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COMPLEX METHOD INTERPOLATION DEFINED USING A HALF-PLANE OR FULL DISC

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The standard definition of the complex interpolation space $[X_0, X_1]_\theta$ due to Calderón [**2**] uses (X_0+X_1)-valued analytic functions in the strip $\{z : 0 < \operatorname{Re} z < 1\}$, see below for details. Is it possible to use only functions that are analytic in the half-plane $\{z : 0 < \operatorname{Re} z\}$?

There is a variant of this question which arises since, as shown by Cwikel [**3**], the space $[X_0, X_1]_\theta$ may also be defined using analytic functions in the annulus $\{z : R_1 < |z| < R_0\}$. Is it possible to use only functions that are analytic in the entire disc $\{z : |z| < R_0\}$?

The main purpose of this paper is to show that, in general, these questions have a negative answer, even if we suppose $X_1 \subseteq X_0$. This is done by an explicit counter example.

Section 1 contains some definitions and an equivalence theorem showing that the two questions above are equivalent.

The counter example is given in Section 2. The reason for considering this particular example is given in Section 3. Section 4 contains some additional results. Some of these concern special cases where the answer to the above question is positive. One such case is when (X_0, X_1) is a couple of Banach lattices with $X_1 \subseteq X_0$. Section 5 is an appendix written by Michael Cwikel which presents another case where the answer is positive.

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1. Preliminaries

We introduce some notation. If X and Y are Banach spaces, then $X = Y$ means that the spaces contain the same elements and that the norms are equivalent (but

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not necessarily equal); similarly, $X \subseteq Y$ means that the inclusion is continuous (but not necessarily isometric). C will be used to denote unspecified constants (possibly depending on some parameters that are kept fixed); the meaning may change from one occurrence to the next.

Suppose that (X_0, X_1) is a Banach couple. We define $\mathcal{F}_S = \mathcal{F}_S(X_0, X_1)$ as the space of all functions F from the closed strip $\overline{S} = \{0 \leq \operatorname{Re} z \leq 1\}$ into $X_0 + X_1$ that are bounded and continuous on $\overline{S}$ and analytic on the interior S, and such that $t \mapsto F(j + it)$ is a bounded and continuous map of the real line into X_j for $j = 0, 1$. We let

$$\|F\|_{\mathcal{F}_S} = \sup_{j=0,1} \sup_{-\infty<t<\infty} \|F(j+it)\|_{X_j}.$$

The complex interpolation space $[X_0, X_1]_\theta$ is defined for $0 < \theta < 1$ by

$$[X_0, X_1]_\theta = \{F(\theta) : F \in \mathcal{F}_S\},$$

equipped with the natural quotient norm $\|x\| = \inf\{\|F\|_{\mathcal{F}_S} : F(\theta) = x\}$, cf. Calderón [**2**] and Bergh and Löfström [**1**].

Remark. It is customary to impose also the condition that $F(j + it) \to 0$ in X_j as $t \to \infty$, which is convenient for some purposes but not for ours; it is easily seen that this yields the same interpolation space. (On the other hand, the continuity condition on the boundary is essential, see [**4**].)

For the half-plane version, we let $\mathcal{F}_H = \mathcal{F}_H(X_0, X_1)$ be the space of bounded continuous functions on the closed half-plane $\overline{H} = \{z : \operatorname{Re} z \geq 0\}$ that are analytic on the open half-plane $H = \{z : \operatorname{Re} z > 0\}$ and such that the restriction to $\overline{S}$ belongs to $\mathcal{F}_S$. We regard $\mathcal{F}_H$ as a subspace of $\mathcal{F}_S$; it is easily seen that $\mathcal{F}_H$ is a closed subspace of $\mathcal{F}_S$, and thus a Banach space. We define, again for $0 < \theta < 1$,

$$C_\theta^+(X_0, X_1) = \{F(\theta) : F \in \mathcal{F}_H(X_0, X_1)\};$$

this is a Banach space with the natural quotient norm. It should be clear that C_θ^+ is an interpolation method.

Moreover, it follows easily, e.g. by applying linear functionals in $(X_0+X_1)^*$, that if $F \in \mathcal{F}_H(X_0, X_1)$ and $z \in H$, then $F(z)$ is given by the integral $\int_{-\infty}^{\infty} F(it)P_z(t)dt$, where $P_z(t)$ denotes the appropriate Poisson kernel. Since $t \to F(it)$ by assumption is a bounded continuous map into X_0, this integral converges in X_0, and $z \mapsto F(z)$ is a bounded continuous map of $\overline{H}$ into X_0. Hence

$$\mathcal{F}_H(X_0, X_1) = \mathcal{F}_H(X_0, X_0 \cap X_1)$$

and thus

$$C_\theta^+(X_0, X_1) = C_\theta^+(X_0, X_0 \cap X_1). \tag{1.1}$$

It is obvious that

$$C_\theta^+(X_0, X_1) \subseteq [X_0, X_1]_\theta \tag{1.2}$$

and (1.1) thus implies

$$C_\theta^+(X_0, X_1) \subseteq [X_0, X_0 \cap X_1]_\theta. \tag{1.3}$$

The first question given in the introduction is whether equality holds in (1.2). We see from (1.1) that it is natural to consider the case $X_1 \subseteq X_0$ only; this is equivalent to asking whether equality always holds in (1.3). A counter example is given in the next section, but first we introduce the annulus and disc versions of the definitions above.

Let R_0 and R_1 be two fixed real numbers with $0 < R_1 < R_0$, and define $R_\theta = R_0^{1-\theta} R_1^\theta$, $0 < \theta < 1$. We consider the annulus $A = \{z : R_1 < |z| < R_0\}$ and the disc $D = \{z : |z| < R_0\}$.

