

# On Symmetry of Uniform and Preferential Attachment Graphs

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**Abstract.** Motivated by the problem of graph structure compression under realistic source models, we study the symmetry behavior of preferential and uniform attachment graphs. These are two dynamic models of network growth in which new nodes attach to a constant number  $m$  of existing ones according to some attachment scheme. We prove symmetry results for  $m = 1$  and  $2$ , and we conjecture that for  $m \geq 3$ , both models yield asymmetry with high probability. We provide new empirical evidence in terms of graph defect. We also prove that vertex defects in the uniform attachment model grow at most logarithmically with graph size, then use this to prove a weak asymmetry result for all values of  $m$  in the uniform attachment model. Finally, we introduce a natural variation of the two models that incorporates preference of new nodes for nodes of a similar age, and we show that the change introduces symmetry for all values of  $m$ .

**Keywords:** preferential attachment, symmetry, automorphism, random graphs

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## 1 Introduction

Study of the asymptotic behavior of the symmetries of random graphs, originally motivated by combinatorial problems, has relatively recently found a new application in the problem of compression of graph structures. The basic problem can be formulated as follows: given a probability distribution on labeled graphs, determine an encoding of graph structures (that is, *unlabeled* graphs) so as to minimize expected description length.

Choi and Szpankowski (2012) studied this problem in the setting of Erdős-Rényi graphs. They showed that, under any distribution giving equal probability to isomorphic graphs, the entropy of the induced distribution on graph structures (i.e., isomorphism classes of graphs) is less than the entropy of the original distribution by an amount proportional to the expected logarithm of the number of automorphisms. Thus the solution to the above problem is intimately connected with the symmetries of the random graph model under consideration.

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Study of symmetries is further motivated by their connection to various measures of information contained in a graph structure. For instance, the *topological entropy* of a random graph, studied by Rashevsky (1955) and Trucco (1956), measures the uncertainty in the orbit class (i.e., the set of nodes having the same long-term neighborhood structure) of a node chosen uniformly at random from the node set of the graph. If the graph is asymmetric with high probability, then the topological entropy is maximized: if  $n$  is the size of the graph, then the topological entropy is, to leading order,  $\log n$ . In general, if the symmetries of the graph can be characterized precisely, then so can the topological entropy. Furthermore, tools developed here will allow us to study and compare topological information of nodes (i.e., by how many bits a graph view of one node differs from another).

The present paper is a first step toward the goal of extending graph structure compression results to other random graph models. In particular, many real-world graphs, such as biological and social networks, exhibit a power law degree distribution (see Durrett (2006)). To explain this phenomenon, Albert and Barabási (2002) proposed the *preferential attachment* mechanism, in which a graph is built one vertex at a time, and each new vertex  $t$  attaches to a given old vertex  $v$  with probability proportional to the current degree of  $v$ . Thus, we study a variant of a preferential attachment model. The primary problem appears to be difficult, so we also study a closely related model in which attachment is uniform, in the hope that the proof techniques used there may be generalized. In both uniform and preferential attachment models, we prove that when each new vertex chooses only one previous vertex as a neighbor, there is symmetry with high probability, and when each new vertex makes two choices, there is a positive probability of symmetry. In addition, we determine the asymptotic behavior of a quantity known as the *defect* of a vertex, introduced by Kim et al. (2002), which measures the extent to which the neighborhood of the vertex contributes to asymmetry of the graph. We then use this to prove a weak asymmetry result for the uniform attachment case.

We also introduce the *sliding window* model, a dynamic model in which new vertices choose neighbors from within windows of expected size uniformly bounded above by a constant, the purpose being to exhibit a “natural” mechanism that, coupled with a quite general attachment scheme that includes preferential and uniform attachment as special cases, results in symmetry with asymptotically positive probability.

Study of the asymptotic behavior of the automorphism group of a random graph started with Erdős and Rényi (1963), wherein Erdős-Rényi graphs with constant connection probability were shown to be asymmetric with high probability, a result motivated by the combinatorial question of determining the asymptotics of the number of unlabeled graphs on  $n$  vertices for  $n \rightarrow \infty$ . A similar question motivated the investigation of symmetry properties of random regular graphs by Bollobás (1982) and Kim et al. (2002). In the latter paper, the authors precisely characterized the range for which Erdős-Rényi graphs are asymmetric by proving concentration results for random variables defined in terms of vertex defect. They then proved an asymmetry result for random regular graphs using the previous result.

