Let us start with a classic result due to Bollobás and Erdős (1976) and Matula (1976). Its proof, based on the second moment method, can be found also in Bollobás (1985, Chapter XI).

**Theorem 7.1.** For $0 < \varepsilon < b = 1/(1 - p)$, set

$$
\hat{k}_{\pm \varepsilon} = \left[2 \log_b n - 2 \log_b \log_b np - 2 \log_b (\varepsilon/2) + 1 \pm \varepsilon/p \right].
$$

Then, for $p = p(n)$ such that $p > n^{-\delta}$ for every $\delta > 0$ but $p \leq c$ for some $c < 1$, a.a.s.

$$
\hat{k}_{-\varepsilon} \leq \alpha(G(n, p)) \leq \hat{k}. 
$$

**Remark 7.2.** In fact, Bollobás and Erdős (1976) and Matula (1976) proved that in the above range of $p(n)$, the stability number $\alpha(G(n, p))$ is asymptotically concentrated on at most two points, that is, there is a sequence $\hat{k}(n)$ such that a.a.s. $k(n) = \alpha(G(n, p)) \leq \hat{k}(n) + 1$.

In this section we will concentrate on the case when $p = p(n) \leq \log^{-2} n$. Then, in order to avoid dealing with logarithms of base $b$, instead of $\hat{k}_{\pm \varepsilon}$ it is convenient to use the functions $k_{\pm \varepsilon}$, defined as

$$
k_{\pm \varepsilon} = \left[2 \left( \log np - \log \log np + 1 - 2 \pm \varepsilon \right) \right].
$$

Elementary calculations show (Exercise!) that for $p \leq \log^{-2} n$, $\varepsilon > 0$, and $n$ large enough, we have $\hat{k}_{-\varepsilon} \leq k_{-\varepsilon} \leq \hat{k}_{-\varepsilon}$ and $\hat{k} \leq k \leq \hat{k}_{3\varepsilon}$, and so it does not matter very much whether we use $\hat{k}_{\pm \varepsilon}$ or $k_{\pm \varepsilon}$ to estimate $\alpha(G(n, p))$.

Let $X(k) = X(k; n, p)$ denote the number of stable sets of size $k$ in $G(n, p)$. Since $\alpha(G(n, p)) \geq k$ if and only if $X(k) > 0$, the most natural way of handling $\alpha(G(n, p))$ is to study the behavior of $X(k)$. First we will estimate the probability $P(X(k) > 0)$ for $k_{-\varepsilon} \leq k \leq k_\varepsilon$, using the second moment method. The following lemma shows that this approach works well for $p = p(n)$ which does not tend to 0 too fast.

**Lemma 7.3.** Let $\varepsilon > 0$, and $k_{\pm \varepsilon}$ be defined as in (7.2). Then there exists a constant $C_\varepsilon > 0$ such that for $C_\varepsilon/n \leq p = p(n) \leq \log^{-2} n$, we have

$$
P(X(k_\varepsilon) > 0) \leq \mathbb{E} X(k_\varepsilon) \to 0
$$

and

$$
\mathbb{E} X(k_{-\varepsilon}) \to \infty
$$

as $n \to \infty$. Furthermore, if $\log^2 n/\sqrt{n} \leq p \leq \log^{-2} n$, then

$$
P(X(k_{-\varepsilon}) > 0) = 1 - o(1)
$$

and if $C_\varepsilon/n \leq p \leq \log^2 n/\sqrt{n}$, then for large $n$.

$$
P(X(k_{-\varepsilon}) > 0) \geq \exp \left( -\frac{k_{-\varepsilon}}{\log^3 np} \right) \geq \exp \left( -\frac{2}{p \log^2 np} \right).
$$

(7.5)

In particular, if $\log^2 n/\sqrt{n} \leq p \leq \log^{-2} n$, then a.a.s.

$$
k_{-\varepsilon} \leq \alpha(G(n, p)) \leq k_\varepsilon.
$$

(7.6)

**Proof.** The first moment of $X(k_\varepsilon)$ is rather easy to handle. For instance, for $np$ large enough,

$$
\mathbb{E} X(k_\varepsilon) \leq \binom{n}{k_\varepsilon} (1-p)^{(k_\varepsilon)/2} \leq \binom{en}{k/2} \exp \left( -\frac{p(k_{-1})}{2} \right) \leq \exp \left( -\frac{p(k_{-1})}{2} \right) \to 0.
$$

We leave to the reader an elementary verification (Exercise!) that if $np \geq C_\varepsilon$, where $C_\varepsilon$ is a sufficiently large constant, then for large $n$

$$
\mathbb{E} X(k_{-\varepsilon}) \geq \exp(\varepsilon k_{-\varepsilon}/2) \to \infty,
$$

(7.7)

and concentrate on the proof of (7.4) and (7.5).

Let us set, for convenience, $k = k_{-\varepsilon}$ and $X = X(k)$, and assume that $C_\varepsilon/n \leq p \leq \log^{-2} n$ with $C_\varepsilon$ large enough. As we have already mentioned, our proof is based on a standard second moment argument, that is, we will estimate $\mathbb{E} X^2$ and then deduce (7.4) and (7.5) from (3.3). Note first that

$$
\frac{\mathbb{E} X^2}{(\mathbb{E} X)^2} - 1 = \frac{\binom{\binom{n}{k} - \binom{k}{k-1} (1-p)^{(k-1)}}{(1-p)^{(k-1)}}}{\binom{\binom{n}{k}}{(k-1)} (1-p)^{(k-1)}} - 1
$$

$$
\leq \sum_{i=1}^{k} \binom{k}{i} \frac{\binom{(n-k)}{i} (1-p)^{(i)}}{(k-1)} (1-p)^{(i)} = \sum_{i=1}^{k} a_i,
$$

where

$$
a_i = \frac{\binom{k}{i} \binom{(n-k)}{i} (1-p)^{(i)}}{(k-1)} (1-p)^{(i)}
$$

for $i = 1, 2, \ldots, k$.

Furthermore, let

$$
b_i = \frac{a_i}{a_i} = \frac{(k-i)^2}{(i+1)(n-2k+i+1)} (1-p)^{-i}.
$$

It is not hard to see that for small $i$, the sequence $b_i$ decreases with $i$ because of the factor $(1-p)^{-i}$ in the denominator, for intermediate $i$ it grows due to the factor $(1-p)^{-i}$ and, finally, when the difference $k-i$ becomes small, $b_i$ declines