We define $\mathcal{F}_A = \mathcal{F}_A(X_0, X_1)$ as the space of all bounded, continuous functions F from $\overline{A}$ into $X_0 + X_1$ that are analytic on A, and such that $t \mapsto F(R_j e^{it})$ is a continuous map of the real line into X_j for $j = 0, 1$; we let

$$\|F\|_{\mathcal{F}_A} = \sup_{j=0,1} \sup_{|z|=R_j} \|F(z)\|_{X_j}.$$

We also define $\mathcal{F}_D = \mathcal{F}_D(X_0, X_1)$ as the space of all bounded continuous functions from $\overline{D}$ into $X_0 + X_1$ that are analytic on D and such that the restriction to $\overline{A}$ belongs to $\mathcal{F}_A(X_0, X_1)$. We regard $\mathcal{F}_D$ as a subspace of $\mathcal{F}_A$, and we use the subspace norm; $\mathcal{F}_D$ is a closed subspace and thus a Banach space. It is easily seen, as for $\mathcal{F}_H$ above, that $\mathcal{F}_D(X_0, X_1) = \mathcal{F}_D(X_0, X_0 \cap X_1)$.

Cwikel [**3**] showed that the complex method may be defined using analytic functions in an annulus; more precisely,

$$[X_0, X_1]_\theta = \{F(R_\theta) : F \in \mathcal{F}_A(X_0, X_1)\}.$$

Cwikel's method also shows the corresponding result for the half-plane and disc.

PROPOSITION 1. *For every Banach couple* (X_0, X_1),

$$C_\theta^+(X_0, X_1) = \{F(R_\theta) : F \in \mathcal{F}_D(X_0, X_1)\}.$$

PROOF. Let $\gamma = \ln(R_0/R_1) > 0$. First, if $F \in \mathcal{F}_D$, then $G(z) = F(R_0 e^{-\gamma z}) \in \mathcal{F}_H$, and thus $F(R_\theta) = G(\theta) \in C_\theta^+(X_0, X_1)$.

Conversely, if $F \in \mathcal{F}_H$, let $F_1(z) = \left(\frac{e^{-\gamma(z-\theta)} - 1}{\gamma(z-\theta)}\right)^2 F(z)$. Then $F_1 \in \mathcal{F}_H$, with $F_1(\theta) = F(\theta)$, and $\|F_1(z)\|_{X_0+X_1} \leq C/(1+|z|^2)$. Hence the sum

$$F_2(z) = \sum_{k=-\infty}^{\infty} F_1(z + 2\pi i k/\gamma)$$

converges for all $z \in \overline{H}$; moreover, $F_2 \in \mathcal{F}_H$ and $F_2(\theta) = F_1(\theta) = F(\theta)$, because $F_1(\theta + 2\pi i k/\gamma) = 0$ when $k \neq 0$, and $\|F_2(z)\|_{X_0+X_1} \leq C/(1 + \operatorname{Re} z)$. Since F_2 is periodic with period $2\pi i/\gamma$, we may define

$$G(z) = F_2\Big(\frac{\ln(R_0/z)}{\gamma}\Big), \qquad 0 < |z| \le R_0,$$

regardless of the branch of the logarithm. Then G is an analytic (X_0+X_1)-valued function in the punctured disc $D \setminus \{0\}$, and since $\|G(z)\|_{X_0+X_1} \to 0$ as $z \to 0$, the origin is a removable singularity and if we define $G(0) = 0$, G becomes analytic in D. It is easily seen that $G \in \mathcal{F}_D$ and $G(R_\theta) = F_2(\theta) = F(\theta)$, which completes the proof. ☐

Consequently, the two questions in the introduction are equivalent.

2. A counter example

Let R_0, R_1 and θ be given with $0 < R_1 < R_0$ and $0 < \theta < 1$, and let $\alpha_0 = \ln R_0$, $\alpha = \ln R_1$. Thus $R_\theta = R_0^{1-\theta} R_1^\theta = e^{(1-\theta)\alpha_0 + \theta\alpha_1}$. We may without loss of generality assume $R_\theta = 1$, and thus $\alpha_0 > 0$, $\alpha_1 < 0$ and

$$(1-\theta)\alpha_0 + \theta\alpha_1 = 0. \tag{2.1}$$

Let FL denote the space of Fourier coefficients of functions in $L^1(\mathbf{T})$:

$$FL = \{(\widehat{f}(n))_{n=-\infty}^{\infty} : f \in L^1(\mathbf{T})\};$$

this is a Banach space with the norm $\|(\widehat{f}(n))_{-\infty}^{\infty}\|_{FL} = \|f\|_{L^1}$.

Define also corresponding weighted spaces by

$$FL_\alpha = \{(x_n)_{-\infty}^{\infty} : (e^{-n\alpha}x_n)_{-\infty}^{\infty} \in FL\},$$

where α is a real number, with $\|(x_n)_{-\infty}^{\infty}\|_{FL_\alpha} = \|(e^{-n\alpha}x_n)_{-\infty}^{\infty}\|_{FL}$. Note that if $(x_n)_{-\infty}^{\infty} \in FL_\alpha$, then

$$|x_n| \le e^{n\alpha}\|(x_k)_{-\infty}^{\infty}\|_{FL_\alpha}. \tag{2.2}$$

It is known that $[FL_{\alpha_0}, FL_{\alpha_1}]_\theta = FL_{(1-\theta)\alpha_0+\theta\alpha_1} = FL$, see e.g. [**7**].

