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Ergodic Theorems for Iterated Function Systems Controlled by Regenerative Sequences

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Iterated function systems are considered, where the function to iterate in each step is determined by a regenerative sequence. Ergodic theorems of distributional and law of large numbers types are obtained under log-average contractivity conditions.

KEY WORDS: Ergodic theorem; stochastic dynamical system; iterated function system; regenerative process.

1. INTRODUCTION

Let X be a Polish space with metric d, and let S be an arbitrary measurable space. Consider a measurable function $w : X \times S \to X$. Assume that, for each $s \in S$, the function $w_s(x) := w(x, s)$ is continuous with respect to x. The set $\{X; w_s, s \in S\}$ is called an iterated function system (IFS).

Let $\{I_n\}_{n=0}^{\infty}$ be a stochastic sequence with state space S. Specify a starting point $x_0 \in X$. The stochastic sequence $\{I_n\}$ then controls the stochastic dynamical system $\{Z_n(x_0)\}_{n=0}^{\infty}$, where

$$Z_n(x_0) := w_{I_{n-1}} \circ w_{I_{n-2}} \circ \cdots \circ w_{I_0}(x_0), \qquad n \geqslant 1, \quad Z_0(x_0) = x_0$$

We call this particular type of stochastic dynamical system an IFS controlled by $\{I_n\}$.

Ergodic theorems are one of the main objects of investigation for stochastic dynamical systems.

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Barnsley and Demko[3] and Barrlund *et al.*,[5] investigated ergodic theorems for the simplest model when $\{I_n\}$ is a sequence of independent identically distributed (i.i.d.) random variables taking a finite number of values. They used the term IFS with probabilities. Letac[17] investigated the same model but with an arbitrary state space of the controlling sequence. Another recent paper related to this model is Åkerlund-Biström.[1]

Generalizations in the direction of a more general controlling sequence has been done by Barnsley *et al.*,[4] to the case of a controlling homogeneous finite Markov chain (they used the term recurrent IFS), Elton[9] to the case of a controlling stationary sequence, and by Stenflo[26] to a controlling finite semi-Markov process with discrete time.

A closely related field of research concerns products of random matrices which correspond to iterations with affine maps or in the case of 2×2 matrices to Möbius maps. In this field the monograph by Högnäs and Mukherjea[12] together with, for instance, the paper by Kaijser[15] can serve as an overview. See also the book by Berger.[6] Other related papers in this context are Mukherjea[21] and Lu and Mukherjea.[19]

We would also like to mention works by Elton,[8] Berger and Soner,[7] and Gadde,[11] related to IFS with place dependent probabilities, the book by Iosifescu and Theodorescu[13] and papers by Kaijser [for instance Kaijser[14]] on the theory of random systems with complete connections, as well as the books by Tong[27] on non-linear time series, and Meyn and Tweedie[20] on the theory of Markov chains with extensive overviews of dynamical models. Some additional references can be found in these works.

In this paper, we consider the model when the controlling sequence is a regenerative process with discrete time and an arbitrary state space. Iterated function systems controlled by sequences of i.i.d. random variables or recurrent Markov chains are particular cases of this model. We obtain ergodic theorems of distributional and law of large numbers types which are uniform with respect to initial points taken in compact sets. Theorem 1, which is the main result in this paper, has also another original feature. It gives the asymptotical behavior for the distribution of the random vector $(Z_n(x_0), I_n)$ in a mixed form implying weak and full convergence for the first and second component, respectively.

2. DISTRIBUTIONAL ERGODIC THEOREMS

Let $\{I_n\}$ be a discrete time regenerative random process with state space S, and let $0 = T_0 < T_1 < \cdots < T_n < \cdots$ denote its regeneration moments (without loss of generality we exclude the possibility for regeneration moments to coincide). Loosely speaking, we consider a process that probabilistically restarts at the regeneration moments; i.e., for all $n \geqslant 0$:

(a) The σ-algebras of random events generated by the sets of random variables $I_0, \ldots, I_{T_n - 1}$ and $I_{T_n}, I_{T_n + 1}, \ldots$ are independent, (b) The finite dimensional distributions of the random sequence $I_{T_n}, I_{T_n + 1}, \ldots$ do not depend on n. For a more detailed definition, see for example Lindvall[18] (Chap. III).

Let us, for a function $h : X \to X$, define the generalized norm

$$\|h\| = \sup_{x, y \in X, x \neq y} \frac{d(h(x), h(y))}{d(x, y)}$$

Assume that

(A) $ET_1 < \infty$.

(B) The distribution of T_1 is nonperiodic.

(C) $E \ln \|Z_{T_1}\| = -c$, where $0 < c \leqslant \infty$.

(D) $E \ln^+ d(y_0, Z_{T_1}(y_0)) = d < \infty$, for some $y_0 \in X$.

The conditions (A) and (B) are the standard conditions of ergodicity for the controlling regenerative sequence $\{I_n\}$. The condition (C) is an "average contraction" condition and (D) is a type of "stochastic boundedness" condition for the dynamical sequence $\{Z_n(x_0)\}$ in one regeneration cycle.

The dynamical system $\{Z_n(x_0)\}$, when the IFS is controlled by a regenerative sequence, is not in general a Markov chain, however, the subsequence taken at the regeneration moments $\{Z_{T_n}(x_0)\}$ is a homogeneous Markov chain. For this Markov chain we have the following lemma.

Lemma 1. Under conditions (C) and (D), the Markov chain $\{Z_{T_n}(x_0)\}$ has a unique stationary probability measure μ, and the distribution of $Z_{T_n}(x_0)$ converges weakly to μ as $n \to \infty$ for all $x_0 \in X$.

The proof is given within the proof of the following distributional ergodic theorem which is the main result of this paper.

