HITTING TIMES FOR RANDOM WALKS WITH RESTARTS

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Abstract. The time it takes a random walker in a lattice to reach the origin from another vertex x, has infinite mean. If the walker can restart the walk at x at will, then the minimum expected hitting time $\gamma(x, 0)$ (minimized over restarting strategies) is finite; it was called the "grade" of x by Dumitriu, Tetali and Winkler. They showed that, in a more general setting, the *grade* (a variant of the "Gittins index") plays a crucial role in control problems involving several Markov chains. Here we establish several conjectures of Dumitriu et al on the asymptotics of the grade in Euclidean lattices. In particular, we show that in the planar square lattice, $\gamma(x, 0)$ is asymptotic to $2|x|^2 \log |x|$ as $|x| \to \infty$. The proof hinges on the local variance of the potential kernel being almost constant on the level sets of h. We also show how the same method yields precise second order asymptotics for hitting times of a random walk (without restarts) in a lattice disk.

1. Introduction

Consider a Markov chain (X_n) on a (countable) state space $\mathcal{V}$, with transition probabilities $\big(p(x, y)\big)_{x,y\in\mathcal{V}}$. We use $\mathbb{P}_x$ and $\mathbb{E}_x$ to denote probability and expectation in the chain with initial state $X_0 = x$.

We assume that the chain is irreducible, i.e. that each state can be reached from any other state.

Dumitriu, Tetali and Winkler [1] defined a function $\gamma(x, z)$ for pairs of states $x, z \in \mathcal{V}$. This function is a version of the Gittins index and is called the *grade*; it can be defined as follows [1, Theorem 6.1]:

Consider a player that starts at x with the goal of reaching z as quickly as possible. Each time the player moves, the state changes randomly according to the transition matrix p of the Markov chain; however, the player then has the option (if she finds the new state to be too bad) to restart by an instantaneous jump back to x. The grade $\gamma(x, z)$ then is the minimum, over all strategies for restarting, of the expected number of moves until z is reached. (Thus, a restart is not counted as a separate move, but the moves already performed are included in the total count. Note that a restart always

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moves back to the original starting state x.) For other equivalent definitions, and applications of the grade to other games, see [1].

Remark 1.1. Once the grade is computed for all starting positions, then the optimal strategies for the game above (with initial state x) can all be described as follows: *If the current state is y, then restart if $\gamma(y, z) > \gamma(x, z)$, but not if $\gamma(y, z) < \gamma(x, z)$; if $\gamma(y, z) = \gamma(x, z)$, then it does not matter whether we restart or not.*

Remark 1.2. The setting in [1] is more general than the one presented here, since that paper allows for a cost of each move that may depend on the present state, while we consider here only the case of constant cost, so that the total cost is the total time.

The purpose of this paper is to answer questions raised by Dumitriu et al [1], on the asymptotics of the grade in $\mathbb{Z}^d$, $d \geq 2$. (The case $d = 1$ is simple; as shown in [1], $\gamma(x, 0) = |x|(|x| + 1)$ for $\mathbb{Z}$.)

By translation invariance it clearly suffices to consider $z = 0$. Denote the Euclidean norm of x by $|x|$.

Theorem 1.3. *For simple random walk on $\mathbb{Z}^2$,*

$$\gamma(x, 0) = 2|x|^2 \log |x| + (2\gamma_e + 3 \log 2 - 1)|x|^2 + O(|x| \log |x|), \qquad |x| \geq 2,$$

where $\gamma_e := \lim_n (\sum_{j=1}^{n} \frac{1}{j} - \log n)$ is Euler's constant.

Theorem 1.4. *For simple random walk on $\mathbb{Z}^d$, $d \geq 3$,*

$$\gamma(x, 0) = \frac{\omega_d}{p_d} |x|^d + O(|x|^{d-1}),$$

where $\omega_d = \pi^{d/2}/\Gamma(d/2 + 1)$ is the volume of the unit ball in $\mathbb{R}^d$, and p_d is the escape probability of the simple random walk in $\mathbb{Z}^d$, i.e., the probability that the walk never returns to its starting point.

The leading terms were conjectured by Dumitriu, Tetali and Winkler in the preprint version of [1]. Based on the heuristic argument that the lattice structure should be unimportant on large scales, they suggested that a near-optimal strategy might be: *Restart if the current state has a larger Euclidean distance to the target 0 than the starting state.* The expected hitting time for this strategy can then be estimated using electrical network theory.

Theorems 1.3 and 1.4 together with Remark 1.1 imply the following corollary, which shows that the optimal restarting strategy is indeed as outlined above, except possibly at some border-line cases.