Let $X_0 = FL_{\alpha_0} + FL_{\alpha_1}$, and $X_1 = FL_{\alpha_1}$; we claim that, cf. (1.2),

$$C_\theta^+(X_0, X_1) \subsetneq [X_0, X_1]_\theta. \tag{2.3}$$

In order to see this, we first observe that

$$[X_0, X_1]_\theta \supseteq [FL_{\alpha_0}, FL_{\alpha_1}]_\theta = FL.$$

On the other hand, we claim that if $(x_n)_{-\infty}^{\infty} \in C_\theta^+(X_0, X_1)$ and $q > 1/\theta$, then

$$\sum_1^\infty |x_n|^q/n < \infty. \tag{2.4}$$

Consequently, in order to verify (2.3) it suffices to show that there is a sequence $(x_n)_{-\infty}^{\infty} \in FL$ such that (2.4) fails. This can be done by explicit examples, such as $1/\ln\ln(3+|n|)$, cf. Zygmund [**14**], Theorem V.1.5, or by observing that otherwise the closed graph theorem would imply $\sum_1^\infty |\widehat{f}(n)|^q/n \le C$ for some $C < \infty$ and all $f \in L^1(\mathbf{T})$ with $\|f\|_{L^1} \le 1$, but that is false as the sequence of Fejér kernels shows.

We turn to the proof of (2.4). We use the characterization with $\mathcal{F}_D$ in Proposition 1.

Suppose that $F \in \mathcal{F}_D(FL_{\alpha_0} + FL_{\alpha_1}, FL_{\alpha_1})$ with $\|F\| \le 1$. We write $F(z) = (f_n(z))_{-\infty}^{\infty}$, thus each f_n is analytic in D and continuous on $\overline{D}$; we further expand each f_n as a Taylor series

$$f_n(z) = \sum_{k=0}^{\infty} a_{nk} z^k,$$

and set $a_{nk} = 0$ for $k < 0$.

If $|z| = R_0$, then $F(z) \in FL_{\alpha_0} + FL_{\alpha_1}$, with norm ≤ 1 and thus, cf. (2.2),

$$|f_n(z)| \le \max(e^{n\alpha_0}, e^{n\alpha_1}) = \begin{cases} e^{n\alpha_0}, & n \ge 0, \\ e^{n\alpha_1}, & n < 0. \end{cases}$$

Consequently, for any k,

$$|a_{nk} R_0^k| \le \sup_{|z|=R_0} |f_n(z)| \le e^{n\alpha_0}, \qquad n \ge 0$$

or, recalling that $R_0 = e^{\alpha_0}$,

$$|a_{n,n+m}| \le R_0^{-n-m} e^{n\alpha_0} = e^{-m\alpha_0}, \qquad n \ge 0,\ -\infty < m < \infty. \tag{2.5}$$

We now turn to z with $|z| = R_1$. By assumption, $F(R_1 e^{it}) = (f_n(R_1 e^{it}))_{-\infty}^{\infty}$ belongs to the unit ball of FL_{α_1} for every real t. Hence there exist $g_t \in L^1(\mathbf{T})$, with $\|g_t\|_{L^1} \le 1$, such that

$$\widehat{g}_t(n) = e^{-\alpha_1 n} f_n(R_1 e^{it}) = \sum_{k=0}^{\infty} e^{-\alpha_1 n} a_{nk} R_1^k e^{ikt} = \sum_{k=0}^{\infty} a_{nk} e^{-\alpha_1(n-k)} e^{ikt}. \tag{2.6}$$

Moreover, the mapping $t \mapsto g_t$ is a continuous map $\mathbf{R} \to L^1(\mathbf{T})$.

Let τ_s be the translation operator on $L^1(\mathbf{T})$ given by $\tau_s g(t) = g(t-s)$ and thus

$$(\tau_s g)^\wedge(n) = e^{-ins}\widehat{g}(n) \tag{2.7}$$

Since, for any t and ε,

$$\begin{aligned} \|\tau_{t+\varepsilon} g_{t+\varepsilon} - \tau_t g_t\|_{L^1} &\le \|\tau_{t+\varepsilon}(g_{t+\varepsilon} - g_t)\|_{L^1} + \|\tau_{t+\varepsilon} g_t - \tau_t g_t\|_{L^1} \\ &= \|g_{t+\varepsilon} - g_t\|_{L^1} + \|\tau_\varepsilon g_t - g_t\|_{L^1}, \end{aligned}$$

it follows that $t \mapsto \tau_t g_t$ is a continuous map into $L^1(\mathbf{T})$. Thus we may define, for $m \in \mathbf{Z}$, the Bochner integral

$$h_m = \tfrac{1}{2\pi} \int_0^{2\pi} e^{-imt} \tau_t g_t \, dt \in L^1(\mathbf{T}),$$

with $\|h_m\|_{L^1} \le 1$. Then, by (2.7) and (2.6),

$$\begin{aligned}\widehat{h}_m(n) &= \tfrac{1}{2\pi}\int_0^{2\pi} e^{-imt}e^{-int}\widehat{g}_t(n)\,dt = \tfrac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{\infty} a_{nk}e^{-\alpha_1(n-k)}e^{i(k-m-n)t}\,dt\\ &= a_{n,n+m}e^{\alpha_1 m}, \qquad n,m\in\mathbf{Z}.\end{aligned}$$

In particular, $\widehat{h}_m(n)=0$ if $n+m<0$. We also see that

$$|a_{n,n+m}| = e^{-\alpha_1 m}|\widehat{h}_m(n)| \le e^{-\alpha_1 m}\|h_m\|_{L^1} \le e^{-\alpha_1 m}. \tag{2.8}$$

Let v_m be the sequence $(a_{n,n+m})_{n=1}^{\infty}$, and let $\ell^q(1/n)$ be the sequence space $\{(x_n)_1^\infty : \sum_1^\infty |x_n|^q/n < \infty\}$.