Theorem 1. Under conditions (A)–(D), for any real-valued bounded and continuous function g on X, any measurable set A in S, and any compact set $K \subseteq X$

$$\sup_{x \in K} \left| Eg(Z_n(x))\, \chi(I_n \in A) - \frac{1}{ET_1} \sum_{k=0}^{\infty} \int_X Eg(Z_k(y))\, \chi(I_k \in A, T_1 > k)\, \mu(dy) \right|$$

$$\to 0 \quad \text{as} \quad n \to \infty \tag{2.1}$$

In the case when $A = S$, (2.1) implies that for an arbitrary point $x \in X$, the distribution of $Z_n(x)$ converges weakly to $\tilde{\mu}$ where

$$\tilde{\mu}(B) := \frac{1}{ET_1} \sum_{k=0}^{\infty} \int_X P(Z_k(y) \in B, T_1 > k)\, \mu(dy)$$

With $g \equiv 1$ (2.1) implies that $P(I_n \in A)$ converges to $\nu(A) = (1/ET_1) \sum_{k=0}^{\infty} P(I_k \in A, T_1 > k)$ for any measurable set A. That is the standard ergodic theorem for the regenerative sequence $\{I_n\}$.

We would like to stress that this latter full convergence for $\{I_n\}$, is stronger than weak convergence for $\{Z_n(x)\}$ which concerns sets, B, with $\tilde{\mu}(\partial B) = 0$, where ∂B denotes the boundary of B.

Thus the relation (2.1) yields a kind of "mixture" of weak convergence and full convergence for the dynamical sequence $\{Z_n(x)\}$ and the controlling sequence $\{I_n\}$, respectively.

We shall now prove Theorem 1.

Proof. Let $\{x_n\}$ be an arbitrary sequence in K. It is equivalent with (2.1) to prove that

$$Eg(Z_n(x_n))\, \chi(I_n \in A) \rightarrow \frac{1}{ET_1} \sum_{k=0}^{\infty} \int_X Eg(Z_k(y))\, \chi(I_k \in A, T_1 > k)\, \mu(dy) \qquad \text{as} \quad n \to \infty \tag{2.2}$$

Let $N_n := \sup\{k : T_k \leqslant n\}$ denote the number of regenerations before time n, and let $\gamma_n := n - T_{N_n}$ be the time since the last regeneration moment before time n. Conditioning on the pair $(Z_{T_{N_n}}(x_n), \gamma_n)$ and using the regeneration property of $\{I_n\}$, we obtain the following equalities

$$\begin{aligned} &Eg(Z_n(x_n))\, \chi(I_n \in A) \\ &= \sum_{k=0}^{\infty} \int_X E(g(Z_n(x_n))\, \chi(I_n \in A) \mid Z_{T_{N_n}}(x_n) = y, \gamma_n = k)\, P(Z_{T_{N_n}}(x_n) \in dy, \gamma_n = k) \\ &= \sum_{k=0}^{\infty} \int_X E(g(Z_k(y))\, \chi(I_k \in A) \mid T_1 > k)\, P(Z_{T_{N_n}}(x_n) \in dy, \gamma_n = k) \\ &= \sum_{k=0}^{\infty} \int_X \frac{Eg(Z_k(y))\, \chi(I_k \in A, T_1 > k)}{P(T_1 > k)}\, P(Z_{T_{N_n}}(x_n) \in dy, \gamma_n = k) \end{aligned} \tag{2.3}$$

The last series is convergent asymptotically uniformly with respect to n which can be seen from the following inequality and well known results from renewal theory based on the conditions (A) and (B)

$$\lim_{N\to\infty} \limsup_{n\to\infty} \left| \sum_{k=N}^{\infty} \int_X \frac{Eg(Z_k(y))\,\chi(I_k \in A,\, T_1 > k)}{P(T_1 > k)} P(Z_{T_{N_n}}(x_n) \in dy,\, \gamma_n = k) \right|$$

$$\leqslant \lim_{N\to\infty} \limsup_{n\to\infty} |g|\, P(\gamma_n \geqslant N)$$

$$= |g| \lim_{N\to\infty} \frac{1}{ET_1} \sum_{k=N} P(T_1 > k) = 0 \tag{2.4}$$

where $|g| := \sup_{x\in X} |g(x)|$.

Thus from (2.4) and since

$$Eg(Z_k(y))\,\chi(I_k \in A,\, T_1 > k) = Eg(w_{I_{k-1}} \circ \cdots \circ w_{I_0}(y))\,\chi(I_k \in A,\, T_1 > k)$$

is a bounded and continuous function of y for each fixed k, (follows from the continuity of $w_s(x)$ for each fixed $s \in S$, and from the dominated convergence theorem), it follows that the theorem will be proved if we can show that for all k and measurable sets, B, with $\mu(\partial B) = 0$,

$$P(Z_{T_{N_n}}(x_n) \in B,\, \gamma_n = k) \to \mu(B) \frac{P(T_1 > k)}{ET_1} \quad \text{as} \quad n \to \infty \tag{2.5}$$

In order to prove (2.5), we represent $Z_{T_n}(x_n)$ as

$$Z_{T_n}(x_n) = W_{\mathbf{I}_{n-1}} \circ \cdots \circ W_{\mathbf{I}_0}(x_n)), \qquad n \geqslant 1, \quad Z_{T_0}(x_0) = x_0 \tag{2.6}$$

where

$$\mathbf{I}_n := (I_{T_n},\ldots, I_{T_{n+1}-1},\, T_{n+1} - T_n) \qquad \text{and} \qquad W_{\mathbf{I}_n} := w_{I_{T_{n+1}-1}} \circ \cdots \circ w_{I_{T_n}}$$

and introduce the random variable

$$\hat{Z}_{T_n}(x_n) := W_{\mathbf{I}_0} \circ \cdots \circ W_{\mathbf{I}_{n-1}}(x_n), \qquad n \geqslant 1, \quad \hat{Z}_{T_0}(x_0) = x_0 \tag{2.7}$$

Since $\{I_n\}$ is a regenerative sequence, $\{\mathbf{I}_n\}$ is a sequence of i.i.d. random variables with values in the space $\mathbf{S} = \bigcup_{k=1}^{\infty} S^k \times \{k\}$. From this fact and the representations (2.6) and (2.7) it follows that the random variables $Z_{T_n}(x_n)$ and $\hat{Z}_{T_n}(x_n)$ have the same distribution. The motivation to introduce $\hat{Z}_{T_n}(x_n)$ is the following lemma.

