ADDENDUM TO "THE MINIMAL SPANNING TREE IN A COMPLETE GRAPH AND A FUNCTIONAL LIMIT THEOREM FOR TREES IN A RANDOM GRAPH"

SVANTE JANSON AND JOHAN WÄSTLUND

In the article "The minimal spanning tree in a complete graph and a functional limit theorem for trees in a random graph" by Janson [6] it is shown that the minimal weight W_n of a spanning tree in a complete graph K_n with independent, uniformly distributed random weights on the edges has an asymptotic normal distribution. The same holds with exponentially distributed weights with mean 1. The mean converges, as shown in the classical paper by A. Frieze [3], to $\zeta(3)$, and the asymptotic variance is σ^2/n for a positive constant σ^2; more precisely, see [6, Theorem 1],

$$n^{1/2}\bigl(W_n - \zeta(3)\bigr) \xrightarrow{\mathrm{d}} N(0,\sigma^2)$$

as $n \to \infty$. The constant σ^2 was given in [6] by the complicated expression

$$\sigma^2 = \frac{\pi^4}{45} - 2\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{(i+k-1)!\,k^k(i+j)^{i-2}j}{i!\,k!\,(i+j+k)^{i+k+2}} \approx 1.6857. \tag{1}$$

This expression has now been evaluated by Wästlund [7], who found the simple result

$$\sigma^2 = 6\zeta(4) - 4\zeta(3). \tag{2}$$

For the proof, see [7]. The proof of Lemma 1 there may be simplified since (3) was shown by N. H. Abel [1]; this has also been noted by Piet Van Mieghem in a personal communication.

In principle, (2) lies within the scope of automated summation techniques. Let

$$a(i,j,k) = \frac{(i+k-1)!\,k^k(i+j)^{i-2}j}{i!\,k!\,(i+j+k)^{i+k+2}}.$$

If we introduce the new summation variable $n = i + j + k$ and write the triple sum as

$$\sum_{n=2}^{\infty}\sum_{j=1}^{n-1}\sum_{i=0}^{n-j-1} a(i,j,n-i-j)$$

(as in the proof in [7]), then provided that intermediate results are simplified, Maple (version 8) correctly returns $2\zeta(3) - \pi^4/45$.

The result (2) fits into an interesting pattern, where the values $\zeta(m)$ for small integer m appear in several similar results, without any obvious

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explanation. A well-known example is the random assignment problem, where D. Aldous [2] proved that the mean converges to $\zeta(2)$ and Wästlund [8] proved that the variance is asymptotically $\big(4\zeta(2) - 4\zeta(3)\big)/n$. Similarly, R. van der Hofstad, G. Hooghiemstra and P. Van Mieghem [4, 5] have studied the shortest path tree from a distinguished root in the complete graph with random exponential weights, finding an asymptotic normal distribution with the asymptotic mean $\zeta(2)$ and variance $4\zeta(3)/n$.

References

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Department of Mathematics, Uppsala University, PO Box 480, SE-751 06 Uppsala, Sweden

E-mail address: `svante.janson@math.uu.se`

URL: `http://www.math.uu.se/~svante/`

Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden

E-mail address: `jowas@mai.liu.se`

URL: `http://www.mai.liu.se/~jowas/`