Consider first the case $m \le 0$. Then $\widehat{h}_m(n) = 0$ when $n<0$, so h_m belongs to the analytic Hardy space $H^1 \subseteq L^1$. By Hardy's inequality, see Zygmund [**14**], Theorem VII.8.7,

$$\sum_1^\infty \frac{1}{n}|a_{n,n+m}| = e^{-\alpha_1 m}\sum_1^\infty \frac{1}{n}|\widehat{h}_m(n)| \le Ce^{-\alpha_1 m}\|h_m\|_{L^1} \le Ce^{-\alpha_1 m}$$

and thus, using (2.8),

$$\sum_1^\infty \frac{1}{n}|a_{n,n+m}|^q \le e^{-(q-1)\alpha_1 m}\sum_1^\infty \frac{1}{n}|a_{n,n+m}| \le Ce^{-q\alpha_1 m},$$

or in the notation just introduced,

$$\|v_m\|_{\ell^q(1/n)} \le Ce^{-\alpha_1 m}, \qquad m \le 0. \tag{2.9}$$

Consider next the case $m>0$. Then $\widehat{h}_m(n)=0$ when $n<-m$, and the shifted sequence $(a_{k-m,k})_{k=-\infty}^{\infty}$ is the Fourier transform of the function $e^{-\alpha_1 m}e^{imt}h_m(t)$ in H^1. Hence Hardy's inequality yields

$$\begin{aligned}\sum_1^\infty \frac{1}{n}|a_{n,n+m}| &\le (m+1)\sum_1^\infty \frac{1}{n+m}|a_{n,n+m}| \le (m+1)\sum_1^\infty \frac{1}{k}|a_{k-m,k}|\\ &\le (m+1)C\|e^{-\alpha_1 m}e^{imt}h_m(t)\|_{L^1} \le C(m+1)e^{-\alpha_1 m}.\end{aligned}$$

We combine this with (2.5) and obtain

$$\sum_1^\infty \frac{1}{n}|a_{n,n+m}|^q \le e^{-(q-1)\alpha_0 m}\sum_1^\infty \frac{1}{n}|a_{n,n+m}| \le C(m+1)e^{-(q-1)\alpha_0 m-\alpha_1 m}. \tag{2.10}$$

From $(1-\theta)\alpha_0+\theta\alpha_1 = 0$ we obtain $q\alpha_0 = q\theta(\alpha_0-\alpha_1)$ and $(q-1)\alpha_0+\alpha_1 = (q\theta-1)(\alpha_0-\alpha_1)$. Hence (2.10) yields

$$\|v_m\|_{\ell^q(1/n)} \le Cm^{1/q}e^{-(\theta-1/q)(\alpha_0-\alpha_1)}, \qquad m>0. \tag{2.11}$$

Since $\alpha_1 < 0$ and $(\theta - 1/q)(\alpha_0 - \alpha_1) > 0$, (2.9) and (2.11) together imply

$$\sum_{m=-\infty}^{\infty} \|v_m\|_{\ell^q(1/n)} < \infty,$$

and thus $\sum_{-\infty}^{\infty} v_m \in \ell^q(1/n)$. But

$$\sum_{-\infty}^{\infty} v_m = \Big(\sum_{m=-\infty}^{\infty} a_{n,n+m}\Big)_{n=1}^{\infty} = \Big(\sum_{k=0}^{\infty} a_{n,k}\Big)_{n=1}^{\infty} = (f_n(1))_{n=1}^{\infty}.$$

This proves (2.4) for the sequence $(x_n)_{-\infty}^{\infty} = (f_n(1))_{-\infty}^{\infty} = F(1) = F(R_\theta)$, and the proof is complete.

3. Background

The example in Section 2 solves our problem, but how was it found? The couple $(FL_{\alpha_0}+FL_{\alpha_1}, FL_{\alpha_1})$ may look rather unnatural at the first sight, but the following argument shows that this is in fact the canonical (counter) example.

If $\widetilde{A}$, $\widetilde{X}$ and A are Banach spaces such that $A \subseteq \widetilde{A}$, and $J \subseteq B(\widetilde{A}, \widetilde{X})$ is a Banach space of bounded linear operators $\widetilde{A} \to \widetilde{X}$, then the orbit of A under J is the subspace of $\widetilde{X}$ given by $\{\sum_1^\infty T_i a_i : \sum_i^\infty \|T_i\|_J \|a_i\|_A < \infty\}$ (with the sum $\sum T_i a_i$ converging in $\widetilde{X}$). This is a Banach space with $\|x\| = \inf \sum_i \|T_i\|_J \|a_i\|_A$ over all such representations of x.

A particularly important example is when (A_0, A_1) and (X_0, X_1) are two Banach couples, $\widetilde{A} = A_0 + A_1$, $\widetilde{X} = X_0 + X_1$ and $J = \{T \in B(\widetilde{A}, \widetilde{X}) \mid T\colon (A_0, A_1) \to (X_0, X_1)\}$ is the space of all linear operators from $A_0 + A_1$ to $X_0 + X_1$ that map A_0 into X_0 and A_1 into X_1. We then denote the orbit of a space $A \subseteq A_0 + A_1$ by $G(A_0, A_1, A; X_0, X_1)$. For fixed A_0, A_1 and $A \neq \{0\}$, this is an interpolation method, and it is easily seen that this is the minimal interpolation method that satisfies $F(A_0, A_1) \supseteq A$.

Now consider the action of this interpolation method on the couple $(X_0, X_0 \cap X_1)$. Since

$$\begin{aligned} T\colon (A_0, A_1) \to (X_0, X_0 \cap X_1) &\iff T\colon A_0 \to X_0,\ T\colon A_1 \to X_0,\ T\colon A_1 \to X_1, \\ &\iff T\colon (A_0 + A_1, A_1) \to (X_0, X_1), \end{aligned}$$

we obtain the following identity.

PROPOSITION 2. *Let (A_0, A_1) and (X_0, X_1) be Banach couples, and $A \subseteq A_0 + A_1$. Then*

$$G(A_0, A_1, A; X_0, X_0 \cap X_1) = G(A_0 + A_1, A_1, A; X_0, X_1).$$

PROPOSITION 3. *Let (A_0, A_1) be a Banach couple and $A \subseteq A_0 + A_1$. If F is an interpolation method, then the following are equivalent.*

(i) *$F(X_0, X_1) \supseteq G(A_0, A_1, A; X_0, X_1)$ for all Banach couples (X_0, X_1) such that $X_0 \supseteq X_1$.*

(ii) $F(X_0, X_0 \cap X_1) \supseteq G(A_0, A_1, A; X_0, X_0 \cap X_1)$ *for all Banach couples* (X_0, X_1).
(iii) $F(A_0 + A_1, A_1) \supseteq A$.