Lemma 2. Under conditions (C) and (D), there exists a random variable $\hat{Z}$, with values in X, such that for any compact set $K \subseteq X$

$$\sup_{x \in K} d(\hat{Z}_{T_n}(x), \hat{Z}) \to 0 \qquad \text{a.s.} \tag{2.8}$$

The distribution μ of the random variable $\hat{Z}$ is also the unique stationary probability measure of the Markov chain $\{Z_{T_n}(x_0)\}$.

Proof. Let y_0 be an arbitrary point in X satisfying condition (D). We shall first prove, modifying the method used in Barnsley *et al.*[4] that $\{\hat{Z}_{T_n}(y_0)\}$ is almost surely (a.s.) a Cauchy sequence. Using the triangle inequality and the definition of the generalised norm we obtain, for $k \leqslant n \leqslant m$,

$$\begin{aligned} d(\hat{Z}_{T_n}(y_0), \hat{Z}_{T_m}(y_0)) &\leqslant \sum_{n=k}^{\infty} d(\hat{Z}_{T_n}(y_0), \hat{Z}_{T_{n+1}}(y_0)) \\ &\leqslant \sum_{n=k}^{\infty} \|\hat{Z}_{T_n}\| \, d(y_0, W_{\mathbf{I}_n}(y_0)) \end{aligned} \tag{2.9}$$

Thus a sufficient condition for the sequence $\{\hat{Z}_{T_n}(y_0)\}$ to be Cauchy a.s. is that the expression in (2.9) converges with probability one to zero as k tends to infinity. Now,

$$\ln \|\hat{Z}_{T_n}\| \, d(y_0, W_{\mathbf{I}_n}(y_0)) \leqslant \ln \|\hat{Z}_{T_n}\| + \ln^+ d(y_0, W_{\mathbf{I}_n}(y_0)) \tag{2.10}$$

From the definition of the norm, condition (C) and the law of large numbers for i.i.d. random variables it follows that

$$\frac{\ln \|\hat{Z}_{T_n}\|}{n} \leqslant \frac{\sum_{i=0}^{n-1} \ln \|W_{\mathbf{I}_i}\|}{n} \to -c \qquad \text{a.s.} \tag{2.11}$$

It also follows from condition (D) and the law of large numbers that,

$$\begin{aligned} &\frac{1}{n} \ln^+ d(y_0, W_{\mathbf{I}_n}(y_0)) \\ &\quad = \frac{\sum_{i=0}^{n} \ln^+ d(y_0, W_{\mathbf{I}_i}(y_0))}{n} - \frac{\sum_{i=0}^{n-1} \ln^+ d(y_0, W_{\mathbf{I}_i}(y_0))}{n} \to 0 \qquad \text{a.s.} \end{aligned} \tag{2.12}$$

Using (2.10)–(2.12), we conclude that

$$\limsup_{n\to\infty} \frac{\ln \|\hat{Z}_{T_n}\| \, d(y_0, W_{\mathbf{I}_n}(y_0))}{n} \leqslant -c \qquad \text{a.s.} \tag{2.13}$$

Thus with probability one, there exist a random integer M_0 such that for all $n \geqslant M_0$, $\|\hat{Z}_{T_n}\| \, d(y_0, W_{\mathbf{I}_n}(y_0)) \leqslant e^{-cn/2}$ and thus the sum in (2.9) is *a.s.* majorized by a convergent sum which proves that

$$\lim_{k\to\infty} \sum_{n=k}^{\infty} \|\hat{Z}_{T_n}\| \, d(y_0, W_{\mathbf{I}_n}(y_0)) = 0 \qquad \text{a.s.} \tag{2.14}$$

This implies that $\{\hat{Z}_n(y_0)\}$ is Cauchy a.s., and therefore converges to, say $\hat{Z}$, a.s., i.e.,

$$d(\hat{Z}_{T_n}(y_0), \hat{Z}) \to 0 \qquad \text{a.s.} \tag{2.15}$$

From the definition of the norm and (2.11) we get that

$$\sup_{x\in K} d(\hat{Z}_{T_n}(x), \hat{Z}_{T_n}(y_0)) \leqslant \|\hat{Z}_{T_n}\| \sup_{x\in K} d(x, y_0) \to 0 \qquad \text{a.s.} \tag{2.16}$$

Since

$$\sup_{x\in K} d(\hat{Z}_{T_n}(x), \hat{Z}) \leqslant \sup_{x\in K} d(\hat{Z}_{T_n}(x), \hat{Z}_{T_n}(y_0)) + d(\hat{Z}_{T_n}(y_0), \hat{Z}) \tag{2.17}$$

it follows from (2.15) and (2.16) used in (2.17) that, (2.8) holds and we have proved the first part of the lemma.

Remark 1. From this proof it follows that condition (D) can be replaced by any condition implying that $(1/n) \ln^+ d(y_0, W_{\mathbf{I}_n}(y_0)) \to 0$ a.s.

Define the measure $\mu(B) = P(\hat{Z} \in B)$. Since $Z_{T_n}(x)$ has the same distribution as $\hat{Z}_{T_n}(x)$ for all n and x, it follows from (2.8) that the distribution of $Z_{T_n}(x)$ converges weakly to μ as $n \to \infty$ for all $x \in X$. Since this limiting measure does not depend on x, and since the Markov chain $\{Z_{T_n}(x_0)\}$ has the Feller property, i.e., for any bounded continuous function $g: X \to R$, $\int_X P(x, dy)\, g(y) = Eg(w_{I_{T_1-1}} \circ \cdots \circ w_{I_0}(x))$ is a bounded continuous function with respect to x, (follows from the continuity of the functions $w_s(x)$, for each fixed $s \in S$, and the dominated convergence theorem), it follows that μ is a unique stationary probability measure for the Markov chain $\{Z_{T_n}(x_0)\}$. For details see Letac.[17] This completes the proof of Lemma 2 and also proves Lemma 1. □

