{5}\pi)^{k+1}} \left(\frac{63\pi}{512} + \frac{61}{160} \right), \text{ for } k > 10. \tag{32}$$

As confirmed by Figure 2, $|\log f(z)|$ has its maxima in the corners of $\mathcal{A}$. This gives,

$$\begin{aligned} a_1 &= |\log f\big(\frac{6}{5}\pi(1+\mathrm{i})\big)| \le 2.96941147, \\ a_2 &= |\log f\big(\frac{6}{5}\pi(-1+\mathrm{i})\big)| \le 4.11528807, \\ a_3 &= |\log f\big(\frac{6}{5}\pi(-1+\mathrm{i})\big)| = a_2, \\ a_4 &= |\log f\big(\frac{6}{5}\pi(-1-\mathrm{i})\big)| = a_2. \end{aligned}$$

From Figure 2, we also see that, by splitting the integral into eight parts, instead of four, we can improve the estimate of (32) to

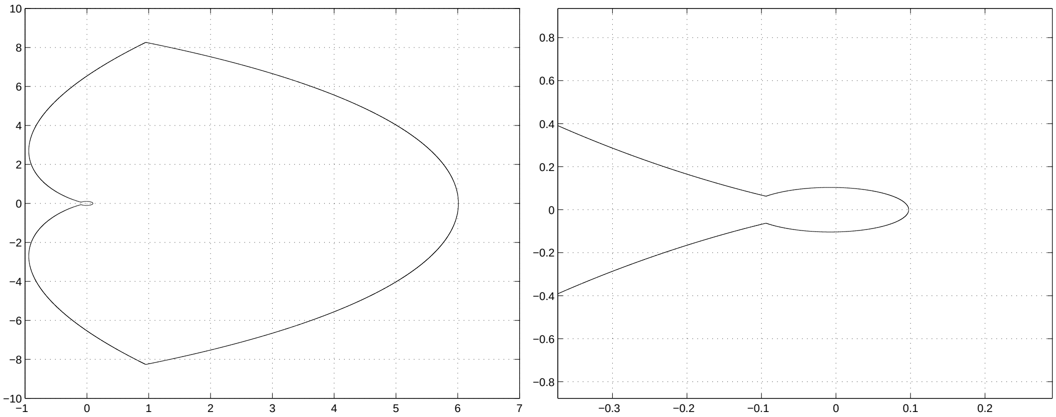
$$\oint_{\mathcal{A}} \frac{|\log f(z)|}{|z|^{k+1}}\, dz < \frac{4a_1 + 4a_2}{(\frac{6}{5}\pi)^{k+1}} \left(\frac{63\pi}{1024} + \frac{61}{320} \right) = \frac{a_1 + a_2}{(\frac{6}{5}\pi)^{k+1}} \left(\frac{63\pi}{256} + \frac{61}{80} \right), \text{ for } k > 10.$$

Thus,

$$|c_k| \le \frac{1}{2\pi} \frac{a_1 + a_2}{(\frac{6}{5}\pi)^{k+1}} \left(\frac{63\pi}{256} + \frac{61}{80} \right) < \frac{0.4593}{(\frac{6}{5}\pi)^{k}}, \text{ for } k > 10,$$

which proves the lemma. □

Figure 3: A plot of $f(z)$ on $\mathcal{D}$ of [3] and zoomed at the origin.



Remark 11. There seems to be a mistake in Fig. 1 of [3], as the figure does not have winding number 0, as claimed, and does not resemble our plot of $f(z)$ on $\mathcal{D}$, see Figure 3. ☐

Using the estimate of Lemma 10 instead of the one given in Lemma 4 of [3] gives a much improved estimate of the remainder $\Delta_{10}(n)$.