For general models, symmetry and asymmetry results can be nontrivial to prove, due to the non-monotonicity of the properties considered. Furthermore, the particular models considered here present more difficulties not seen in the Erdős-Rényi case: there is significant dependence between edge events, and graph sparseness makes derivation of concentration results difficult.

The rest of the paper is organized as follows: in Section 2, we formally state the models and the main problem; we then state the main results, along with a discussion of their significance. We also present some empirical validation of the symmetry results, as well as evidence in support of the asymmetry conjecture. Finally, in Section 3, we give sketches of some proofs.

## 2 Main Results

In this section, we state the main problems, introduce the models that we consider, and formulate the main results. First, we introduce some standard graph-theoretic terminology and notation. We start with the notion of structure-preserving transformations between labeled graphs: given two graphs  $G_1$  and  $G_2$  with vertex sets  $V(G_1)$  and  $V(G_2)$ , a mapping  $\phi : V(G_1) \rightarrow V(G_2)$  is said to be an *isomorphism* if it is bijective and preserves edge relations; that is, for any  $x, y \in V(G_1)$ , there is an edge between  $x$  and  $y$  if and only if there is an edge in  $G_2$  between  $\phi(x)$  and  $\phi(y)$ . When such a  $\phi$  exists,  $G_1$  and  $G_2$  are said to be *isomorphic*; that is, they have the same structure.

An isomorphism from the vertices of a graph  $G$  to itself is called an *automorphism* or *symmetry*. The set of automorphisms of  $G$ , together with the operation of function composition, forms a group, which is called the *automorphism group* of  $G$ , denoted by  $\text{Aut}(G)$ . Note that the image of  $G$  under any of its symmetries is  $G$ , the same labeled graph.

We then say that  $G$  is *symmetric* if it has at least one *nontrivial symmetry* and that  $G$  is *asymmetric* if the only symmetry of  $G$  is the identity permutation. Intuitively,  $G$  is symmetric if and only if there are at least two vertices whose graph perspectives are the same; that is, their neighborhoods at any distance have the same structure.

The main problem can then be stated as follows: given a random graph process  $\{G_n\}_{n \geq 1}$ , characterize the behavior of its automorphism group for  $n \rightarrow \infty$ .

### 2.1 Definitions of Models

In what follows, vertices of an  $n$ -vertex graph are the elements of the set  $[n] := \{1, 2, \dots, n\}$ .

A preferential attachment model is a dynamic model of network growth in which new vertices, when they choose vertices already in the graph as neighbors, have a preference for a given vertex that is proportional to its current degree, see Albert and Barabási (2002). Thus, nodes with high degree tend to be preferred for new connections. The following definition formalizes this. (A slightly different formalization of the Barabási–Albert model is given by Bollobás and Riordan (2004).)

**Definition 1 (Preferential attachment model)** *A preferential attachment graph  $\mathcal{P}(n, m)$  on  $n$  vertices, with parameter  $m$ , is constructed as follows: at time  $t = 1$ , a single vertex with name 1 and attractiveness  $\text{att}_1(1) = 0$  is added. For each time  $1 < t \leq n$ , a vertex with name  $t$  is added, and  $m$  vertices  $c_{t1}, \dots, c_{tm}$  in  $[t - 1]$  are chosen with replacement such that*

$$\Pr[c_{tj} = v] = \frac{\text{att}_v(t-1)}{\sum_{w=1}^{t-1} \text{att}_w(t-1)} = \frac{\text{att}_v(t-1)}{2m(t-1)}.$$

(Here we adopt the convention that  $0/0 = 1$ .) An edge between  $t$  and  $v$  is added if and only if  $c_{ti} = v$  for some  $i$ . For each  $v \in [t - 1]$ , we set  $\text{att}_v(t) = \text{att}_v(t - 1) + |\{j | c_{tj} = v\}|$ . Finally, we set  $\text{att}_t(t) = m$ .