Corollary 1.5. *For simple random walk on $\mathbb{Z}^d$, with target $z = 0$, there exists a constant $C = C(d)$, independent of the starting position x, such that every optimal strategy restarts from every position y with $|y| > |x| + C$, but never when $|y| < |x| - C$.* □

For intermediate cases with $|x| - C \le |y| \le |x| + C$, we cannot prescribe explicitly the optimal startegy; numerical calculations indicate that the simple heuristic strategy is not always optimal, i.e. we cannot take $C = 0$ in the corollary. (Peter Winkler, personal communication).

To prove the theorems above, we state and prove in Section 2 a result, Theorem 2.1, yielding bounds on the grade for general Markov chains. This theorem is applied to $\mathbb{Z}^d$ in Sections 3 and 4. (We separate the recurrent case $d = 2$ from the transient case $d \ge 3$ since the details are somewhat different.) In Section 5 we present analogous results for a continuous version of the problem, with the random walk replaced by Brownian motion in $\mathbb{R}^d$. In this case we obtain exact results, analogous to the asymptotic results in Theorems 1.3 and 1.4. In the final section we show how our method yields precise second order asymptotics for hitting times of a random walk (without restarts) in a lattice disk.

2. A general estimate

We state a theorem yielding upper and lower bounds on the grade. The theorem applies in principle to any Markov chain, but its usefulness depends on the existence of a suitable harmonic function for the Markov chain. Recall that a function $h : \mathcal{V} \to \mathbb{R}$ is **harmonic** at $x \in \mathcal{V}$ if $\mathbb{E}_x\, h(X_1) = h(x)$, i.e. if $\sum_y p(x, y)h(y) = h(x)$.

For a function $h : \mathcal{V} \to \mathbb{R}$ and $x \in \mathcal{V}$, define the **local variance**,

$$V_h(x) := \mathbb{E}_x\, |h(X_1) - h(x)|^2 = \sum_y p(x, y)|h(y) - h(x)|^2.$$

Theorem 2.1. *Let $z \in \mathcal{V}$ and suppose that $h : \mathcal{V} \to [0, \infty)$ is a non-negative function that is harmonic on $\mathcal{V} \setminus \{z\}$ and satisfies $h(z) = 0$.*

Suppose that g_+, g_- are positive functions defined on $[0, \sup h)$, such that for every $x, y \in \mathcal{V}$ with $p(x, y) > 0$, and every real number ξ between $h(x)$ and $h(y)$, the local variance satisfies

$$g_-(\xi) \le V_h(x) \le g_+(\xi)\,. \tag{2.1}$$

Then, for every $x \in \mathcal{V}$,

$$\int_0^{h(x)} \frac{2s}{g_+(s)}\, \boldsymbol{d}s \le \gamma(x, z) \le \int_0^{h^*(x)} \frac{2s}{g_-(s)}\, \boldsymbol{d}s\,, \tag{2.2}$$

where

$$h^*(x) = \sup\{h(y) : p(w, y) > 0 \textit{ for some } w \in \mathcal{V} \textit{ with } h(w) \le h(x)\}.$$

Proof. Fix a starting position $x_0 \in \mathcal{V}$. To prove the **lower bound** in (2.2), define a function $F = F_+ : [0, \infty) \to [0, \infty)$ by

$$F(s) := \int_0^{s\wedge h(x_0)} \int_t^{h(x_0)} \frac{2}{g_+(u)}\, \boldsymbol{d}u\, \boldsymbol{d}t. \tag{2.3}$$

Thus $F(0)=0$, and by Fubini's theorem,

$$F(h(x_0)) = \iint_{0<t<u<h(x_0)} \frac{2}{g_+(u)}\,\boldsymbol{d}t\,\boldsymbol{d}u = \int_0^{h(x_0)} \frac{2u}{g_+(u)}\,\boldsymbol{d}u\,. \tag{2.4}$$

For all $s \geq 0$,

$$0 \leq F(s) \leq F(h(x_0)). \tag{2.5}$$

Moreover,

$$F'(s) = \begin{cases} \int_s^{h(x_0)} \frac{2}{g_+(u)}\,\boldsymbol{d}u, & s \leq h(x_0),\\ 0, & s \geq h(x_0), \end{cases}$$

and, a.e.,

$$F''(s) = \begin{cases} -\frac{2}{g_+(u)}, & s \leq h(x_0),\\ 0, & s > h(x_0). \end{cases} \tag{2.6}$$

Let $\widehat{X}_n$, $n=0,1,\dots,$ be the process obtained by starting at $\widehat{X}_0 = x_0$, choosing successive states by running the Markov chain and restarting according to some non-anticipating strategy Λ. (Formally, Λ is a $\{0,1\}$-valued function on finite sequences of states.) That is, suppose that a step of the Markov chain takes $\widehat{X}_n$ to $X^\#_{n+1}$. If $\Lambda(\widehat{X}_1,\dots,\widehat{X}_n,X^\#_{n+1})=0$, then we let $\widehat{X}_{n+1}=X^\#_{n+1}$, while if $\Lambda(\widehat{X}_1,\dots,\widehat{X}_n,X^\#_{n+1})=1$, then we let $\widehat{X}_{n+1}=x_0$.