Let us now return to the proof of the theorem. From the representations (2.6) and (2.7) we obtain the following equality for all n and k,

$$\begin{aligned}
P(Z_{T_{N_n}}(x_n) \in B, \gamma_n = k) \\
&= \sum_{m=0}^{\infty} P(Z_{T_m}(x_n) \in B, \gamma_n = k, N_n = m) \\
&= \sum_{m=0}^{\infty} P(Z_{T_m}(x_n) \in B, T_m = n-k, T_{m+1} - T_m > k) \\
&= \sum_{m=0}^{\infty} P(\hat{Z}_{T_m}(x_n) \in B, T_m = n-k, T_{m+1} - T_m > k) \\
&= \sum_{m=0}^{\infty} P(\hat{Z}_{T_m}(x_n) \in B, \gamma_n = k, N_n = m) \\
&= P(\hat{Z}_{T_{N_n}}(x_n) \in B, \gamma_n = k)
\end{aligned} \tag{2.18}$$

Due to (2.18) we can instead of proving (2.5) complete the proof of Theorem 1 by proving that for all k and measurable sets, B, with $\mu(\partial B) = 0$,

$$P(\hat{Z}_{T_{N_n}}(x_n) \in B, \gamma_n = k) \to \mu(B)\, \pi(k) \qquad \text{as} \quad n \to \infty \tag{2.19}$$

where

$$\pi(k) = \frac{P(T_1 > k)}{E(T_1)} \tag{2.20}$$

Under conditions (A) and (B), the sequence $\{\gamma_n\}$ is an ergodic Markov chain with stationary probability distribution π, where, see for example Feller[10] [Chap. XV], π is given by (2.20), and $P(\gamma_n = k \mid \gamma_r = m) \to \pi(k)$ as $n \to \infty$ for all k and m. Therefore, see Rényi,[23] the sequence $\{\gamma_n\}$ is strongly mixing in the sense that $P(C, \gamma_n = k) \to P(C)\, \pi(k)$, for an arbitrary random event C. Thus,

$$P(\hat{Z} \in B, \gamma_n = k) \to P(\hat{Z} \in B)\, \pi(k) \qquad \text{as} \quad n \to \infty \tag{2.21}$$

Since T_n are sums of i.i.d. random variables, from renewal theory,

$$N_n / n \to 1/ET_1 \qquad \text{a.s.} \tag{2.22}$$

Therefore $N_n \to \infty$ a.s., which together with Lemma 2 implies that,

$$\hat{Z}_{T_{N_n}}(x_n) \to \hat{Z} \qquad \text{a.s.} \tag{2.23}$$

Using (2.23) it follows that for all sets B with $\mu(\partial B) = 0$,

$$\begin{aligned} &|P(\hat{Z}_{T_{N_n}}(x_n) \in B, \gamma_n = k) - P(\hat{Z} \in B, \gamma_n = k)| \\ &\quad \leqslant P(\{\hat{Z}_{T_{N_n}}(x_n) \in B\} \, \Delta \{\hat{Z} \in B\}) \to 0 \qquad \text{as} \quad n \to \infty \end{aligned} \tag{2.24}$$

From (2.21) and (2.24) it follows that (2.19) holds. This completes the proof of Theorem 1. □

We can also give a version of Theorem 1 when the regenerative process is periodic, i.e., condition (B) does not hold.

Theorem 2. Suppose the distribution of T_1 has period p. Under conditions (A), (C), and (D), for any real-valued bounded continuous function g on X, any measurable set A in S, any compact set $K \subseteq X$, and any $r \in \{0, 1, ..., p-1\}$,

$$\begin{aligned} &\sup_{x \in K} \Bigg| Eg(Z_{np+r}(x)) \, \chi(I_{np+r} \in A) \\ &\qquad - \frac{p}{ET_1} \sum_{k=0}^{\infty} \int_X Eg(Z_{kp+r}(y)) \, \chi(I_{kp+r} \in A, T_1 > kp + r) \, \mu(dy) \Bigg| \\ &\quad \to 0 \qquad \text{as} \quad n \to \infty \end{aligned} \tag{2.25}$$

The proof is analogous with that of Theorem 1, but now we need to take into account the periodicity of the Markov chain $\{\gamma_n\}$, and replace the limits in (2.4), (2.5), (2.19) and then consequently in (2.1) by the corresponding subsequential limits.

3. INDIVIDUAL ERGODIC THEOREMS

In this section, we will give some individual ergodic theorems in the case of a regenerative controlling sequence.

For a function $h: X \to X$ define, for $r > 0$

$$\|h\|_r := \sup_{x, y \in X, 0 < d(x, y) \leqslant r} \frac{d(h(x), h(y))}{d(x, y)}$$

Obviously $\|h\|_r$ is nondecreasing in r, and $\lim_{r \to \infty} \|h\|_r = \|h\|$.

Assume that

(E) $E \max_{0 \leqslant k < T_1} \ln^+ \|Z_k\|_r < \infty$, for some $r > 0$.

Lemma 3. Under conditions (A), (C) and (E), the following relation holds for any $q > 0$

$$\limsup_{n \to \infty} \frac{\ln \|Z_n\|_q}{n} \leqslant \frac{-c}{ET_1} < 0 \qquad \text{a.s.} \tag{3.1}$$

Proof. Using the definition of the generalized norm, condition (C), (2.6) and the law of large numbers we obtain,

$$\limsup_{n \to \infty} \frac{\ln \|Z_{T_n}\|}{n} \leqslant \limsup_{n \to \infty} \frac{\sum_{i=0}^{n-1} \ln \|W_{\mathbf{I}_i}\|}{n} = -c \qquad \text{a.s.} \tag{3.2}$$