Another way to express this is to first construct a growing multigraph, where we at each step add one new vertex and  $m$  edges from it, with the other endpoints chosen at random with replacement as above; then  $\text{att}_v(t)$  equals the degree of  $v$  at time  $t$ . We then reduce any set of multiple edges to a single edge to obtain the simple graph  $\mathcal{P}(n, m)$ .

We will also consider a variation, which we call the *uniform attachment* model, with the only change being that vertex choices are now equiprobable; that is, we now fix  $\text{att}_v(t-1) = 1$  for all  $t$  and  $v \in [t-1]$ . (For  $m = 1$  this yields the well-known *random recursive tree* (see Smythe and Mahmoud (1995)).) The

rationale for studying this simpler model is that solving our symmetry problems poses many of the same challenges for both models, but we need not deal with the dependence resulting from the preferential attachment mechanism.

We also study another practical variant of the attachment model called the sliding window model that we define next.

**Definition 2 (Sliding window model)** *The sliding window model with random window size works as follows: at time 1, vertex 1 is added. At time  $t > 1$ , vertex  $t$  is added, and a window size  $W_t$ , taking values in  $\{2, \dots, t-1\}$ , is chosen according to the distribution function  $F_t$ , independent of anything else. Then,  $m$  vertices are chosen with replacement from the set  $[t - W_t, t - 1]$  (which we call the window of vertex  $t$ ), with the distribution of each choice  $c_{tj}$  determined by the ratio of the attractiveness of any node in the window to the total attractiveness of the window. That is,*

$$\Pr[c_{tj} = v | W_t = w] = \frac{\text{att}_v(t-1)}{\sum_{k=t-w}^{t-1} \text{att}_k(t-1)}.$$

Here,  $\text{att}_v(x)$  denotes the attractiveness of vertex  $v$  at time  $x$ . In the preferential attachment sliding window model,  $\text{att}_v(x)$  is given by

$$\text{att}_v(x) = m + \sum_{k=1}^{x-1} \sum_{j=1}^m [c_{kj} = v].$$

In the uniform attachment version,  $\text{att}_v(x) = 1$ .

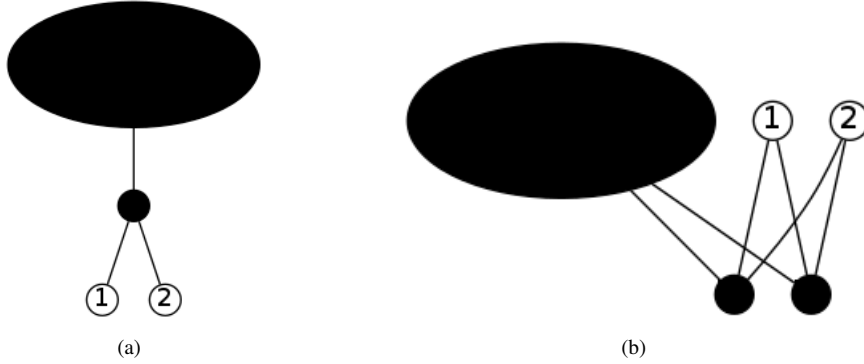
## 2.2 Statement of Results

The first result characterizes the expected vertex defect for the uniform attachment model. Vertex, permutation, and graph defect were introduced by Kim et al. (2002) in order to prove asymmetry for Erdős-Rényi graphs. The definitions are as follows.

**Definition 3 (Defect)** *Fix a graph  $G$  on  $n$  vertices. Given a permutation  $\pi \in S_n$  and  $u \in [n]$ , we define the defect of  $u$  with respect to  $\pi$  to be  $D_\pi(u) = |N(\pi(u)) \Delta \pi(N(u))|$ , where  $N(x)$  denotes the set of neighbors of vertex  $x$ , and  $\Delta$  denotes the symmetric difference of two sets. We define the defect of  $\pi$  to be  $D_\pi(G) = \max_{u \in [n]} D_\pi(u)$ . Finally, we define the defect of  $G$  to be  $D(G) = \min_{\pi \neq \text{ID}} D(\pi)$ .*

Some simple consequences of these definitions are as follows: for a graph  $G$  on  $n$  vertices and a permutation  $\pi \in [n]$ ,  $\pi$  is an automorphism of  $G$  if and only if  $D_\pi(G) = 0$ , which is equivalent to non-existence of a vertex  $u \in [n]$  such that  $D_\pi(u) \neq 0$ ;  $G$  has nontrivial symmetries if and only if  $D(G) = 0$ . Thus, vertex defect measures the extent to which a given vertex's neighborhood structure breaks the symmetry of  $\pi$ , permutation defect measures the number of edges adjacent to any particular vertex that need to be modified in order to make  $\pi$  a symmetry of  $G$ , and graph defect measures the number of edges adjacent to any vertex that need to be modified in order to introduce a nontrivial symmetry into  $G$ .