Proof. The equivalence of (i) and (ii) is clear.

By the comments before Proposition 2, (iii) holds if and only if

$$G(A_0 + A_1, A_1, A; X_0, X_1) \subseteq F(X_0, X_1)$$

for every couple (X_0, X_1), which by Proposition 2 easily is seen to be equivalent to (ii). $\square$

The complex method of interpolation $[X_0, X_1]_\theta$ can be characterized as the orbit $G(FL_{\alpha_0}, FL_{\alpha_1}, FL_{(1-\theta)\alpha_0+\theta\alpha_1}; X_0, X_1)$ for any real α_0 and α_1 with $\alpha_0 \neq \alpha_1$, see [**7**]. We thus have the following corollaries.

Corollary 1. *Let* $\alpha_0 \neq \alpha_1$ *and* $0 < \theta < 1$. *Then, for any Banach couple* (X_0, X_1),

$$[X_0, X_0 \cap X_1]_\theta = G(FL_{\alpha_0} + FL_{\alpha_1}, FL_{\alpha_1}, FL_{(1-\theta)\alpha_0+\theta\alpha_1}; X_0, X_1).$$

Corollary 2. *Let* $\alpha_0 \neq \alpha_1$ *and* $0 < \theta < 1$ *and let* F *be an interpolation method. Then* $F(X_0, X_1) \supseteq [X_0, X_1]_\theta$ *for all Banach couples* (X_0, X_1) *with* $X_0 \supseteq X_1$, *if and only if*

$$F(FL_{\alpha_0} + FL_{\alpha_1}, FL_{\alpha_1}) \supseteq FL_{(1-\theta)\alpha_0+\theta\alpha_1}.$$

Corollary 2 thus shows that if $C_\theta^+(X_0, X_1) \supseteq [X_0, X_1]_\theta$ fails for any couple (X_0, X_1), then it fails for $(FL_{\alpha_0} + FL_{\alpha_1}, FL_{\alpha_1})$.

4. Further comments

The interpolation method C_θ^+ is probably not of much practical use, but let us nevertheless give a couple of results for it. First, as another example of the propositions in Section 3, consider the $\pm$-method defined by Peetre [**12**]. It was shown in [**7**] that this method, there and here denoted by G_1, can be characterized as $G(c_{0,\alpha_0}, c_{0,\alpha_1}, c_{0,(1-\theta)\alpha_0+\alpha_1}; X_0, X_1)$, where $c_{0,\alpha}$ is a weighted version of $c_0 = \{(x_n)_{-\infty}^{\infty} : x_n \to 0 \text{ as } |n| \to \infty\}$.

It is easily verified (we omit this) that $C_\theta^+(c_{0,\alpha_0} + c_{0,\alpha_1}, c_{0,\alpha_1}) \supseteq c_{0,(1-\theta)\alpha_0+\theta\alpha_1}$. Proposition 3 thus implies, together with (1.2), the following.

Proposition 4. *For any Banach couple* (X_0, X_1) *such that* $X_0 \supseteq X_1$,

$$G_1(X_0, X_1) \subseteq C_\theta^+(X_0, X_1) \subseteq [X_0, X_1]_\theta.$$

In particular, this shows that if (X_0, X_1) is a Banach couple such that $X_0 \supseteq X_1$ and

$$G_1(X_0, X_1) = [X_0, X_1]_\theta, \tag{4.1}$$

then $[X_0, X_1]_\theta$ may be defined using X_0-valued functions F that are analytic in a half-plane or disc, as defined in detail in Section 1.

REMARK. One important case when (4.1) holds, pointed out to me by Michael Cwikel, is for a couple of lattices on the same measure space (Ω, Σ, μ) i.e. when $X_j = Y_j(\mathbf{C})$ is the complexification of a Banach lattice Y_j of real valued measurable functions on Ω for $j = 0, 1$. This fact has been observed by a number of authors. (Sometimes they impose additional conditions.) For the reader's convenience let us list some results which can be combined to immediately prove it:
(i) the continous inclusion $[X_0, X_1]_\theta \subset Y_0^{1-\theta} Y_1^\theta(\mathbf{C})$ (see [**2**] section 13.6 (i) p. 125),
(ii) the continuous inclusion $Y_0^{1-\theta} Y_1^\theta(\mathbf{C}) \subset G_2(X_0, X_1)$ (see [**11**] Lemma 8.2.1 p. 453.)

The rest of these "ingredients" also hold for arbitrary Banach couples (X_0, X_1).
(iii) the density of $X_0 \cap X_1$ in $[X_0, X_1]_\theta$ ([**2**] p. 116) and the fact that the closure of $X_0 \cap X_1$ in $G_2(X_0, X_1)$ is $G_1(X_0, X_1)$. ([**7**] Theorem 8 p. 60.)
(iv) the continuous inclusion $G_1(X_0, X_1) \subset [X_0, X_1]_\theta$. (See [**12**] p. 176 or [**7**] p. 67).

The interpolation method C_θ^+ can also be represented as an orbit method. Define $P_+(x_n)_{-\infty}^\infty = (x_n)_0^\infty$, the restriction of a doubly infinite sequence to non-negative indices, and let $FL_\alpha^+ = P_+(FL_\alpha)$, equipped with the quotient norm.