Let us fix $q > 0$ and let $r > 0$ be a fixed number such that condition (E) is satisfied. From (3.2) it follows that there exists a random integer M_1, finite with probability one, such that $\|Z_{T_n}\| \leqslant r/q$ if $n \geqslant M_1$. Since $N_n \to \infty$ a.s., there exists a random integer M_2, finite with probability one, such that, if $n \geqslant M_2$ then $N_n \geqslant M_1$ and thus $\|Z_{T_{N_n}}\| \leqslant r/q$. Therefore, for $n \geqslant M_2$,

$$\sup_{d(x, y) \leqslant q} d(Z_{T_{N_n}}(x), Z_{T_{N_n}}(y)) \leqslant \|Z_{T_{N_n}}\| \sup_{d(x, y) \leqslant q} d(x, y) \leqslant r \tag{3.3}$$

and thus for $n \geqslant M_2$

$$\begin{aligned} \|Z_n\|_q = & \sup_{0 < d(x, y) \leqslant q} \left\{ \frac{d(w_{I_{n-1}} \circ \cdots \circ w_{I_{T_{N_n}}}(Z_{T_{N_n}}(x)), w_{I_{n-1}} \circ \cdots \circ w_{I_{T_{N_n}}}(Z_{T_{N_n}}(y)))}{d(Z_{T_{N_n}}(x), Z_{T_{N_n}}(y))} \right. \\ & \left. \times \frac{d(Z_{T_{N_n}}(x), Z_{T_{N_n}}(y))}{d(x, y)} \right\} \\ \leqslant & \|w_{I_{n-1}} \circ \cdots \circ w_{I_{T_{N_n}}}\|_r \cdot \|Z_{T_{N_n}}\| \end{aligned} \tag{3.4}$$

(We interpret $\|w_{I_m} \circ \cdots \circ w_{I_n}\|_r$ as 1 if $m < n$, and the left-hand side of the inequality as 0 for those points x, y for which $d(Z_{T_{N_n}}(x), Z_{T_{N_n}}(y)) = 0$.) Taking logs we obtain, for $n \geqslant M_2$,

$$\ln \|Z_n\|_q \leqslant \ln^+ \|w_{I_{n-1}} \circ \cdots \circ w_{I_{T_{N_n}}}\|_r + \ln \|Z_{T_{N_n}}\| \tag{3.5}$$

and thus,

$$\limsup_{n\to\infty} \frac{\ln \|Z_n\|_q}{n} \leqslant \limsup_{n\to\infty} \frac{\ln^+ \|w_{I_{n-1}} \circ \cdots \circ w_{I_{T_{N_n}}}\|_r}{n} + \limsup_{n\to\infty} \frac{\ln \|Z_{T_{N_n}}\|}{n}$$

$$\leqslant \limsup_{n\to\infty} \frac{\Delta_{N_n}}{n} + \limsup_{n\to\infty} \frac{\ln \|Z_{T_{N_n}}\|}{n} \qquad \text{a.s.} \tag{3.6}$$

where

$$\Delta_n := \max_{T_n \leqslant j < T_{n+1}} \ln^+ \|w_{I_j} \circ \cdots \circ w_{I_{T_n}}\|_r$$

Since $\{\Delta_n\}$ is a sequence of i.i.d. random variables, it follows from condition (E) and the law of large numbers that

$$\frac{\Delta_n}{n} = \frac{\sum_{i=0}^{n} \Delta_i}{n} - \frac{\sum_{i=0}^{n-1} \Delta_i}{n} \to 0 \qquad \text{a.s.} \tag{3.7}$$

From (2.22), (which implies that $N_n \to \infty$ a.s.), and (3.7) we obtain that

$$\frac{\Delta_{N_n}}{n} = \frac{\Delta_{N_n}}{N_n} \frac{N_n}{n} \to 0 \qquad \text{a.s.} \tag{3.8}$$

In the same way as in (3.8) it follows from (3.2) and (2.22) that

$$\limsup_{n\to\infty} \frac{\ln \|Z_{T_{N_n}}\|}{n} = \limsup_{n\to\infty} \frac{\ln \|Z_{T_{N_n}}\|}{N_n} \cdot \frac{N_n}{n} \leqslant -\frac{c}{ET_1} \qquad \text{a.s.} \tag{3.9}$$

Using (3.8) and (3.9) in (3.6) completes the proof of the lemma. □

Remark 2. If we replace condition (E) in Lemma 3 by the stronger condition

(E′) $E \max_{0 \leqslant k < T_1} \ln^+ \|Z_k\| < \infty$

we can sharpen (3.1) to

$$\limsup_{n\to\infty} \frac{\ln \|Z_n\|}{n} \leqslant -\frac{c}{ET_1} < 0 \qquad \text{a.s.}$$

In this case (3.4) is replaced by the following inequality which holds for all n

$$\|Z_n\| \leqslant \|w_{I_{n-1}} \circ \cdots \circ w_{I_{T_{N_n}}}\| \cdot \|Z_{T_{N_n}}\|$$

The rest of the proof follows in analogy with the previous proof.

Corollary 1. Under conditions (A), (C) and (E), for any compact set $K \subseteq X$,

$$\limsup_{n \to \infty} \frac{\ln(\sup_{x, y \in K} d(Z_n(x), Z_n(y)))}{n} \leqslant -\frac{c}{ET_1} < 0 \qquad \text{a.s.} \qquad (3.10)$$

In particular it follows from (3.10) that

$$\sup_{x, y \in K} d(Z_n(x), Z_n(y)) \to 0 \qquad \text{a.s.} \qquad (3.11)$$

Proof. This follows from Lemma 3 since for $q > \sup_{x, y \in K} d(x, y)$

$$\begin{aligned} &\limsup_{n \to \infty} \frac{\ln(\sup_{x, y \in K} d(Z_n(x), Z_n(y)))}{n} \\ &\quad \leqslant \frac{1}{n} \limsup_{n \to \infty} \ln \left(\sup_{x, y \in K, x \neq y} \frac{d(Z_n(x), Z_n(y))}{d(x, y)} d(x, y) \right) \\ &\quad \leqslant \limsup_{n \to \infty} \frac{\ln \|Z_n\|_q}{n} + \limsup_{n \to \infty} \frac{\ln(\sup_{x, y \in K} d(x, y))}{n} \\ &\quad \leqslant -\frac{c}{ET_1} \qquad \text{a.s.} \end{aligned}$$