We claim that

$$Y_n := F\bigl(h(\widehat{X}_n)\bigr) + n$$

is a submartingale for any choice of restarting strategy Λ.

To see this, start by observing that

$$F\bigl(h(\widehat{X}_{n+1})\bigr) \geq F\bigl(h(X^\#_{n+1})\bigr)\,,$$

since F attains its maximum at $h(x_0)$. Hence, denoting $\widehat{X}_n = x$, we find that

$$\begin{aligned} \mathbb{E}(Y_{n+1} \mid \widehat{X}_1,\dots,\widehat{X}_n) &\geq \mathbb{E}\bigl(F(h(X^\#_{n+1})) + n + 1 \mid \widehat{X}_1,\dots,\widehat{X}_n\bigr)\\ &= \mathbb{E}_x\, F(h(X_1)) + n + 1. \end{aligned} \tag{2.7}$$

Denote $Z = h(X_1)-h(x)$. A Taylor expansion of F (with error in integral form), followed by an application of (2.6) and (2.1), yields

$$\begin{aligned} F\bigl(h(X_1)\bigr) &= F\bigl(h(x)+Z\bigr)\\ &= F\bigl(h(x)\bigr) + ZF'\bigl(h(x)\bigr) + \int_0^1 (1-t)F''\bigl(h(x)+tZ\bigr)Z^2\,\boldsymbol{d}t\\ &\geq F\bigl(h(x)\bigr) + ZF'\bigl(h(x)\bigr) - Z^2\int_0^1 (1-t)\frac{2}{g_+\bigl(h(x)+tZ\bigr)}\,\boldsymbol{d}t\\ &\geq F\bigl(h(x)\bigr) + ZF'\bigl(h(x)\bigr) - Z^2\int_0^1 (1-t)\frac{2}{V_h(x)}\,\boldsymbol{d}t\\ &\geq F\bigl(h(x)\bigr) + ZF'\bigl(h(x)\bigr) - \frac{Z^2}{V_h(x)}. \end{aligned}$$

If $x \neq z$, then h is harmonic at x, so $\mathbb{E}_x Z = 0$ and $\mathbb{E}_x Z^2 = V_h(x)$. Therefore, taking the expectation in the last displayed inequality, we find that

$$\mathbb{E}_x F\big(h(X_1)\big) \geq F\big(h(x)\big) - 1\,.$$

This also holds, trivially, when $x = z$. Thus by (2.7),

$$\mathbb{E}(Y_{n+1} \mid \widehat{X}_1, \ldots, \widehat{X}_n) \geq F\big(h(x)\big) + n = Y_n,$$

which proves that (Y_n) is a submartingale.

We stop this submartingale at

$$\tau := \inf\{n : \widehat{X}_n = z\}. \tag{2.8}$$

Note that $Y_\tau = F\big(h(z)\big) + \tau = \tau$. Moreover, by (2.5),

$$\sup_{n\leq\tau} |Y_n| = \sup_{n\leq\tau} Y_n \leq F\big(h(x_0)\big) + \tau.$$

Hence, if $\mathbb{E}\,\tau < \infty$, the stopped submartingale is uniformly integrable, and thus by the optional sampling theorem

$$\mathbb{E}\,\tau = \mathbb{E}\,Y_\tau \geq \mathbb{E}\,Y_0 = F\big(h(x_0)\big).$$

This is trivially true if $\mathbb{E}\,\tau = \infty$ too.

In other words, for any restarting strategy, the expected hitting time of z by $(\widehat{X}_n)$ is at least $F\big(h(x_0)\big)$, i.e. $\gamma(x_0, z) \geq F\big(h(x_0)\big)$, and the left hand side of (2.2) follows by (2.4), since x_0 is arbitrary.