Lemma 12.

$$\Delta_{10}(n) < 0.0474\, n^{3/2}\, 2^{-n/2} + \frac{0.29596}{n^5}.$$

Proof. The lemma is obtained by a straightforward modification of Lemma 5 of [3], and of the estimate of μ_{10} of Lemma 6 of [3], by simply replacing the estimate of $|c_k|$. ☐

It is sufficient to bound $\Delta_{10}(n)$ by C/n^4, provided that the constant C is sufficiently small; less than the coefficient d_4 of $1/n^4$ in (27).

Lemma 13. For $n \geq 208$,

$$\Delta_{10}(n) \leq \frac{C_\Delta}{n^4},$$

where $C_\Delta = 0.0014229$.

Proof. $n^{11/2} \cdot 2^{-n/2}$ is decreasing for $n \geq 16$, so that, for $n \geq n_0 \geq 16$,

$$\Delta_{10}(n) \leq \frac{1}{n^4}\left(0.0474\, n_0^{11/2}\, 2^{-n_0/2} + \frac{0.29596}{n_0}\right).$$

Choosing $n_0 = 208$ gives the lemma. ☐

The next lemma gives the necessary upper and lower bounds for θ_n.

Lemma 14. For $n \geq 208$,

$$\begin{aligned}\theta_n &\leq \frac{1}{3} + \frac{4}{135\,n} - \frac{8}{2835\,n^2} - \frac{16}{8505\,n^3} + \frac{C_1}{n^4},\\ \theta_n &\geq \frac{1}{3} + \frac{4}{135\,n} - \frac{8}{2835\,n^2} - \frac{16}{8505\,n^3} + \frac{C_2}{n^4}\\ &\geq \frac{1}{3} + \frac{4}{135\,n} - \frac{8}{2835\,n^2} - \frac{C_3}{n^3},\end{aligned}$$

with $C_1 = 0.001427$, $C_2 = 0.0000005$ and $C_3 = \dfrac{16}{8505}$.

Proof. By (26) and Lemma 13,

$$2\,\theta_n = D(n) < D_{10}(n) + \Delta_{10}(n) < \sum_{k=0}^{9} \frac{d_k}{n^k} + \frac{C_\Delta}{n^4}.$$

Here, $d_6 < 0$, $d_9 < 0$ and $nd_7 + d_8 < 0$, so that

$$D(n) < \sum_{k=0}^{3} \frac{d_k}{n^k} + \frac{1}{n^4}(d_4 + C_\Delta + \frac{d_5}{n}).$$

Choosing $C_1 > \dfrac{d_4 + C_\Delta + \frac{d_5}{208}}{2}$ proves the first inequality.

Similarly,

$$D(n) > D_{10}(n) - \Delta_{10}(n) > \sum_{k=0}^{9} \frac{d_k}{n^k} - \frac{C_\Delta}{n^4}.$$

Here, $n\,d_8 + d_9 > 0$ and $n^2\,d_5 + n\,d_6 + d_7 > 0$, so that

$$D(n) > \sum_{k=0}^{3} \frac{d_k}{n^k} + \frac{1}{n^4}(d_4 - C_\Delta).$$

Choosing $C_2 < \dfrac{d_4 - C_\Delta}{2}$ proves the second inequality, and the third follows immediately as $C_2 > 0$. □

Proof. **(Theorem 8)** First, assume that $n \geq 208$. Using (24), we get

$$\begin{aligned}
\Delta k_n = k_n - k_{n+1} &= \frac{4}{135(\theta_n - \frac{1}{3})} - n - \left(\frac{4}{135(\theta_{n+1} - \frac{1}{3})} - (n+1)\right) \\
&= 1 + \frac{4}{135(\theta_n - \frac{1}{3})} - \frac{4}{135(\theta_{n+1} - \frac{1}{3})} = 1 + \frac{4}{135}\,\frac{\theta_{n+1} - \theta_n}{(\theta_n - \frac{1}{3})(\theta_{n+1} - \frac{1}{3})} \\
&= 1 - \frac{4}{135}\,\frac{\theta_n - \theta_{n+1}}{(\theta_n - \frac{1}{3})(\theta_{n+1} - \frac{1}{3})}. \qquad (33)
\end{aligned}$$