PROPOSITION 5. *Let $\alpha_0 > \alpha_1$. Then for any Banach couple (X_0, X_1),*

$$C_\theta^+(X_0, X_1) = G(FL_{\alpha_0}^+, FL_{\alpha_1}^+, FL_{(1-\theta)\alpha_0+\theta\alpha_1}^+; X_0, X_1).$$

PROOF. We may assume $(1-\theta)\alpha_0 + \theta\alpha_1 = 0$. We use Proposition 1, choosing $R_0 = e^{\alpha_0}$ and $R_1 = e^{\alpha_1}$, and thus $R_\theta = 1$. First, suppose that $x = (x_n)_{-\infty}^\infty$ is a finite sequence of complex numbers, i.e. a sequence with all but finitely many elements 0. Then $F(z) = (x_n z^n)_0^\infty$ defines an entire analytic function into $FL_{\alpha_0}^+ \cap FL_{\alpha_1}^+$ and if $z = R_j e^{it}$, $j = 0$ or 1, then

$$\|F(z)\|_{FL_{\alpha_j}^+} = \|(x_n e^{int})_0^\infty\|_{FL^+} \leq \|(x_n e^{int})_{-\infty}^\infty\|_{FL} = \|(x_n)_{-\infty}^\infty\|_{FL}.$$

Hence $F(z) \in \mathcal{F}_D(FL_{\alpha_0}^+, FL_{\alpha_1}^+)$ with norm $\leq \|x\|_{FL}$, and thus $P_+x = F(1) \in C_\theta^+(FL_{\alpha_0}^+, FL_{\alpha_1}^+)$ with norm $\leq \|x\|_{FL}$. The set of all finite sequences is dense in FL, so by continuity

$$FL^+ = P_+(FL) \subseteq C_\theta^+(FL_{\alpha_0}^+, FL_{\alpha_1}^+),$$

which implies, by the minimality of the interpolation functor G,

$$G(FL_{\alpha_0}^+, FL_{\alpha_1}^+, FL^+; X_0, X_1) \subseteq C_\theta^+(X_0, X_1)$$

for every Banach couple (X_0, X_1).

In order to prove the converse, suppose that $F \in \mathcal{F}_D(X_0, X_1)$, and expand F as a Taylor series $F(z) = \sum_0^\infty x_k z^k$, with $x_k \in X_0 \cap X_1$. If $(a_n)_{-\infty}^\infty \in FL_{\alpha_0}$, then

$a_n = e^{n\alpha_0}\widehat{f}(n)$ for some $f \in L^1(\mathbf{T})$ and (with X_0-valued integrals)

$$\begin{aligned}\tfrac{1}{2\pi}\int_0^{2\pi} f(e^{-it})F(R_0e^{it})\,dt &= \lim_{r\nearrow 1}\tfrac{1}{2\pi}\int_0^{2\pi} f(e^{-it})F(rR_0e^{it})\,dt = \lim_{r\nearrow 1}\sum_{n=0}^{\infty}\widehat{f}(n)r^nR_0^nx_n \\ &= \lim_{r\nearrow 1}\sum_{n=0}^{\infty} r^n a_n x_n. \end{aligned} \tag{4.2}$$

Hence the mapping $T : (a_n)_{-\infty}^{\infty} \to \lim_{r\nearrow 1}\sum_{n=0}^{\infty} r^n a_n x_n$ is a well-defined linear map $FL_{\alpha_0} \to X_0$, and

$$\|T((a_n)_{-\infty}^{\infty})\|_{X_0} \le \|f\|_{L^1}\sup_t \|F(R_0e^{it})\|_{X_0} \le \|(a_n)_{-\infty}^{\infty}\|_{FL_{\alpha_0}}\|F\|_{\mathcal{F}_D}.$$

Since Ta obviously depends on P_+a only, we can also regard T as a bounded linear map $FL^+_{\alpha_0} \to X_0$.

Similarly if $(a_n)_0^{\infty} \in FL^+_{\alpha_1}$, then $a_n = e^{n\alpha_1}\widehat{f_1}(n)$, $n \ge 0$, for some $f_1 \in L^1(\mathbf{T})$ and, by the same argument as in (4.2),

$$T((a_n)_0^{\infty}) = \tfrac{1}{2\pi}\int_0^{2\pi} f_1(e^{-it})F(R_1e^{it})\,dt,$$

with the integral convergent in X_1. It follows that $T\colon FL^+_{\alpha_1} \to X_1$. Thus

$$T\colon (FL^+_{\alpha_0}, FL^+_{\alpha_1}) \to (X_0, X_1),$$

and we see also that $\|T\| \le \|F\|_{\mathcal{F}_D}$. Furthermore, if $f \in L^1(\mathbf{T})$, then by the same argument again,

$$\tfrac{1}{2\pi}\int_0^{2\pi} f(e^{-it})F(e^{it})\,dt = T((\widehat{f}(n))_{-\infty}^{\infty}).$$

Thus, if we let $f * F$ denote the X_0-valued function

$$f * F(z) = \tfrac{1}{2\pi}\int_0^{2\pi} f(e^{-it})F(ze^{it})\,dt,$$

and $Y = G(FL^+_{\alpha_0}, FL^+_{\alpha_1}, FL^+; X_0, X_1)$, then by the definition of the latter space,

$$f * F(1) = T(\widehat{f}) \in Y \quad \text{with} \quad \|f * F(1)\|_Y \le \|f\|_{L^1}\|F\|_{\mathcal{F}_D}. \tag{4.3}$$

