A NOTE ON THE VARIANCE CALCULATION FOR GENERALIZED POLYA URNS

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1. Introduction

Limit theorems for generalized Pólya urns are given in [3]. In particular, a central limit theorem for the composition is shown under very general conditions, with an explicit but rather complicated formula for the covariance matrix of the asymptotic multi-dimensional normal distribution.

When computing the covariance matrix numerically in some applications (see [2] and [1]), Cecilia Holmgren and Axel Heimbirger found a minor simplification. The purpose of this note is to explain why this simplification works, in a general setting.

We use the assumptions and notation of [3]. In particular, all vectors are column vectors. Furthermore,

- $\xi_i = (\xi_{ij})_{j}$ is the (possibly random) replacement vector when a ball of type (colour) $i$ is drawn;
- $a = (a_i)_i$ is the vector of activities of the different types;
- $A := (a_j E \xi_{ji})_{i,j}$; $\lambda_1, \ldots$ are the eigenvalues of $A$, ordered with $\lambda_1 \geq \text{Re} \lambda_2 \geq \text{Re} \lambda_3 \ldots$;
- $u'_1$ and $v_1$ are left and right eigenvectors corresponding to the largest eigenvalue $\lambda_1$; these are normalized by $a \cdot v_1 = a' v_1 = 1$ and $u_1 \cdot v_1 = u'_1 v_1 = 1$ and are then uniquely defined under the assumptions in [3].

2. Results

Lemma 1. Suppose that $a \cdot \xi_i = m$ deterministically for some $m > 0$ and every $i$. (In other words, the activity increases deterministically by a fixed amount every time a ball is drawn.) Then

$$Bu_1 = m^2 v_1.$$  (1)

Proof. Note first that the condition implies $m = \lambda_1$ and $u_1 = a$, see [3, Lemma 5.4]. By [3, (2.13)], $B_1 := E(\xi_i \xi'_i)$, and thus, since $\xi'_i a = \xi \cdot a = m$,

$$B_1 u_1 = B_1 a = E(\xi_i \xi'_i a) = m E \xi_i.$$  (2)

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Furthermore, the matrix $B$ is by [3, (2.14)] defined by $B := \sum_i v_i a_i B_i$, and thus (2) yields

$$B u_1 = \sum_i v_i a_i m \mathbb{E} \xi_i,$$

where the $j$:th component is

$$(Bu_1)_j = \sum_i v_i a_i m \mathbb{E} \xi_{ij} = m \sum_i v_i A_{ji} = m(A v_1)_j = m \lambda_1(v_1)_j. \quad (4)$$

Hence,

$$Bu_1 = m \lambda_1 v_1 = m^2 v_1. \quad \Box$$

**Lemma 2.** Suppose that $a \cdot \xi_i = m$ deterministically for some $m > 0$ and every $i$. Suppose further that $\text{Re} \lambda_2 < \frac{1}{2} \lambda_1$. Then

$$P_1 B P_1' = P_1 B = B P_1' = B - m^2 v_1 v_1'. \quad (5)$$

**Proof.** When $\text{Re} \lambda_2 < \frac{1}{2} \lambda_1$, we have by definition and [3, (2.7)]

$$P_1 = I - P_{\lambda_1} = I - v_1 u_1'.$$

Consequently, $P_1' = I - u_1 v_1'$ and, by Lemma 1,

$$B - B P_1' = B(I - P_1') = B u_1 v_1' = m^2 v_1 v_1', \quad (6)$$

which yields

$$P_1 B - P_1 B P_1' = m^2 P_1 v_1 v_1' = 0,$$

since $P_1 v_1 = v_1 - v_1 u_1' v_1 = 0$ by the construction of $P_1$. Thus $P_1 B = P_1 B P_1'$, and taking the transpose yields $B P_1' = P_1 B P_1'$.

The final equation in (5) follows by (6). \quad \Box

Under the conditions in Lemma 2, the asymptotic covariance matrix $\Sigma$ in [3, Theorem 3.22] equals by [3, Lemma 5.4] $m \Sigma_1$, where by [3, (2.15)] and the fact that $P_1$ commutes with $A$,

$$\Sigma_1 := \int_0^\infty P_1 e^{sA} B e^{sA'} P_1' e^{-\lambda_1 s} \, ds = \int_0^\infty e^{sA} P_1 B P_1' e^{sA'} e^{-\lambda_1 s} \, ds. \quad (7)$$

**Theorem 3.** Suppose that $a \cdot \xi_i = m$ deterministically for some $m > 0$ and every $i$. Suppose further that $\text{Re} \lambda_2 < \frac{1}{2} \lambda_1$. Then we may drop either $P_1$ or $P_1'$ (but not both) from (7). Furthermore,

$$\Sigma_1 = \int_0^\infty \left( e^{-\lambda_1 s} e^{sA} B e^{sA'} - m^2 e^{\lambda_1 s} v_1 v_1' \right) \, ds. \quad (8)$$

**Proof.** That we can drop $P_1$ or $P_1'$ is an immediate consequence of (5) in Lemma 2. To see that we cannot drop both $P_1$ and $P_1'$, note that by $A' u_1 = \lambda_1 u_1$ and thus $e^{sA'} u_1 = e^{\lambda_1 s} u_1$, which by transposing also yields $u_1' e^{sA} = e^{\lambda_1 s} u_1'$. Hence, by Lemma 1,

$$u_1'(e^{sA} B e^{sA'} e^{-\lambda_1 s}) u_1 e^{-\lambda_1 s} e^{\lambda_1 s} u_1' B(e^{\lambda_1 s} u_1) = e^{\lambda_1 s} u_1' B u_1
$$

$$= m^2 e^{\lambda_1 s} u_1' v_1 = m^2 e^{\lambda_1 s}. \quad (9)$$

Thus the integral (7) diverges without $P_1$ or $P_1'$. 

Finally, (8) follows from (7) and (5), recalling that $e^{sA}v_1 = e^{s\lambda_1}v_1$ and $v'_1 e^{sA'} = e^{s\lambda_1'}v'_1$.

□

Remark 4. It is easily seen that, for some $q \geq 0$, $P_I e^{sA} = O\left((1+s^q)e^{Re \lambda_2 s}\right)$, and thus the integrand in (7) is $O\left((1+s^{2q})e^{(2Re \lambda_2 - \lambda_1)s}\right)$, which is integrable because $2Re \lambda_2 - \lambda_1 < 0$. The same holds for (8), since its integrand is the same, by the proof above.

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References