For a nontrivial permutation  $\pi \in S_n$  and any  $u \in [n]$ , define  $\omega(\pi, u) = \min\{u, \pi(u)\}$ . We also define  $\omega(\pi)$  to be the minimum vertex not fixed by  $\pi$  (where we say that  $\pi$  fixes a vertex  $w$  if  $\pi(w) = w$ ).



**Fig. 1:** The shapes on which we focus for the proof of Theorem 3.

**Theorem 1 (Expected defect for a vertex)** Fix  $m \in \mathbb{N}$  in the uniform attachment model. For any  $n$  sufficiently large,  $\pi \neq \text{ID}$ ,  $\pi \in S_n$ , and  $u \in [n]$  not fixed by  $\pi$ ,

$$\log \left( \frac{n}{\max\{\omega(\pi, u) + 2, (2m + 2)\}} \right) \leq \mathbb{E}[D_\pi(u)] \leq 1 + 4m \left( 2 + \log \left( \frac{n}{\omega(\pi, u)} \right) \right).$$

This theorem is significant for two reasons: it plays a central role in the proof of Theorem 2, and it gives an indication that an approach to an asymmetry proof via defects, as used in the setting of Erdős-Rényi graphs (see Kim et al. (2002)), may not be fruitful. A key difference between the Erdős-Rényi model and the uniform and preferential attachment ones is that the expected defect in the former is  $\Theta(np(1-p))$  for  $p, 1-p \gg \frac{\log n}{n}$ , which is essential for the proof technique used for that model to work.

The previous theorem can be used to derive a *weak asymmetry* result for the uniform attachment model as follows: for a given sequence of permutations  $\pi_n \neq \text{ID}$ , to show that  $\pi_n \notin \text{Aut}(G_n)$  with high probability, it is sufficient to exhibit a sequence of vertices  $u_n$  such that  $\lim_{n \rightarrow \infty} \Pr[D_{\pi_n}(u_n) = 0] = 1$ . In particular, we can choose  $u_n = \omega(\pi_n)$ , the minimum non-fixed vertex of  $\pi_n$ . We prove the following result.

**Theorem 2 (Probability of vertex defect being 0)** Fix  $m \geq 1$  and consider a sequence of graphs in the uniform attachment model  $G_n \sim \mathcal{U}(n, m)$ . Let  $\{\pi_n\}_{n=1}^\infty, \pi_n \in S_n - \{\text{ID}\}$ , and, for each  $n$ , let  $u_n = \omega(\pi_n)$ . Then

$$\Pr[D_{\pi_n}(u_n) = 0] \xrightarrow{n \rightarrow \infty} 0,$$

so that the asymptotic probability that  $\pi_n \in \text{Aut}(G_n)$  is 0.

Observe that Theorem 2 does not prove asymmetry of a uniform attachment model. For this we would need to prove that the *graph defect*  $D(G) > 0$  whp. However, we are able to make some statements about symmetry/asymmetry of this model. We discuss it next.

In the case  $m = 1$ , both the uniform and preferential attachment process yield trees. In such trees, we find  $\Theta(n)$  leaves with high probability, so that the probability of vertex  $n$  choosing a parent of a leaf, thereby forming a pair of sibling leaves which may be swapped (which some authors have called a *cherry* as shown in Figure 1(a)), is positive. Results on random recursive trees allow this argument to be strengthened to symmetry with high probability in the uniform attachment case.

The case  $m = 2$  is midway between the high-probability symmetry of the  $m = 1$  case and the conjectured asymmetry of the  $m \geq 3$  case. Examining the asymptotic probability of two vertices making the same choices and being unchosen by subsequent vertices yields the following results as shown in Figure 1(b).