Using Lemma 14, we get

$$\begin{aligned}
\theta_n - \theta_{n+1} &< \frac{1}{3} + \frac{4}{135\,n} - \frac{8}{2835\,n^2} - \frac{16}{8505\,n^3} + \frac{C_1}{n^4} \\
&\quad - \left(\frac{1}{3} + \frac{4}{135\,(n+1)} - \frac{8}{2835\,(n+1)^2} - \frac{16}{8505\,(n+1)^3} + \frac{C_2}{(n+1)^4}\right) \\
&= \frac{4}{135\,n(n+1)} - \frac{8}{2835}\,\frac{2n+1}{n^2\,(n+1)^2} - \frac{16}{8505}\,\frac{3n^2+3n+1}{n^3\,(n+1)^3} + \frac{C_1}{n^4} - \frac{C_2}{(n+1)^4} \\
&< \frac{4}{135\,n(n+1)} - \frac{16}{2835}\,\frac{1}{n^2\,(n+1)} + \frac{8-16}{2835}\,\frac{1}{n^2\,(n+1)^2} + \frac{C_1}{n^4} - \frac{C_2}{(n+1)^4} \\
&< \frac{4}{135\,n(n+1)} - \frac{16}{2835}\,\frac{1}{n^2\,(n+1)} - \frac{C_\theta}{n^3(n+1)}, \qquad (34)
\end{aligned}$$

where

$$C_\theta = \frac{208}{209}\,\frac{8}{2835} - \frac{209}{208}\,C_1 + \left(\frac{208}{209}\right)^3 \cdot C_2 > 0.$$

Further,

$$\begin{aligned}
&(\theta_n - \frac{1}{3}) \cdot (\theta_{n+1} - \frac{1}{3}) \\
&\qquad > \left(\frac{4}{135\,n} - \frac{8}{2835\,n^2} - \frac{C_3}{n^3} \right) \cdot \left(\frac{4}{135\,(n+1)} - \frac{8}{2835\,(n+1)^2} - \frac{C_3}{(n+1)^3} \right) \\
&\qquad = \left(\frac{4}{135} \right)^2 \frac{1}{n(n+1)} \left(1 - \frac{2}{21\,n} - \frac{135\,C_3}{4\,n^2} \right) \cdot \left(1 - \frac{2}{21(n+1)} - \frac{135\,C_3}{4(n+1)^2} \right) \\
&\qquad = \left(\frac{4}{135} \right)^2 \frac{1}{n(n+1)} (1 - g(n)),
\end{aligned}$$

where, recalling that $C_3 = 16/8505 > 0$,

$$\begin{aligned}
g(n) &= \frac{2}{21\,n} + \frac{2}{21(n+1)} + \frac{135\,C_3}{4\,n^2} - \frac{4}{441\,n(n+1)} + \frac{135\,C_3}{4(n+1)^2} \\
&\quad - \frac{45\,C_3}{14\,n^2(n+1)} - \frac{45\,C_3}{14\,n(n+1)^2} - \left(\frac{135}{4} \right)^2 \frac{C_3^2}{n^2(n+1)^2} \\
&< \frac{4}{21\,n} - \frac{2}{21\,n(n+1)} + \frac{4}{63\,n^2} - \frac{4}{441\,n(n+1)} + \frac{4}{63(n+1)^2} \\
&= \frac{4}{21\,n} + \frac{4}{63} \left(\frac{1}{n} - \frac{1}{n+1} \right)^2 + \frac{8}{63\,n(n+1)} - \frac{46}{441\,n(n+1)} \\
&= \frac{4}{21\,n} + \frac{4}{63\,n^2(n+1)^2} + \frac{10}{441\,n(n+1)} < \frac{4}{21\,n} + \frac{C_g}{n^2},
\end{aligned}$$