It is easily seen that $f * F \in \mathcal{F}_D(X_0, X_1)$ for all $f \in L^1(\mathbf{T})$, and that, if K_n denotes the n:th Fejér kernel, $K_n * F \to F$ in $\mathcal{F}_D(X_0, X_1)$ as $n \to \infty$; moreover, $(f * g) * F = f * (g * F)$ for all $f, g \in L^1(\mathbf{T})$. Define $y_n = K_n * K_n * F(1)$. By (4.3), $y_n \in Y$. Furthermore,

$$y_n - y_m = (K_n + K_m) * (K_n - K_m) * F(1)$$

and thus, by (4.3) for $(K_n - K_m) * F$,

$$\|y_n - y_m\|_Y \le \|K_n + K_m\|_{L^1}\|(K_n - K_m) * F\|_{\mathcal{F}_D} \le 2\|K_n * F - K_m * F\|_{\mathcal{F}_D} \to 0,$$

as $n, m \to \infty$. Hence $\{y_n\}$ is a Cauchy sequence in Y, so $y_n \to y \in Y$. But $K_n * K_n * F \to F$ in $\mathcal{F}_D(X_0, X_1)$, and thus $y_n \to F(1)$ in $C_\theta^+(X_0, X_1)$. Consequently $F(1) = y \in Y$, which completes the proof. □

REMARK. It is also possible to define interpolation methods using *harmonic* functions in a half-plane or disc, see Janson and Peetre [**9**]. The resulting interpolation spaces contain the ones given by analytic functions, and are in general larger.

For harmonic interpolation, we do not know if the half-plane and and disc versions always yield the same interpolation spaces, or even if the disc interpolation spaces are independent of the ratio R_0/R_1 as in the analytic case, cf. Proposition 1.

For the disc version, with fixed $R_0 = e^{\alpha_0}$ and $R_1 = e^{\alpha_1}$, the following analogues of the results above hold. (We omit the proofs.)

First, a somewhat more complicated version of the argument in Section 2 shows that, at least for the disc version, the harmonic interpolation space for the couple $(FL_{\alpha_0} + FL_{\alpha_1}, FL_{\alpha_1})$ does not contain the standard complex method space.

Furthermore, the method has an orbit description as $G(A_{\alpha_0}, A_{\alpha_1}, A_{\alpha_\theta}; X_0, X_1)$, where $A_\alpha = \{(a_n)_{-\infty}^{\infty} : (e^{-\alpha|n|} a_n)_{-\infty}^{\infty} \in FL\}$.

An argument similar to the one in Section 2 shows that for the couple $(A_{\alpha_0}, A_{\alpha_1})$, the standard complex method space does not contain the harmonic interpolation space. Consequently, even assuming $X_0 \supset X_1$, the harmonic method and the standard complex method are not comparable.

Finally, the harmonic method space is included in the one given by Ovchinnikov's method ϕ_u [**10**], denoted by H_1 in [**7**]. It follows that for 'tame' couples, the harmonic and analytic disc methods coincide with the standard complex method.

5. Appendix:

ANOTHER CASE WHERE $C_\theta^+(X_0, X_1) = [X_0, X_1]_\theta$.

by Michael Cwikel

Here we consider couples (X_0, X_1) which can be obtained by "one sided reiteration".

THEOREM 5.1. *Let (X_0, X_1) be a Banach couple satisfying $X_1 \subseteq X_0$. Suppose that there exists another Banach space B such that (X_0, B) forms a Banach couple and such that*

$$X_1 = [X_0, B]_\beta \text{ for some } \beta \in (0, 1).$$

Then

$$C_\theta^+(X_0, X_1) = [X_0, X_1]_\theta \text{ for all } \theta \in (0, 1).$$

PROOF. Let (Y_0, Y_1) be the Banach couple obtained by setting $Y_0 = X_1$ and $Y_1 =$

X_0, and let $\alpha = 1 - \beta$. I.e. we have $Y_0 \subseteq Y_1$ and

$$Y_0 = [B, Y_1]_\alpha \tag{5.2}$$

(We introduce Y_0 and Y_1 and α because this is the easiest way of adapting our original version of this proof, which was written without access to a definitive version of the previous sections of this paper, to the format of the notation used in those sections.)

The proof of this theorem amounts to showing that $[X_0, X_1]_\theta \subseteq C_\theta^+(X_0, X_1)$ for all $\theta \in (0,1)$. This is of course equivalent to showing that

$$[Y_0, Y_1]_\theta \subseteq C_{1-\theta}^+(Y_1, Y_0) \text{ for all } \theta \in (0,1). \tag{5.3}$$

We may suppose without loss of generality that $B \cap Y_1$ is dense in B, since if not we can simply replace B by $B^\circ = [B, Y_1]_0$, the closure of $B \cap Y_1$ in B, which satisfies $[B^\circ, Y_1]_\alpha = [B, Y_1]_\alpha$. (Cf. [**2**] Paragraph 9.3 p. 116.) It follows immediately from (5.2) that

$$\|a\|_{Y_0} \le \|a\|_B^{1-\alpha} \|a\|_{Y_1}^\alpha$$

for every $a \in B \cap Y_1$. But since also $\|a\|_{Y_1} \le C \|a\|_{Y_0}$ for some constant C, we deduce that $\|a\|_{Y_0} \le C^{\alpha/(1-\alpha)} \|a\|_B$ for all such a. Using the density of $B \cap Y_1$ in B we deduce that the previous estimate holds for all $a \in B$ and so obtain the continuous embedding

$$B \subset Y_0. \tag{5.4}$$

By [**2**] Paragraph 9.3 p. 116, the spaces $[B, Y_1]_1$ and $[Y_0, Y_1]_1$ are the completions of B and of Y_0 respectively in Y_1. Since $B = B \cap Y_1$ is dense in Y_0, it follows that $[B, Y_1]_1 = [Y_0, Y_1]_1$. Applying [**2**] Paragraph 9.3 p. 116 once more, we have that $[Y_0, [Y_0, Y_1]_1]_\theta = [Y_0, Y_1]_\theta$ and so $[Y_0, Y_1]_\theta = [[B, Y_1]_\alpha, [B, Y_1]_1]_\theta$. Applying the reiteration formula (see [**3**] p. 1005) to this latter space we deduce that

$$[Y_0, Y_1]_\theta = [B, Y_1]_{(1-\theta)\alpha + \theta}. \tag{5.5}$$

Let us now consider an arbitrary fixed element $a \in [Y_0, Y_1]_\theta$. By (5.5) it can be represented in the form $a = g_1((1-\theta)\alpha + \theta)$ for some $g_1 \in \mathcal{F}_S(B, Y_1)$. Now let $g_2(z) = g_1((1-z)\alpha + z)$. Then g_2 is a continuous and bounded $B + Y_1 = Y_1$ valued function on the strip $\operatorname{Re} z \in [\frac{-\alpha}{1-\alpha}, 1]$ and it is analytic in the interior of this strip. Furthermore $g_2(\theta) = a$ and the restrictions of g_2 to the lines $\operatorname{Re} z = \frac{-\alpha}{1-\alpha}$ and $\operatorname{Re} z = 0$ are continuous bounded B valued and $[B, Y_1]_\alpha = Y_0$ valued functions respectively.