**Theorem 3 (Symmetry results for  $m = 1, 2$ )** Fix  $m = 1, 2$ , and let  $G_n \sim \mathcal{U}(n, m)$  or  $G_n \sim \mathcal{P}(n, m)$ . Then there exists a constant  $C > 0$  such that, for  $n$  sufficiently large,

$$\Pr[|\text{Aut}(G_n)| > 1] > C.$$

For the uniform attachment model, the result for  $m = 1$  can be strengthened to symmetry with high probability.

We conjecture that symmetry with high probability also holds in the preferential attachment model for  $m = 1$  and that, for  $m = 2$ , in both models,  $\Pr[|\text{Aut}(G_n)| > 1]$  converges to a constant strictly less than 1.

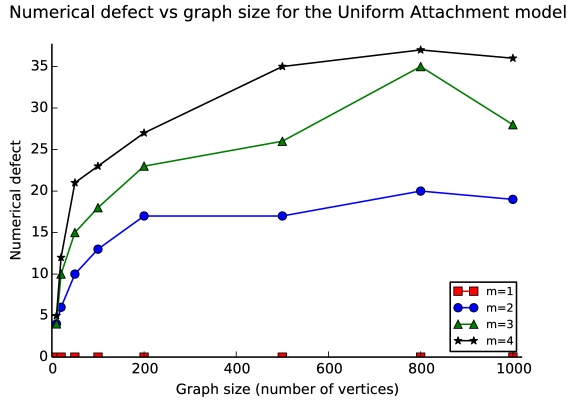
The result for  $m = 2$  is particularly interesting in light of the fact that empirical investigations of the symmetries of  $\mathcal{U}(n, 2)$  graphs with insufficiently many samples may lead to the incorrect conclusion that there is asymmetry with high probability in this case.

For fixed  $m \geq 3$ , we propose the following conjecture.

**Conjecture 1 (Asymmetry conjecture)** Fix  $m \geq 3$  and let  $G_n \sim \mathcal{U}(n, m)$  or  $G_n \sim \mathcal{P}(n, m)$ . Then

$$\Pr[|\text{Aut}(G_n)| > 1] \xrightarrow{n \rightarrow \infty} 0.$$

That is, graphs drawn according to the specified distributions are asymmetric with high probability.



**Fig. 2:** Plots showing minimum/maximum vertex defects for certain classes of permutations.

Empirical evidence in support of this conjecture abounds. For instance, MacArthur and Anderson (2006) give plots of number of automorphisms as  $n$  increases for sampled graphs, which show initial increase and then swift decay to 1. We contribute defect-based evidence here.

Figure 2 shows growth of a graph defect estimate as  $n$ , the number of vertices of the sampled graphs, grows large, for a few values of  $m$ . As only a small subset of permutations could be sampled due to time and space constraints, the pictured defect estimates only give upper bounds on the true defects. For  $m = 1$ , the estimate quickly drops to 0, due to the presence of automorphisms that are swaps of two vertices, as the proof of Theorem 3 indicates. For  $m = 2$ , the estimate grows away from 0, but this does not give a complete picture of the situation in this case: it fails to capture the phenomenon of symmetry with asymptotically nonzero (but quite small) probability predicted by Theorem 3. For  $m \geq 3$ , the graph defect exhibits logarithmic growth, which is in keeping with the statement of Theorem 1. Furthermore, since the defect grows away from 0, the evidence is in keeping with the asymmetry conjecture (though the small permutation sample size prevents us from claiming it as strong evidence of the conjecture).

We also give some weak supporting evidence in the form of a theorem about probability of automorphism group membership for sequences of permutations, that of Theorem 2.

Finally, we discuss the sliding window model, that could naturally capture the behavior of dynamic networks in which new nodes are very unlikely to attach to old ones, but whose attachment policy is otherwise quite general (for example, one may think of a social network whose nodes are people admitted to a university and whose edges represent friendships formed after admission, and then, except with small probability, nodes will choose neighbors in a window of bounded size.). The next result deals with symmetry in the sliding window model. If windows are restricted to be of expected length less than a constant bound, then considering the event that nodes  $n - 1$  and  $n$  form a cherry shows that symmetry results with nonzero probability.

