SOME SYSTEMS OF ORTHOGONAL POLYNOMIALS

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Abstract. In this note we shall present two systems of polynomials, that are orthogonal systems for two different but related inner product spaces. We also describe three operators that are related to the systems.

1. Introduction

Let $ch(x) = 2\cosh\frac{\pi x}{2}$ and let $\omega(x) = 1/ch(x)$. The function ω is then the density function of a probability measure and it has furthermore two interesting properties that make it useful as a weight function. The first is that it is the fourier transform of the function $1/\cosh t$, and the second is that it is essentially the Poisson kernel for a strip of width 2. The first property makes it possible to interpret its moments as values at 0 of successive derivatives, while the second can be used for direct computations of many integrals. In the following we shall investigate the system of orthogonal polynomials obtained by applying the Gram-Schmidt procedure to the sequence $\{x^k\}_{k=0}^{\infty}$. It turns out that the system has a simple recursion formula, so that the exponential generating function can easily be computed. Using this the orthogonality can easily be proved. It turns out that the system is closely related to a system of orthogonal polynomials in the strip $S = \{z \mid -1 < Im(z) < 1\}$, and that there are some simple operators that connect the systems with each other.

Finally I wish to conclude this introduction by expressing my gratitude to Maciej Klimek for persuading me to fulfill this investigation.

2. Basic properties

We shall prove the following

Theorem 2.1. *Let the system $\{\tau_k\}_{k=0}^{\infty}$ be given by the following recursion procedure.*

$$\tau_{-1} = 0, \ \tau_0 = 1, \ \text{and } \tau_{k+1}(x) = x\tau_k(x) - k^2\tau_{k-1}(x). \tag{1}$$

Then
(i) the function τ_k is a monic polynomial of degree k;
(ii) The polynomials $\{(k!)^{-1}\tau_k\}_{k=0}^{\infty}$ are an orthonormal sequence in the Hilbert space $L^2(\omega)$;

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(iii) The exponential generating function $G_\tau(x,s) = \sum \frac{\tau_k(x)}{k!} s^k$ *is given by the function*

$$G\tau(x,s) = \frac{e^{x \arctan s}}{\sqrt{1+s^2}}. \tag{2}$$

Proof. Since (i) is obvious we shall first prove (iii). Summing the recursion we see that

$$\sum_{k=0}^{\infty} (\tau_{k+1}(x) - x\tau_k(x) + k^2 \tau_{k-1}(x)) \frac{s^k}{k!} = 0 \tag{3}$$

and this leads to the differential equation

$$G'_\tau(x,s) - xG_\tau(x,s) + sG_\tau(x,s) + s^2 G'_\tau(x,s) \tag{4}$$

(where all derivatives are with respect to s). Solving the differential equation and using the fact that $G_\tau(0,0) = 1$ (iii) follows. To prove (ii) we shall show that

$$\int_{-\infty}^{\infty} G_\tau(x,s)\overline{G_\tau(x,t)} \frac{dx}{2\cosh\frac{\pi x}{2}} = \frac{1}{1 - s\bar{t}}. \tag{5}$$

This is most easily done by residue calculus over a rectangle with vertices at $-R$, R, $R+i$ and $-R+i$. □

Remark 1. The idea of using the integral (5) was suggested by Svante Janson.

Remark 2. It is a well-known fact that polynomials are dense in $L^2(\omega)$, and therefore the system is not only orthonormal but also a basis in the space.

3. A related system and some useful operators

The most useful property of the weight function $\frac{1}{2\cosh\frac{\pi x}{2}}$ is that it can be interpreted as a Poisson kernel. We have namely the following

Proposition 3.1. *Let the function f be continuous and harmonic in the strip $S = -1 \le Im(z) \le 1$ and suppose further that $|f(z)| < Ce^{a|z|}$ for some a, $0 \le a < \frac{\pi}{2}$. Then*

$$f(0) = \int_{-\infty}^{\infty} \frac{f(x+i) + f(x-i)}{2} \frac{dx}{2\cosh\frac{\pi x}{2}}. \tag{6}$$

Proof. This is simply the Poisson integral. □

The preceding proposition makes it natural to consider the following three operators, which are all densely defined in $L^2(\omega)$.

$$Rf(x) = \frac{1}{2}(f(x+i) + f(x-i)) \tag{7}$$

$$Jf(x) = \frac{1}{2i}(f(x+i) - f(x-i)) \tag{8}$$

$$Qf(x) = xf(x) \tag{9}$$

The notation for the last operator is inspired by analogies with quantum mechanics, an analogy which seems natural in the light of the following easily verified relations between the operators.

Proposition 3.2. *The operators R, J and Q satisfy the following relations:*

$$RQ - QR = -J \tag{10}$$

$$JQ - QJ = R \tag{11}$$

$$RJ - JR = 0 \tag{12}$$

$$R^2 + J^2 = I \tag{13}$$

where I is the identity operator.