□

Let us consider a measurable function $f: X \times S \to R$, such that

(F) $\int_X E \sum_{k=0}^{T_1-1} |f(Z_k(y), I_k)| \, \mu(dy) < \infty$

Define

$$m_f := \frac{1}{ET_1} \int_X E \sum_{k=0}^{T_1-1} f(Z_k(y), I_k) \, \mu(dy)$$

Theorem 3. Under conditions (A), (C), (D), and (F) there exists a set B with $\mu(B) = 1$ such that for all $x \in B$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(Z_k(x), I_k) = m_f \qquad \text{a.s.} \qquad (3.12)$$

Proof. Let Z_0 be a random variable with distribution μ which is independent of the regenerative sequence $\{I_n\}$, and define $Z_n := Z_n(Z_0)$, $n = 1, 2, ...$. Let $\xi_n := \sum_{k=0}^{n} f(Z_k, I_k)$, $\alpha_n := \sum_{k=T_{n-1}}^{T_n-1} f(Z_k, I_k)$, $\delta_n := \sum_{k=T_{n-1}}^{T_n-1} |f(Z_k, I_k)|$, and $\beta_n := \sum_{k=T_{N_n}}^{n} f(Z_k, I_k)$. Using this notation, we can represent the process ξ_n as

$$\xi_n = \sum_{k=1}^{N_n} \alpha_k + \beta_n, \qquad n = 0, 1, ... \tag{3.13}$$

We shall show, repeating the way in Silvestrov,[24] that

$$\frac{\xi_n}{n} \to m_f \qquad \text{a.s.} \tag{3.14}$$

Using arguments identical as for $\{\alpha_n\}$ below, we see that $\{\delta_n\}$ is a stationary sequence. Due to condition (F) we can use Birkhoff's ergodic theorem to obtain,

$$\frac{\delta_n}{n} = \frac{\sum_{k=1}^{n} \delta_k}{n} - \frac{\sum_{k=1}^{n-1} \delta_k}{n} \to 0 \qquad \text{a.s.} \tag{3.15}$$

Since, $|\beta_n| \leq \delta_{N_n+1}$ we get from (2.22) and (3.15) that

$$\left|\frac{\beta_n}{n}\right| \leq \frac{\delta_{N_n+1}}{n} = \frac{\delta_{N_n+1}}{N_n+1} \frac{N_n+1}{n} \to 0 \qquad \text{a.s.} \tag{3.16}$$

If we can prove that,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \alpha_k = E\alpha_1 \qquad \text{a.s.} \tag{3.17}$$

then (3.14) follows. This can be seen since from (3.13), (3.16), (3.17) and (2.22) we get

$$\lim_{n\to\infty} \frac{\xi_n}{n} = \lim_{n\to\infty} \frac{1}{n}\left(\sum_{k=1}^{N_n} \alpha_k + \beta_n\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{N_n} \alpha_k$$

$$= \lim_{n\to\infty} \frac{N_n}{n} \frac{1}{N_n} \sum_{k=1}^{N_n} \alpha_k = \frac{E\alpha_1}{ET_1} = m_f \qquad \text{a.s.}$$

So it remains to prove that the sequence $\{\alpha_n\}$ is ergodic in the sense of relation (3.17). In order to prove this, we consider the sequence $\{Z_{T_n}, \alpha_n\}_{n=1}^{\infty}$. This sequence is a Markov renewal process, i.e., a homogeneous two component Markov chain with transition probabilities not depending on the second component $(P(Z_{T_{n+1}} \in B,\ \alpha_{n+1} \in D \mid Z_{T_n} = x, \alpha_n = s) = P_x(B, D))$.

According to Lemma 1, the Markov chain $\{Z_{T_n}\}$ has the unique invariant probability measure μ. As is easily verified this implies that $\{Z_{T_n}, \alpha_n\}$ has the unique invariant probability measure given by $\pi(B, D) = \int P_y(B, D)\, \mu(dy)$. Since Z_0 has distribution μ, the Markov chain $\{Z_{T_n}, \alpha_n\}$ forms a stationary sequence.

A stationary Markov chain with a unique invariant probability measure is ergodic in the sense of a trivial tail σ-algebra, see for example Elton.[8] Thus we obtain, using Birkhoff's ergodic theorem, that for every measurable function $g : X \times R \to R$ satisfying $E\,|g(Z_{T_1}, \alpha_1)| < \infty$,

$$\frac{1}{n} \sum_{k=1}^{n} g(Z_{T_n}, \alpha_n) \to Eg(Z_{T_1}, \alpha_1) \qquad \text{a.s.} \tag{3.18}$$

So in particular with $g(x, s) = s$ we obtain using condition (F) that (3.17) and thus (3.14) holds. Obviously (3.14) implies that $\xi_{n-1}/n \to m_f$ a.s. and therefore

$$\begin{aligned} 1 &= P\left(\lim_{n \to \infty} \frac{\sum_{k=0}^{n-1} f(Z_k)}{n} = m_f\right) \\ &= \int_X P\left(\lim_{n \to \infty} \frac{\sum_{k=0}^{n-1} f(Z_k(y))}{n} = m_f\right) \mu(dy) \end{aligned}$$

From this it follows that the probabilities under the integral sign equal 1 for almost all y with respect to μ. This completes the proof of the theorem. □

In order to allow an arbitrary initial point we must impose some additional restrictions on the function f. For simplicity let us consider functions only depending on the first variable, $f(x, y) = f(x)$. The condition (F) now takes the form

(F) $\int_X E \sum_{k=0}^{T_1-1} |f(Z_k(y))|\, \mu(dy) < \infty$

and

$$m_f := \frac{1}{ET_1} \int_X E \sum_{k=0}^{T_1-1} f(Z_k(y))\, \mu(dy) = \int f\, d\tilde{\mu}$$

Theorem 4. Suppose that X is a locally compact Polish space and that $f: X \to R$ is a function which can be represented as $f = f_1 + f_2$, where f_1 is uniformly continuous and f_2 is bounded and continuous. Under conditions (A) and (C)–(F), for any compact set $K \subseteq X$,

$$\lim_{n \to \infty} \left(\sup_{x \in K} \left| \frac{\sum_{k=0}^{n-1} f(Z_k(x))}{n} - m_f \right| \right) = 0 \qquad \text{a.s.} \tag{3.19}$$