where

$$C_g = \frac{4}{63 \cdot (209)^2} + \frac{10}{441}.$$

Using the identity

$$\frac{1}{1-x} = 1 + x + \frac{x^2}{1-x},$$

we get

$$\begin{aligned}
\frac{1}{1 - g(n)} &< 1 + \frac{4}{21\,n} + \frac{C_g}{n^2} + \frac{\left(\frac{4}{21\,n} + \frac{C_g}{n^2} \right)^2}{1 - \frac{4}{21\,n} - \frac{C_g}{n^2}} \\
&< 1 + \frac{4}{21\,n} + \frac{1}{n^2} \left(C_g + \frac{\left(\frac{4}{21} + \frac{C_g}{n} \right)^2}{1 - \frac{4}{21\,n} - \frac{C_g}{n^2}} \right) \\
&< 1 + \frac{4}{21\,n} + \frac{C_0}{n^2},
\end{aligned}$$

where

$$C_0 = C_g + \frac{\left(\frac{4}{21} + \frac{C_g}{208} \right)^2}{1 - \frac{4}{21 \cdot 208} - \frac{C_g}{208^2}}.$$

Thus,

$$\frac{1}{(\theta_n - \frac{1}{3}) \cdot (\theta_{n+1} - \frac{1}{3})} < \left(\frac{135}{4} \right)^2 \cdot n(n+1) \cdot \left(1 + \frac{4}{21\,n} + \frac{C_0}{n^2} \right), \tag{35}$$

so that, inserting (34) and (35) into (33),

$$\begin{aligned}\Delta k_n &> 1-\frac{4}{135}\left(\frac{4}{135}\,\frac{1}{n(n+1)}-\frac{16}{2835}\,\frac{1}{n^2(n+1)}-\frac{C_\theta}{n^3(n+1)}\right)\cdot\\&\qquad\left(\frac{135}{4}\right)^2\cdot n(n+1)\cdot\left(1+\frac{4}{21\,n}+\frac{C_0}{n^2}\right)\\&=1-\left(1-\frac{4}{21\,n}-\frac{135\,C_\theta}{4\,n^2}\right)\cdot\left(1+\frac{4}{21\,n}+\frac{C_0}{n^2}\right)\\&=1-\left(1+\frac{4}{21\,n}+\frac{C_0}{n^2}-\frac{4}{21\,n}-\frac{16}{441\,n^2}-\frac{4\,C_0}{21\,n^3}-\frac{135\,C_\theta}{4\,n^2}-\frac{45\,C_\theta}{7\,n^3}-\frac{135\,C_\theta\,C_0}{4\,n^4}\right)\\&=\frac{1}{n^2}\left(\frac{16}{441}-C_0+\frac{135\,C_\theta}{4}\right)+\frac{1}{n^3}\left(\frac{4\,C_0}{21}+\frac{45\,C_\theta}{7}\right)+\frac{1}{n^4}\cdot\frac{135\,C_\theta\,C_0}{4}\\&>\frac{A_2}{n^2},\end{aligned}$$

where

$$A_2=\frac{16}{441}-C_0+\frac{135\,C_\theta}{4}>0.0236>0,$$

so that $\Delta k_n > 0$ for all $n \geq 208$.

It only remains to verify that $\Delta k_n > 0$ also for $n < 208$. The first few k_n and Δk_n ($n \leq 10$) are given in Table 3. For $10 < n \leq 210$, we see from the plot in Figure 4 that $\Delta k_n > 0$, which finishes the proof. □

Table 3: k_n and Δk_n

n	k_n	Δk_n
0	0.177778	0.029680
1	0.148098	0.021166
2	0.126932	0.009370
3	0.117562	0.005163
4	0.112399	0.003245
5	0.109155	0.002221
6	0.106933	0.001613
7	0.105320	0.001224
8	0.104096	0.000960
9	0.103136	0.000773
10	0.102363	0.000635