Next, we prove the **upper bound** in (2.2). We denote the initial state by x_0, and use the simple restarting strategy: *Restart to x_0 from all points $y = X_n^{\#}$ with $h(y) > h(x_0)$.*

Denote the resulting process by $(\widehat{X}_n)$ and observe that

$$h(\widehat{X}_n) \leq h(x_0) \text{ and } h(\widehat{X}_{n+1}) \leq h(X_{n+1}^{\#}) \leq h^*(x_0) \text{ for all } n.$$

Consider

$$F^*(s) = F^*_-(s) := \int_0^{s\wedge h^*(x_0)} \int_t^{h^*(x_0)} \frac{2}{g_-(u)}\, \boldsymbol{du}\, \boldsymbol{dt}\,. \tag{2.9}$$

By an argument similar to the one above, we find that

$$\mathbb{E}\big(F^*(h(\widehat{X}_{n+1})) \mid \widehat{X}_1, \ldots, \widehat{X}_n\big) \leq F^*\big(h(\widehat{X}_n)\big) - 1.$$

Denote $Y_n^* = F^*\big(h(\widehat{X}_n)\big) + n$ and let τ be defined by (2.8). Then $(Y^*_{n\wedge\tau})_{n\geq 0}$ is a positive supermartingale, whence by the optional sampling theorem,

$$\gamma(x_0, z) \leq \mathbb{E}\,\tau = \mathbb{E}\,Y_\tau \leq \mathbb{E}\,Y_0 = F^*\big(h(x_0)\big) \leq F^*\big(h^*(x_0)\big). \qquad \square$$

Remark 2.2. The proof above suggests that a reasonable strategy is to restart from every state y with $h(y) > h(x)$, as in the second part of the proof. For $\mathbb{Z}^2$, $d \geq 2$, with h as described in Sections 3 and 4, this is close (but not identical) to the strategy based on Euclidean distance, and Corollary 1.5 shows that it is, in some sense, close to optimal.

Remark 2.3. To obtain matching upper and lower bounds from Theorem 2.1, we want $g_- \approx g_+$. It is thus essential that we can find a harmonic function h such that $V_h(x)$ is approximately a function of $h(x)$, i.e. such that $V_h(x)$ is roughly constant in sets where $h(x)$ is.

Remark 2.4. The applications of Theorem 2.1 below follow a common pattern, here given as a heuristic guide to later precise calculations. Suppose that $r(x)$ is a function on $\mathcal{V}$ such that $h(x) \approx \varphi(r(x))$ and $V_h(x) \approx \psi(r(x))$ for some φ and ψ with φ increasing and differentiable. Suppose further that $h(x) - h(y)$ is sufficiently small when $p(x, y) > 0$. We then can take $g_\pm(s) \approx \psi(\varphi^{-1}(s))$ and obtain

$$\gamma(x, z) \approx \int_0^{\varphi(r(x))} \frac{2s}{\psi(\varphi^{-1}(s))} \boldsymbol{ds} = \int_{\varphi^{-1}(0)}^{r(x)} \frac{2\varphi(t)\varphi'(t)}{\psi(t)} \boldsymbol{dt}.$$

3. Two dimensions: Proof of Theorem 1.3

In this section the underlying Markov chain (X_n) is simple random walk on $\mathbb{Z}^2$. We choose $h(x) = \frac{\pi}{2} a(x)$, where

$$a(x) := \sum_{n=0}^{\infty} \Big[\mathbb{P}_0(X_n = 0) - \mathbb{P}_0(X_n = x) \Big]$$

is the potential kernel studied in [8, 7, 4, 2]. A complete asymptotic expansion of $a(x)$ is presented in [2, 3]; here we only quote the second order expansion given, e.g. in [8, 2] and [4, Section 1.6]:

$$h(x) = \tfrac{\pi}{2} a(x) = \log |x| + b + O(|x|^{-2}), \tag{3.1}$$

where $b = \gamma_e + \frac{3}{2} \log 2$; moreover, if $\vec{e}$ is a unit vector along one of the coordinate axis, then

$$a(x + \vec{e}) - a(x) = \vec{e} \cdot \nabla\big(\tfrac{2}{\pi} \log |x|\big) + O(|x|^{-2})$$

and thus

$$V_h(x) = \tfrac{1}{2} \big|\nabla(\log |x|)\big|^2 + O(|x|^{-3}) = \tfrac{1}{2} |x|^{-2} + O(|x|^{-3}).$$

If $p(x, y) > 0$, then $|x - y| = 1$ and thus by (3.1)

$$h(y) = h(x) + O(|x|^{-1}) = \log |x| + b + O(|x|^{-1}). \tag{3.2}$$

It is now easily seen that (2.1) is satisfied with

$$g_\pm(s) = \tfrac{1}{2} e^{-2(s-b)} (1 \pm C e^{-s}) \tag{3.3}$$

if C is a sufficiently large constant. For small s this could make $g_-(s) \le 0$, but we redefine $g_-(s)$ to be a small positive constant in these cases. We then have

$$\frac{1}{g_\pm(s)} = 2 e^{2(s-b)} \big(1 + O(e^{-s})\big) \quad \text{as } s \to \infty. \tag{3.4}$$