Proof. From Theorem 3 it follows that there exists a point, $x_0 \in X$ such that

$$\frac{\sum_{k=0}^{n-1} f(Z_k(x_0))}{n} \to m_f \qquad \text{a.s.} \tag{3.20}$$

Suppose f is uniformly continuous. First note, by (3.20) that

$$\begin{aligned} &\limsup_{n \to \infty} \left(\sup_{x \in K} \left| \frac{\sum_{k=0}^{n-1} f(Z_k(x))}{n} - m_f \right| \right) \\ &\quad \leqslant \limsup_{n \to \infty} \left(\sup_{x \in K} \frac{\sum_{k=0}^{n-1} |f(Z_k(x)) - f(Z_k(x_0))|}{n} \right) \\ &\qquad + \limsup_{n \to \infty} \left| \frac{\sum_{k=0}^{n-1} f(Z_k(x_0))}{n} - m_f \right| \\ &\quad = \limsup_{n \to \infty} \left(\sup_{x \in K} \frac{\sum_{k=0}^{n-1} |f(Z_k(x)) - f(Z_k(x_0))|}{n} \right) \qquad \text{a.s.} \end{aligned} \tag{3.21}$$

From Corollary 1 and since f is uniformly continuous we obtain that

$$\limsup_{n \to \infty} (\sup_{x \in K} |f(Z_k(x)) - f(Z_k(x_0))|) = 0 \qquad \text{a.s.} \tag{3.22}$$

This implies that

$$\limsup_{n \to \infty} \left(\sup_{x \in K} \frac{\sum_{k=0}^{n-1} |f(Z_k(x)) - f(Z_k(x_0))|}{n} \right) = 0 \qquad \text{a.s.} \tag{3.23}$$

(since convergence of a sequence implies convergence in a Cesaro sense) and we have completed the proof if f is uniformly continuous.

The idea of the proof with f_2 being bounded and continuous originates from Elton.[8] Let $\varepsilon > 0$ be given and let f_2 be a bounded continuous function, with $|f_2| := \sup_{x \in X} |f_2(x)|$. Let C be a compact set such that $\tilde{\mu}(C) \geqslant \max(1 - \varepsilon/2, 1 - \varepsilon/(2\,|f_2|))$. (Recall that X is a locally compact Polish space.) By Urysohn's lemma, there exists a continuous function g with compact support, $C_1 \supseteq C$, such that $g(x) = 1$ for $x \in C$ and $0 \leqslant g(x) \leqslant 1$ for $x \notin C$.

We shall use the following inequality

$$\sup_{x \in K} \left| \frac{\sum_{k=0}^{n-1} f_2(Z_k(x))}{n} - m_{f_2} \right| \leqslant \left| \int f_2(y)\, g(y)\, \tilde{\mu}(dy) - m_{f_2} \right| + \sup_{x \in K} \left| \frac{\sum_{k=0}^{n-1} f_2(1-g)(Z_k(x))}{n} \right| + \sup_{x \in K} \left| \frac{\sum_{k=0}^{n-1} f_2 \cdot g(Z_k(x))}{n} - \int f_2(y)\, g(y)\, \tilde{\mu}(dy) \right| \tag{3.24}$$

Now,

$$\left| \int f_2(y)\, g(y)\, \tilde{\mu}(dy) - m_{f_2} \right| \leqslant |f_2| \int (1 - g(y))\, \tilde{\mu}(dy) \leqslant |f_2|\, (1 - \tilde{\mu}(C)) \leqslant \varepsilon/2 \tag{3.25}$$

Since g and so $1 - g$ are bounded they also satisfy condition (F) due to condition (A). These functions are also uniformly continuous and thus from the first part of this proof it follows that

$$\begin{aligned} \limsup_{n \to \infty} \left(\sup_{x \in K} \left| \frac{\sum_{k=0}^{n-1} f_2(1-g)(Z_k(x))}{n} \right| \right) &\leqslant |f_2| \limsup_{n \to \infty} \left(\sup_{x \in K} \frac{\sum_{k=0}^{n-1} (1-g)(Z_k(x))}{n} \right) \\ &\leqslant |f_2| \left[\limsup_{n \to \infty} \left(\sup_{x \in K} \left| \frac{\sum_{k=0}^{n-1} (1-g)(Z_k(x))}{n} - \int (1 - g(y))\, \tilde{\mu}(dy) \right| \right) + \int (1 - g(y))\, \tilde{\mu}(dy) \right] \\ &= |f_2| \int (1 - g(y))\, \tilde{\mu}(dy) \leqslant \varepsilon/2 \qquad \text{a.s.} \end{aligned} \tag{3.26}$$

Since $f_2 \cdot g$ is bounded it follows from condition (A) that (F) holds for this function. This function is also continuous and has support within the compact set C_1. Therefore it is uniformly continuous and we obtain from the first part of this proof that

$$\limsup_{n \to \infty} \left(\sup_{x \in K} \left| \frac{\sum_{k=0}^{n-1} f_2 \cdot g(Z_k(x))}{n} - \int f_2(y)\, g(y)\, \tilde{\mu}(dy) \right| \right) = 0 \qquad \text{a.s.} \quad (3.27)$$

It follows from (3.25), (3.26), and (3.27) used in (3.24) that

$$\limsup_{n \to \infty} \left(\sup_{x \in K} \left| \frac{\sum_{k=0}^{n-1} f_2(Z_k(x))}{n} - m_{f_2} \right| \right) \leqslant \varepsilon \qquad \text{a.s.} \qquad (3.28)$$

for an arbitrary $\varepsilon > 0$. This completes the proof of the theorem. □

4. COMMENTS CONCERNING THE CONDITIONS

Here we give some sufficient conditions for the conditions used in this paper.