**Theorem 4 (Symmetry results for sliding window model)** *In the sliding window model with random window size, for any  $m$ , if there exists a constant  $w$  such that  $\mathbb{E}[W_i] \leq w$  for all  $i$ , then the probability of symmetry is asymptotically positive. If there exists  $w$  such that, for all  $i$ ,  $W_i \leq w$  with probability 1, then a graph drawn according to this distribution is symmetric with high probability.*

### 3 Some Proofs

In this section, we fix some useful notation, then give proofs only of Theorems 1 and 3 leaving the other proofs to the full version of the paper. Given two vertices  $u$  and  $v$ , we write  $E[u, v]$  for the event that there is an (undirected) edge between  $u$  and  $v$ . For vertex  $u \in [n]$  and permutation  $\pi$ , we can write the defect  $D_\pi(u)$  as:

$$D_\pi(u) = \sum_{v=1}^n B_{u,\pi}(v),$$

where we define  $B_{u,\pi}(v)$  to be 1 if  $v \in N(\pi(u)) \Delta \pi(N(u))$  and 0 otherwise. We can express each such indicator in terms of edge events:

$$B_{u,\pi}(v) = (E[v, \pi(u)] \cap \neg E[\pi^{-1}(v), u]) \cup (\neg E[v, \pi(u)] \cap E[\pi^{-1}(v), u]).$$

Note that the two conjunctions are disjoint.

#### 3.1 Proof of Theorem 1

We now assume uniform attachment model. First, we state some useful lemmas about probabilities of edge events. We omit the simple proofs.

**Lemma 1** For all  $i, q, j, r$  such that  $i < j$  and  $q < r$ , if  $j < r$ , then  $\Pr[E[i, j]] > \Pr[E[q, r]]$ .

**Lemma 2** For all  $i < j$ ,

$$\frac{1}{j} \leq \Pr[E[i, j]] \leq \frac{2m}{j}.$$

**Lemma 3** For all  $i, q, j, r$  such that  $i < j$ ,  $q < r$ ,  $r \geq 2m + 1$ , and either  $i \neq q$  or  $j \neq r$ ,

$$\Pr[E[i, j] \cap \neg E[q, r]] \geq \Pr[E[i, j] \cap E[q, r]].$$

**Lemma 4** For all  $x \geq 1$ ,  $\sum_{i=1}^x \Pr[E[i, x]] \leq 2m$ .

**Lemma 5 (Harmonic Sum Log Sandwich)** For all  $n$  and  $j \in \mathbb{Z}$  such that  $1 \leq j \leq n$ ,

$$\log \frac{n}{j} \leq \sum_{i=j}^n \frac{1}{i} \leq \frac{1}{j} + \log \frac{n}{j}.$$

Now we move on to the proof of the main result. Throughout, we assume that  $u < \pi(u)$ ; the case  $u > \pi(u)$  follows from this by noting that  $D_{\pi^{-1}}(\pi(u)) = D_{\pi}(u)$ . First, we derive the lower bound. We start by lower bounding the probability of event  $B_{u, \pi}(i)$  by the probability of an edge. For any vertex  $i$  such that  $\pi^{-1}(i) \geq 2m + 1$  and  $i \neq u, \pi(u)$  (so all but a constant number of them), we have

$$\begin{aligned} \Pr[B_{u, \pi}(i)] &= \Pr[E[i, \pi(u)] \cap \neg E[\pi^{-1}(i), u]] + \Pr[\neg E[i, \pi(u)] \cap E[\pi^{-1}(i), u]] \\ &\stackrel{(a)}{\geq} \Pr[E[i, \pi(u)] \cap E[\pi^{-1}(i), u]] + \Pr[E[\pi^{-1}(i), u] \cap \neg E[i, \pi(u)]] \\ &= \Pr[E[\pi^{-1}(i), u]]. \end{aligned}$$