Furthermore, (3.2) implies that

$$h^*(x) = \log |x| + b + O(|x|^{-1}).$$

Hence Theorem 2.1 yields, for $|x| \geq 2$,

$$\begin{aligned}\gamma(x,0) &= \int_0^{\log|x|+b+O(|x|^{-1})} 4se^{2(s-b)}\big(1+O(e^{-s})\big)\,\boldsymbol{ds}\\ &= \big[2se^{2(s-b)} - e^{2(s-b)} + O(se^s + e^s)\big]_0^{\log|x|+b+O(|x|^{-1})}\\ &= 2(\log|x|+b)|x|^2 - |x|^2 + O(|x|\log|x|). \qquad \square\end{aligned}$$

4. Transient case: Proof of Theorem 1.4

For simple random walk on $\mathbb{Z}^d$, $d \geq 3$, we employ the Green function $G(x) := G(x,0) = \sum_{n=0}^{\infty}\Big[\mathbb{P}_x(X_n = 0)\Big]$. We have [4, Section 1.5]

$$G(x) = a_d|x|^{2-d} + O(|x|^{-d}),$$

where $a_d = \frac{2}{(d-2)\omega_d}$, and

$$V_G(x) = \frac{1}{d}\big|\nabla(a_d|x|^{2-d})\big|^2 + O(|x|^{1-2d}).$$

Let

$$h(x) := a_d^{-1}\big(G(0) - G(x)\big) = a_d^{-1}G(0) - |x|^{2-d} + O(|x|^{-d})$$

and write $h(\infty) = a_d^{-1}G(0)$. Thus

$$h(y) = h(\infty) - |x|^{2-d} + O(|x|^{1-d}), \qquad p(x,y) > 0,$$

and

$$h^*(x) = h(\infty) - |x|^{2-d} + O(|x|^{1-d}).$$

Moreover,

$$\begin{aligned}V_h(x) = a_d^{-2}V_G(x) &= \frac{1}{d}\big|\nabla(|x|^{2-d})\big|^2 + O(|x|^{1-2d})\\ &= \frac{(d-2)^2}{d}|x|^{2-2d}\big(1+O(|x|^{-1})\big).\end{aligned}$$

Hence we can take, for some large constant C and with a modification for small s to keep the values positive,

$$g_\pm(s) = \frac{(d-2)^2}{d}\big(h(\infty)-s\big)^{\frac{2d-2}{d-2}}\big(1 \pm C(h(\infty)-s)^{\frac{1}{d-2}}\big).$$

Consequently, Theorem 2.1 yields

$$\gamma(x,0) = \int_0^{\widetilde{h}(x)} \frac{2sd}{(d-2)^2}\big(h(\infty)-s\big)^{\frac{2-2d}{d-2}}\big(1+O\big(h(\infty)-s\big)^{\frac{1}{d-2}}\big)\,\boldsymbol{ds}\,, \quad (4.1)$$

where $\widetilde{h}(x) = h(\infty) - |x|^{2-d} + O(|x|^{1-d})$. (Recall that $\widetilde{h}(x)$ is $h(x)$ in the lower bound for $\gamma(x,0)$, and $h^*(x)$ in the upper bound.)

Next, we change variables to $u = u(s) := (h(\infty) - s)^{-1/(d-2)}$. Observe that $u(\widetilde{h}(x)) = |x| + O(1)$ and $\boldsymbol{d}s = (d-2)u^{1-d}\,\boldsymbol{d}u$. If we denote $u_0 = h(\infty)^{-1/(d-2)}$, then

$$\begin{aligned}\gamma(x,0) &= \int_{u_0}^{|x|+O(1)} \frac{2\bigl(h(\infty) - u^{2-d}\bigr)d}{(d-2)^2} u^{2d-2}\bigl(1+O(u^{-1})\bigr)(d-2)u^{1-d}\,\boldsymbol{d}u \\ &= 2h(\infty)\frac{d}{d-2}\int_{u_0}^{|x|+O(1)} \bigl(u^{d-1} + O(u^{d-2})\bigr)\,\boldsymbol{d}u \\ &= \frac{2h(\infty)}{d-2}|x|^d + O(|x|^{d-1}).\end{aligned}$$

The result follows because $G(0) = 1/p_d$ and thus

$$\frac{2h(\infty)}{d-2} = \frac{2G(0)}{(d-2)a_d} = \frac{\omega_d}{p_d}. \qquad \square$$

5. Brownian motion

In this section we consider a continuous analogue of the problem studied above. We consider Brownian motion in $\mathbb{R}^d$, starting at some given $x \in \mathbb{R}^d$, and again we are allowed to restart at x at any given time. Since, when $d \geq 2$, the Brownian motion a.s. never will hit 0, we now let our target be a small ball $B_{r_0} = \{y : |y| \leq r_0\}$, where $r_0 > 0$ is some arbitrary fixed number. (For $d = 1$ we could take $r_0 = 0$ too.) The grade then is defined as in the discrete case, by taking the infimum of the expected hitting time over all restarting strategies.