Remark 15. Computations were performed with Maple and Matlab. k_n and Δk_n were computed by Maple with 100 digits precision. Figures 1–3 were produced by Matlab, where the zoom option was most useful for Figures 1 and 3, whereas Figure 4 was produced with Maple. □

Remark 16. Table 3, Figure 4 and (25) indicate that also the sequence $\{\Delta k_n\}$ is decreasing for all n. □

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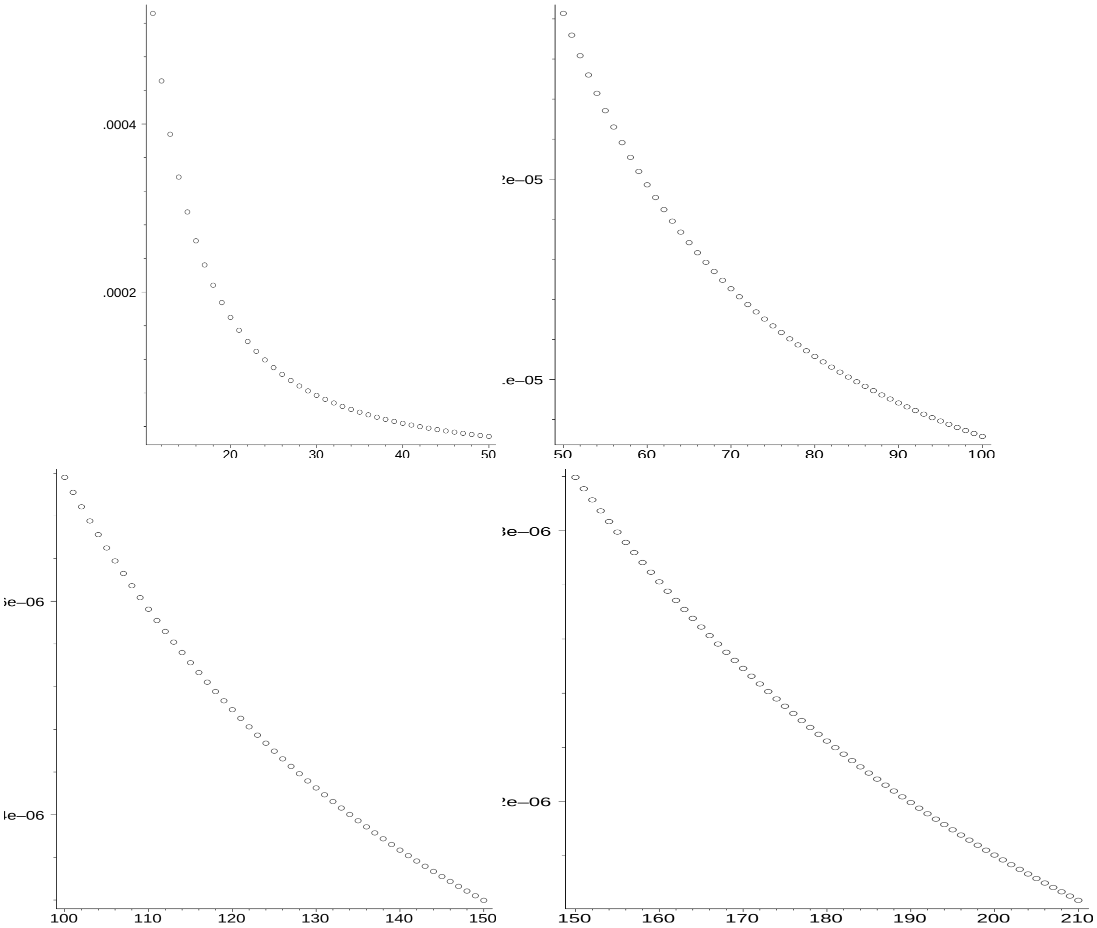
Figure 4: A plot of Δk_n for $10 < n \leq 210$.



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