(C) An application of Jensen's inequality shows that (C′): $E \|Z_{T_1}\| < 1$ is sufficient for condition (C). The requirement that all functions w_s, $s \in S$ are contractions with contraction coefficients $\rho_s \leqslant \rho < 1$, implies that condition (C′) is satisfied.

(D) A sufficient condition for (D) is (D′): $Ed(y_0, Z_{T_1}(y_0)) < \infty$, for some $y_0 \in X$. If condition (C′) also holds, (D′) can only hold for all $x \in X$ simultaneously. This follows since $Ed(x, Z_{T_1}(x)) \leqslant Ed(x, y_0) + Ed(y_0, Z_{T_1}(y_0)) + Ed(Z_{T_1}(y_0), Z_{T_1}(x)) < \infty$. Condition (D′) is satisfied when X is a compact set.

(E) If all maps $\{w_s\}$ are Lipschitz continuous with the same Lipschitz constant a, for all $s \in S$, then (A) is sufficient for (E). This follows since for every k, $\|Z_k\|_r \leqslant a^k r$, and thus $E \max_{0 \leqslant k < T_1} \ln^+ \|Z_k\|_r \leqslant ET_1 \ln^+ a + \ln^+ r < \infty$.

(F) If f is bounded then condition (A) is sufficient for (F).

We would also like to note that all conditions (A), and (C)–(F) can be replaced by some sufficient conditions which require the existence of expectations for some functional of additive type accumulated in one regeneration cycle. Condition (A) and (F) are of this type. The following inequalities can be used for the other conditions: $\ln \|Z_{T_1}\| \leqslant \sum_{0 \leqslant k < T_1} \ln \|w_{I_k}\|$; $d(y, Z_{T_1}(y)) \leqslant \sum_{0 \leqslant k < T_1} d(Z_k(y), Z_{k+1}(y))$; and $\max_{0 \leqslant k < T_1} \ln^+ \|Z_k\|_r \leqslant \sum_{0 \leqslant k < T_1} \ln^+ \|w_{I_k}\|$.

Conditions based on expectations of such kind of additive functionals can be effectively checked for various classes of regenerative processes.

Finally we would also like to note that all theorems and lemmas formulated here can be generalized to the case of a controlling regenerative process $\{I_n\}$ with delay. The only change we need to do is to replace all quantities in condition (A)–(F) calculated for the first regeneration cycle (now delayed) by the corresponding quantities calculated for the second (standard) regeneration cycle.

5. IFS CONTROLLED BY A SEQUENCE OF I.I.D. RANDOM VARIABLES

We shall here consider the model with the controlling sequence $\{I_n\}$ being a sequence of i.i.d. random variables. Obviously $\{I_n\}$ can be considered as a regenerative sequence with regeneration moments $T_n = nn_0$, $n = 0, 1,...$ for any fixed integer $n_0 \geqslant 1$. Condition (A) then obviously holds, and the other conditions simplify to

(C*) $E \ln \|Z_{n_0}\| = -c < 0.$

(D*) $E \ln^+ d(y_0, Z_{n_0}(y_0)) < \infty$, for some $y_0 \in X$.

(E*) $E \ln^+ \|Z_k\|_r < \infty$, for $k = 1, 2,..., n_0 - 1$.

(F*) $\int_X |f| \, d\mu < \infty$.

In this case it can be easily shown that under conditions (C*) and (D*), $\{Z_n(x_0)\}$ is a Markov chain with a unique stationary probability measure μ coinciding with the unique stationary probability measure of the Markov chain $\{Z_{nn_0}(x_0)\}$.

As corollaries of Theorems 2 and 4 we obtain the following ergodic theorems.

Corollary 2. Suppose the conditions (C*) and (D*) hold for some n_0. Then for any real-valued bounded continuous function g on X and any compact set $K \subseteq X$,

$$\sup_{x \in K} \left| Eg(Z_n(x)) - \int_X g \, d\mu \right| \to 0 \quad \text{as} \quad n \to \infty \tag{5.1}$$

Corollary 3. Suppose that X is a locally compact Polish space, $f: X \to R$ is a function which can be represented as $f = f_1 + f_2$, where f_1 is uniformly continuous and f_2 is bounded and continuous and that there exist an n_0 such that the conditions (C*)–(F*) hold. Then for any compact set $K \subseteq X$,

$$\lim_{n\to\infty}\left(\sup_{x\in K}\left|\frac{\sum_{k=0}^{n-1} f(Z_k(x))}{n}-\int_X f\,d\mu\right|\right)=0 \qquad \text{a.s.} \tag{5.2}$$

6. IFS CONTROLLED BY A MARKOV CHAIN

If $\{I_n\}$ is an ergodic Markov chain with finite or countable state space, it can be considered as a regenerative sequence with regeneration moments T_n which are return times to some fixed state. The theorems which we obtain in this case however differ from similar theorems in previous papers in that our conditions are of "cyclic" type.

The results in this paper can also be translated to the model in which $\{I_n\}$ is a Harris recurrent Markov chain with a general state space. Here the method of artificial regeneration, developed by Athreya and Ney,(2) Kovalenko,(16) and Nummelin,(22) can be used. According to this method, the Markov chain $\{I_n\}$ can be "embedded" in a two-component Markov chain $\{\tilde{I}_n\}$, (where $\tilde{I}_n=(I_n, I'_n)$ and the random variables I'_n are $\{0, 1\}$-valued), in such a way that $\{\tilde{I}_n\}$ is a regenerative sequence with regenerations at the return moments, T_n, of the second component I'_n to state 1. If we redefine the map w in an obvious way, $\{Z_n(x_0)\}$ can be considered as an IFS controlled by the regenerative sequence $\{\tilde{I}_n\}$.

It is then natural to express the conditions related to the regeneration moment T_1 in terms of transition probabilities of the Markov chain $\{Z_n(x_0), \tilde{I}_n\}$. This can effectively be done using techniques based on stochastic Lyapunov functions [see for example Meyn and Tweedie(20) or Silvestrov(25)]. These techniques may be used to obtain upper bounds for the functionals of additive type mentioned in section 4. This however is not an obvious task, and can be an object of further investigations.

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