Let

$$h(x) := \begin{cases} |x| - r_0, & d = 1, \\ \log\bigl(|x|/r_0\bigr), & d = 2, \\ r_0^{2-d} - |x|^{2-d}, & d \geq 3. \end{cases} \tag{5.1}$$

Then h is harmonic and positive in the complement of B_{r_0}, with $h(x) = 0$ when $|x| = r_0$. Moreover,

$$|\bigtriangledown h(x)|^2 = \begin{cases} 1 & d = 1, \\ |x|^{-2}, & d = 2, \\ (d-2)^2|x|^{2-2d}, & d \geq 3 \end{cases} \tag{5.2}$$

is now exactly a function of $h(x)$, say $g(h(x))$.

Let the starting point be x_0 and denote the process, using some non-anticipating restarting rule, by $\widehat{X}_t$. Let further $\tau := \inf\{t : |\widehat{X}_t| = r_0\}$. If F is defined by (2.3) (with $g_+ = g$), we see as in the proof of Theorem 2.1, now using Itô's formula instead of a Taylor expansion, that $F\bigl(h(\widehat{X}_t)\bigr) + t$ is a local submartingale and, again by the optional sampling theorem, that $\mathbb{E}\,\tau \geq F\bigl(h(x_0)\bigr)$. Since this holds for any restarting strategy,

$$\gamma(x_0, \partial B_{r_0}) \geq F\bigl(h(x_0)\bigr).$$

Conversely, using the strategy *restart when* $h(\widehat{X}_t) \geq h(x_0) + \varepsilon$ for some $\varepsilon > 0$, we find that if F^* is defined by (2.9) with $h^*(x_0) = h(x_0) + \varepsilon$ and $g_- = g$, then $\mathbb{E}\, T \leq F^*\big(h^*(x_0)\big)$. Letting $\varepsilon \to 0$, this and the lower bound above show, together with (2.4), that the grade is given by

$$\gamma(x_0, \partial B_{r_0}) = F\big(h(x_0)\big) = \int_0^{h(x_0)} \frac{2s}{g(s)}\, \boldsymbol{d}s. \tag{5.3}$$

Remark 5.1. We see that the optimal strategy is to restart whenever the current position is more distant from the origin than the starting point x_0, which is the intuitively obvious strategy. Some care has to be taken interpreting this, however, since this a.s. entails infinitely many restarts in any interval $(0, \delta)$. The resulting process can be obtained by taking a limit as in the proof of (5.3) above, or by utilizing a reflected Brownian motion (for the radial part).

Write $h(x) = \varphi(|x|)$ and $|\bigtriangledown h(x)|^2 = \psi(x)$, so that $g(s) = \psi(\varphi^{-1}(s))$, We obtain, cf. Remark 2.4, that

$$\gamma(x, \partial B_{r_0}) = \int_0^{\varphi(|x|)} \frac{2s}{\psi(\varphi^{-1}(s))}\, \boldsymbol{d}s = \int_{r_0}^{|x|} \frac{2\varphi(r)\varphi'(r)}{\psi(r)} \boldsymbol{d}r.$$

Taking φ and ψ from (5.1) and (5.2), we easily evaluate this integral and deduce the following result.

Theorem 5.2. *For Brownian motion in* $\mathbb{R}^d$*, if* $|x| \geq r_0 > 0$*,*

$$\gamma(x, \partial B_{r_0}) = \begin{cases} (|x| - r_0)^2, & d = 1, \\ |x|^2 \log |x| - |x|^2(\frac{1}{2} + \log r_0) + \frac{1}{2} r_0^2, & d = 2, \\ \frac{2 r_0^{2-d}}{d(d-2)} |x|^d - \frac{1}{d-2}|x|^2 + \frac{1}{d} r_0^2, & d \geq 3. \end{cases} \qquad \square$$