The next step will be to use the construction defined in [**3**] p. 1008. (Cf. also the proof of Proposition 1 above.) This enables us to construct from g_2 a new function g_3 which has all the properties listed above for g_2 and also the additional property that $g_3(z + 2\pi i) = g_3(z)$ for all z in the strip $\operatorname{Re} z \in [\frac{-\alpha}{1-\alpha}, 1]$. (The functions $w(z)$ and $e^{\delta(z-\theta)^2}$ which were used in that construction on the strip $\operatorname{Re} z \in [0,1]$ of course also satisfy estimates of the required form on the wider strip which we need here.)

Now we define yet another continuous and bounded Y_1 valued function g_4, this time on the annulus $|z| \in [e^{-\alpha/(1-\alpha)}, e]$ by setting $g_4(z) = g_3(\log z)$. Here, because of the $2\pi i$ periodicity of g_3, the choice of branch of $\log z$ is irrelevant. Clearly g_4 is analytic in the interior of the annulus, and furthermore its restrictions to each of the three circles of radius $\gamma := e^{-\alpha/(1-\alpha)}$, 1 and e respectively are continuous maps into the Banach spaces B, Y_0 and Y_1 respectively.

It will be convenient, for each $r > 0$, to let C_r denote the circle $|z| = r$ oriented in the anticlockwise direction. We can now introduce the function

$$g_5(z) = g_4(z) + \frac{1}{2\pi i}\left(\int_{C\gamma} g_4(\zeta)\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - e^\theta}\right) d\zeta\right). \tag{5.6}$$

Obviously $g_5(e^\theta) = a$. Furthermore, since the integral in (5.6) is a continuous B valued function of z in the region $|z| > \gamma$ and is also analytic in that region, we deduce that g_5 is a continuous Y_1 valued function on the annulus $|z| \in [1, e]$ and is analytic in its interior. Furthermore the restrictions of the g_5 to the circles C_1 and C_e are continuous maps into Y_0 and Y_1 respectively. By Cauchy's integral formula we also have, whenever $|z| \in (\gamma, e)$, that

$$g_5(z) = \frac{1}{2\pi i}\left(\int_{C_e} \frac{g_4(\zeta)}{\zeta - z} d\zeta - \int_{C_\gamma} \frac{g_4(\zeta)}{\zeta - e^\theta} d\zeta\right).$$

But the formula on the right defines an analytic Y_1 valued function in the disk $|z| < e$.

Let us now define $\mathcal{F}_D(Y_1, Y_0) = \mathcal{F}_D(X_0, X_1)$ as in Section 1, for the particular choice of radii $R_1 = 1$ and $R_0 = e$. The preceding discussion shows that g_5 extends to a function in $\mathcal{F}_D(Y_1, Y_0)$. Consequently, by Proposition 1, $a = g_5(e^\theta) = g_5(R_0^\theta R_1^{1-\theta})$ is an element of $C^+_{1-\theta}(Y_1, Y_0)$. This establishes (5.3) and so completes the proof of the theorem. ☐

Remark. The norms $\|a\|_{C^+_\theta(X_0,X_1)}$ and $\|a\|_{[X_0,X_1]_\theta}$ must of course be equivalent. By making obvious appropriate norm estimates at each step of the above proof it is possible to show that the constants of equivalence of these norms depend only on θ, β and the norms of the continuous embeddings of B into X_0 and X_1.

Remark. The preceding theorem shows that the scale of spaces used in the counter example of Section 2 provides an apparently new example of a scale of complex interpolation spaces which cannot be continued beyond a certain value of the parameter. Other examples of related phenomena have been considered by Kalton [??], by N. and V. Zobin [**13**] and also by Zafran (see the Appendix on pp. 297-298 of [**8**]).

For a discussion of a related phenomenon, where the continuation of the scale exists but is not unique, see [**8**] pp. 295-297.

In view of Theorem 11 of [**5**] pp. 273–274 (cf. also Theorem 10 on p. 272) this "non-continuability" of the complex interpolation scale for the Banach couple $(FL_{\alpha_0} + FL_{\alpha_1}, FL_{\alpha_1})$ suggests the possibility that an appropriate modification of this couple might provide the setting for a counterexample to settle a long standing open question about compact operators and the complex interpolation method. See also [**6**] for various simplifications and reductions of this problem. The reader who wishes to consider this possibility should probably keep in mind that if there is a counterexample for the above question then there is necessarily a counterexample in the context of domain couples of the form $(\ell^1(FL^1_{\alpha_0}), \ell^1(FL^1_{\alpha_1}))$ and/or range couples $(\ell^\infty(FL^\infty_{\alpha_0}), \ell^\infty(FL^\infty_{\alpha_1}))$ (See [**6**] Proposition 3, p. 356) and furthermore it should be possible to show via arguments of equicontinuity etc. or from a special case treated in [**2**] p. 118 that compactness of operators *is* preserved by the complex method in the context of domain couples of the form $(FL^1_{\alpha_0}, FL^1_{\alpha_1})$ and/or range couples $(FL^\infty_{\alpha_0}, FL^\infty_{\alpha_1})$.

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