Since the weight ω is so closely related to the strip S, we shall also describe an orthogonal basis for the space $H^2(S, \mathcal{P})$ where $\mathcal{P}$ is the Poisson measure for 0. We shall therefore besides the system $\{\tau_k\}$ defined above, also consider the system of polynomials described in the following

Proposition 3.3. *Let the system $\{\sigma_k\}_{k=0}^{\infty}$ be given by the following recursion procedure.*

$$\sigma_{-1} = 0,\ \sigma_0 = 1,\ \textit{and}\ \sigma_{k+1}(z) = z\sigma_k(z) - k(k-1)\sigma_{k-1}(z). \tag{14}$$

Then
(i) the function σ_k is a monic polynomial of degree k;
(ii) The polynomials $\{(k!)^{-1}\sigma_k\}_{k=0}^{\infty}$ are an orthogonal basis in the Hilbert space $H^2(S, \mathcal{P})$;
(iii) The norm of the polynomial $(k!)^{-1}\sigma_k$ is 1 for $k = 0$ and 2 for $k \geq 1$.
(iv) The exponential generating function $G_\sigma(z, s) = \sum \frac{\sigma_k(z)}{k!} s^k$ is given by the function

$$G_\sigma(z, s) = e^{z \arctan s} \tag{15}$$

Proof. Since (i) is obvious we shall first prove (iv) and then use that to prove (ii) and (iii). Summing the recursion we see that

$$\sum_{k=0}^{\infty} (\sigma_{k+1}(z) - x\sigma_k(z) + k(k-1)\sigma_{k-1}(z)) \frac{s^k}{k!} = 0 \tag{16}$$

and this leads to the differential equation

$$G'_\sigma(z, s) - xG_\sigma(z, s) + s^2 G'_\sigma(z, s) \tag{17}$$

(where all derivatives are with respect to s). Solving the differential equation and using the fact that $G_\sigma(0, 0) = 1$ (iv) follows. (ii) and (iii) follow from the integral

$$\int_{\partial S} G_\sigma(z, s)\overline{G_\sigma(z, t)}\, d\mathcal{P}z = \frac{1 + s\bar{t}}{1 - s\bar{t}}, \tag{18}$$

which follows immediately from the corresponding integral of the generating function for the τ-system. $\square$

Remark 3. The completeness of the system $\{\tilde{\sigma}_k\}$ can be deduced from the completeness of the previous system.

4. Some connections between the systems

We shall conclude by stating some useful connections between the systems, in terms of the operators R, J and Q.

Theorem 4.1. *The following connections between the two systems of orthogonal polynomials $\{\tau_k\}$ and $\{\sigma_k\}$ hold:*

$$R\sigma_k = \tau_k \tag{19}$$

$$QR\tau_k = \sigma_{k+1} \tag{20}$$

$$J\sigma_k = k\tau_{k-1} \tag{21}$$

$$QJ\tau_k = k\sigma_k \tag{22}$$

Since all these relations are easily proved by induction we leave the proof to the interested reader. Instead we shall state the following consequences of the relations.

Theorem 4.2. *Let the operators S, T, A, B and C be defined as follows: (1) $S = QRR$, $T = RQR$, $A = JQR$, $B = RQJ$ and $C = QJR$. Then the following relations hold:*

$$S^k(\sigma_0) = \sigma_k \tag{23}$$

$$T^k(\tau_0) = \tau_k \tag{24}$$

$$A(\tau_k) = (k+1)\tau_k \tag{25}$$

$$B(\tau_k) = k\tau_k \tag{26}$$

$$C(\sigma_k) = k\sigma_k \tag{27}$$

5. A remarkable relation

We shall in the following denote the polynomials $\tau_k/k!$ by $\tilde{\tau}_k$ and likewise $\sigma_k/k!$ by $\tilde{\sigma}_k$. It follows that the system $\{\tilde{\tau}_k\}_{k=0}^{\infty}$ is an orthonormal basis for the Hilbert space $L^2(\omega)$, and that the system $\{\tilde{\sigma}_k\}_{k=0}^{\infty}$ is an orthogonal basis for the Hilbert space $H^2(S, \mathcal{P})$.

The fact that the generating function for the S-system, i.e. the system $\{\tilde{\sigma}\}$ is so simple has some interesting consequences which will be investigated below. First we shall however observe that the following general fact holding for large classes of systems of polynomials, gets a very pleasant form in the present case.

Theorem 5.1. *Let $h(s) = \sum a_k s^k$ be analytic in the unit disc and let $\{p_n(x)\}_{n=0}^{\infty}$ be a set of polynomials such that the generating function*

$$G_p(x,s) = \sum_{n=0}^{\infty} p_n(x)s^n = \varphi(s)e^{xh(s)}.$$

Then

$$\sum_{n=0}^{n} p_n'(x)s^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k p_{n-k}(x)\right)s^n$$

and in particular if $h(s) = \arctan(s)$ it follows that

$$p_n'(x) = \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k \frac{p_{n-1-2k}(x)}{2k+1}.$$

Proof. This follows immediately from the relation

$$\frac{\partial}{\partial x} G_p(x, s) = h(s) G_p(x, s)$$

□

A more interesting and also more specific relation is given by the following

Theorem 5.2. *Let $\{p_n(x)\}_{n=0}^{\infty}$ be a set of polynomials such that the generating function*

$$G_p(x, s) = \sum_{n=0}^{\infty} p_n(x) s^n = \varphi(s) e^{x \arctan s}.$$

Then

$$p_n(x + y) = \sum_{k=0}^{n} p_{n-k}(x) \tilde{\sigma}_k(y).$$

Proof. This follows immediately from the relation

$$G_p(x + y, s) = G_p(x, s) G_\sigma(y, s)$$

□

The following special case of the preceding fact seems worth observing.

Corollary 5.1. *The following identity holds*

$$p_n(x) = \sum_{k=0}^{n} p_{n-k}(0) \tilde{\sigma}_k(x)$$

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