It is instructive to compare these exact results for $\mathbb{R}^d$ and the asymptotic results for $\mathbb{Z}^d$ in Theorems 1.3 and 1.4. Note first that the time scales differ by a factor d, since in the simple random walk, each coordinate of a step has variance $1/d$. With this adjustment we see that we obtain the same leading term for $\mathbb{Z}^d$ and $\mathbb{R}^d$ when $d \leq 2$; in this case, the choice of r_0 affects only lower order terms. When $d \geq 3$, however, we obtain the same $|x|^d$ rate, but the constant for Brownian motion depends on the choice of r_0, and there is no reasonable way to obtain the right constant for $\mathbb{Z}^d$ from the continuous limit. This reflects the fact that the constant for $\mathbb{Z}^d$ involves the escape probability p_d, which depends on the local lattice structure near 0 that is lost in the continuous limit.

6. Hitting times for random walk in a disk

The method above can also be used to estimate expected hitting times in reversible Markov chains. For simplicity, we consider only simple random walk on a graph $(\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$. We thus assume $p(x, y) = 1/\deg_{\mathcal{V}}(x)$ when $x \sim y$ (and 0 otherwise), where $\deg_{\mathcal{V}}(x) := \{y \in \mathcal{V} : y \sim x\}$.

Theorem 6.1. *Let z, h, g_+ and g_- be as in Theorem 2.1, for simple random walk on a graph $(\mathcal{V}, \mathcal{E})$, and let D be a finite connected subset of $\mathcal{V}$ with $z \in D$. Define*

$$\begin{aligned} \partial D &:= \{x \in D : x \sim y \text{ for some } y \notin D\}, \\ \partial^2 D &:= \partial D \cup \{x \in D : x \sim y \text{ for some } y \in \partial D\}, \\ h_1 &:= \min\{h(x) : x \in \partial^2 D\}, \\ B &:= \{x \in D : x \sim y \text{ for some } y \text{ with } h(y) \geq h_1\}. \end{aligned}$$

Let $\{X_n\}_{n=0}^{\infty}$ be a simple random walk on D. Let $\tau := \min\{n : X_n = z\}$. Then, for any $X_0 = x_0 \in D$,

$$\int_0^{h_1} \frac{2(u \wedge h(x_0))}{g_+(u)} \boldsymbol{d}u \leq \mathbb{E}_{x_0} \tau \leq \int_0^{h_1} \frac{2(u \wedge h(x_0))}{g_-(u)} \boldsymbol{d}u + \Delta, \tag{6.1}$$

where $\Delta := \mathbb{E}_{x_0} \#\{n \leq \tau : X_n \in B\}$.

Note that h is harmonic on all of $\mathcal{V}$, while X_n is defined on D with transition probabilities $p_D(x, y) := 1/\deg_D(x)$ when $x \sim y$ and $x, y \in D$.

The error term Δ can be estimated in several ways. One of them is to bound $\tau = \tau_z$ by the time τ_* that it takes the random walk to visit z and return to x_0. Then (see, e.g., Lemma 10.5 and Proposition 10.6 in [5])

$$\Delta \leq \mathbb{E}_{x_0} \#\{n \leq \tau_* : X_n \in B\} = \frac{\mu(B)}{\mu(D)} \mathbb{E}_{x_0}(\tau_*) = \mu(B) \mathcal{R}(x_0 \leftrightarrow z), \tag{6.2}$$

where $\mu(B) = \sum_{x \in B} \deg_D(x)$ and $\mathcal{R}(x_0 \leftrightarrow z)$ is the resistance between x_0 and z in D, regarded as an electrical network.

Proof. We define, in analogy with (2.3) and (2.9),

$$F_\pm(s) := \int_0^{s \wedge h_1} \int_t^{h_1} \frac{2}{g_\pm(u)} \boldsymbol{d}u \boldsymbol{d}t = \int_0^{h_1} \frac{2(s \wedge u)}{g_\pm(u)} \boldsymbol{d}u. \tag{6.3}$$

We thus integrate only up to h_1, and we may redefine $g_+(u) = \infty$ for $u > h_1$. The right hand inequality in (2.1) then still holds for all $x \in \mathcal{V}$, and we obtain from $x \in D \backslash \partial D$, exactly as in Section 2,

$$\mathbb{E}(F_+(h(X_{n+1})) | X_n = x) = \mathbb{E}_x F_+(h(X_1)) \geq F_+(h(X)) - 1. \tag{6.4}$$

On the other hand, if $X_n = x \in \partial D$, then $X_{n+1} \in \partial^2 D$, and thus $h(X_n)$, $h(X_{n+1}) \geq h_1$ and $F_+(h(X_{n+1})) = F_+(h(X_n))$, so (6.4) holds in this case too. Consequently, $Y_n := F_+(h(X_n)) + n$ is a submartingale, and as in Section 2

$$\mathbb{E}_{x_0} \tau = \mathbb{E}_{x_0} Y_\tau \geq \mathbb{E}_{x_0} Y_0 = F_+(h(x_0)),$$

which is the left hand inequality of (6.1).

For an upper bound, we assume that $x \neq z$. The argument in Section 2 works for $x \in D \backslash B$, and we obtain

$$\mathbb{E}_x F_-(h(X_1)) \leq F_-(h(x)) - 1, \qquad x \in D \backslash B. \tag{6.5}$$

For $x \in B \backslash \partial D$, the same argument yields only, using $F_-'' \le 0$,

$$\mathbb{E}_x F_-(h(X_1)) \le F_-(h(x)), \qquad x \in B \backslash \partial D. \tag{6.6}$$

Finally, if $x \in \partial D$, then $h(x), h(X_1) \ge h_1$ for every $X_1 \sim x$, and

$$F_-(h(X_1)) = F_-(h(x)), \qquad x \in \partial D. \tag{6.7}$$

We define $N_n := \#\{k < n : X_k \in B\}$ and find from (6.5)–(6.7) that if $Y_n^* := F_-(h(X_n)) + n - N_n$, then $Y_{n\wedge\tau}^*$, $n \ge 0$, is a supermartingale and thus

$$\mathbb{E}_{x_0} \tau - \mathbb{E}_{x_0} N_\tau = \mathbb{E}_{x_0} Y_\tau^* \le Y_0^* = F_-(h(x_0)),$$

which completes the proof of (6.1). □

6.1. **Application.** Take $\mathcal{V} = \mathbb{Z}^2$ with edges between vertices at distance 1. Consider simple random walk on the disk $D = \{x \in \mathbb{Z}^2 : |x| \le R\}$. Let $z = 0 \in D$ and take h and $g_\pm$ as in (3.1) and (3.3). Then by (3.4) and (6.3),

$$\begin{aligned} F_\pm(x_0) &= \int_0^{h_1} \frac{2(u \wedge h(x_0))}{g_\pm(u)} \boldsymbol{d}u \\ &= \int_0^{h_1} 4(u \wedge h(x_0))(e^{2(u-b)} \pm O(e^u)) \boldsymbol{d}u \\ &= \big[2(u \wedge h(x_0)) e^{2(u-b)}\big]_0^{h_1} - \int_0^{h(x_0)\wedge h_1} 2e^{2(u-b)} + O\Big(\int_0^{h_1} h(x_0) e^u \boldsymbol{d}u\Big) \\ &= 2(h(x_0) \wedge h_1) e^{2(h_1 - b)} - e^{2(h(x_0)\wedge h_1 - b)} + 1 + O(h(x_0) e^{h_1}), \end{aligned}$$

where $b = \gamma_e + \frac{3}{2}\log 2$. By (3.1),

$$\begin{aligned} h_1 &= \log(R + O(1)) + b + O(R^{-2}) = \log R + b + O(R^{-1}), \\ h(x_0) &= \log|x_0| + b + O(|x_0|^{-2}). \end{aligned}$$

Thus

$$\begin{aligned} F_\pm(x_0) &= 2\big(h(x_0) + O(R^{-1})\big) e^{2\log R + O(R^{-1})} - e^{2\log|x_0| + O(|x_0|^{-2} + R^{-1})} + 1 + O(R \log|x_0|) \\ &= 2R^2 h(x_0) + O(R\log R) - |x_0|^2. \end{aligned}$$

Further, it is easily seen that $x \in B$ implies $|x| = R - O(1)$, and thus $\mu(B) = O(R)$. Also, it is easy to see that for x_0 in D we have

$$\mathcal{R}(x_0 \leftrightarrow 0) = O(\log R)\,.$$

(This follows, e.g., from the method of random paths [6] by picking a uniform point u on the chord bisecting the segment $x_0 z$, and considering the lattice path closest to the union of the segments $x_0 u$ and uz.) Thus (6.2) yields

$$\Delta = O(\mathbb{E}_{x_0} \tau_*/R) = O(R \cdot \mathcal{R}(x_0 \leftrightarrow 0)) = O(R\log R).$$

Consequently, Theorem 6.1 yields for the hitting time τ of the origin, that

$$\mathbb{E}_{x_0} \tau = 2R^2 h(x_0) + O(R\log R) - |x_0|^2.$$

For $|x_0| \ge R^{1/2}$ say, we can write this as

$$\mathbb{E}_{x_0} \tau = 2R^2 \log|x_0| + 2bR^2 - |x_0|^2 + O(R\log R).$$

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