Here, (a) is a result of Lemma 3 since  $\max\{\pi^{-1}(i), u\} \geq 2m + 1$ . Hence,

$$\begin{aligned} \sum_{i=1}^n \Pr[B_{u, \pi}(i)] &\geq \sum_{\pi^{-1}(i) = \max\{\omega(\pi, u) + 1, (2m+1)\}}^n \Pr[B_{u, \pi}(i)] \\ &\stackrel{(a)}{\geq} \sum_{\pi^{-1}(u) \neq \pi^{-1}(i) = \max\{\omega(\pi, u) + 1, (2m+1)\}}^n \Pr[E[\pi^{-1}(i), u]] \\ &\stackrel{(b)}{\geq} \sum_{\pi^{-1}(u) \neq \pi^{-1}(i) = \max\{\omega(\pi, u) + 1, (2m+1)\}}^n \frac{1}{\pi^{-1}(i)} \\ &\stackrel{(c)}{\geq} \log \left( \frac{n}{\max\{\omega(\pi, u) + 2, (2m+2)\}} \right), \end{aligned}$$

where (a) is a consequence of the previous inequality, (b) is an invocation of Lemma 2, and (c) is a result of Lemma 5 and the observation that, if  $\pi^{-1}(u) \geq \max\{\omega(\pi, u) + 1, 2m + 1\}$ , then its contribution to the sum is  $\frac{1}{\pi^{-1}(u)} \leq \frac{1}{\max\{\omega(\pi, u) + 1, 2m + 1\}}$ . This completes the proof of the lower bound.

Now we prove the upper bound. We start by upper bounding the probability of  $B_{u, \pi}(i)$ :

$$\begin{aligned} \Pr[B_{u, \pi}(i)] &= \Pr[E[i, \pi(u)] \cap \neg E[\pi^{-1}(i), u]] + \Pr[\neg E[i, \pi(u)] \cap E[\pi^{-1}(i), u]] \\ &\stackrel{(a)}{\leq} \Pr[E[i, \pi(u)]] + \Pr[E[\pi^{-1}(i), u]], \end{aligned}$$



where (a) is a consequence of two applications of monotonicity of probabilities. Hence

$$\begin{aligned}
\sum_{i=1}^n \Pr[B_{u,\pi}(i)] &\stackrel{(a)}{\leq} \sum_{i=1}^n \Pr[E[i, \pi(u)]] + \sum_{\pi^{-1}(i)=1}^n \Pr[E[\pi^{-1}(i), u]] \\
&\stackrel{(b)}{\leq} 1 + 2 \sum_{\pi^{-1}(i)=1}^n \Pr[E[\pi^{-1}(i), u]] \\
&\leq 1 + 2 \sum_{\pi^{-1}(i)=1}^{\omega(\pi,u)} \Pr[E[\pi^{-1}(i), u]] + 2 \sum_{\pi^{-1}(i)=\omega(\pi,u)}^n \Pr[E[\pi^{-1}(i), u]] \\
&\stackrel{(c)}{\leq} 1 + 4m + 2 \sum_{\pi^{-1}(i)=\omega(\pi,u)}^n \Pr[E[\pi^{-1}(i), u]],
\end{aligned}$$

where (a) follows from the previous inequality, (b) from Lemma 1, and (c) from Lemma 4. The justification for (b) is slightly more complicated: it follows from the fact that,

$$\Pr[E[i, \pi(u)]] \leq \Pr[E[i, u]], \quad i \neq u,$$

which can be seen as follows: for  $u \neq i < \pi(u)$ , it follows from Lemma 1. If  $i = \pi(u)$ , then the left-hand side is 0, so the inequality holds. Finally, if  $i > \pi(u) > u$ , then the two probabilities are equal, due to the uniformity of the attachment process. For the case  $i = u$ , the inequality fails, and we instead upper bound that term by 1.

Thus, we can upper bound some more:

$$\sum_{\pi^{-1}(i)=\omega(\pi,u)}^n \Pr[E[\pi^{-1}(i), u]] \stackrel{(a)}{\leq} 2m \sum_{\pi^{-1}(i)=\omega(\pi,u)}^n \frac{1}{\pi^{-1}(i)} \stackrel{(b)}{\leq} 2m \left( 1 + \log \frac{n}{\omega(\pi,u)} \right),$$

where (a) follows from Lemma 2 and (b) from Lemma 5. This completes the proof.

### 3.2 Proof of Theorem 3

CASE  $m = 1$ .

In order to prove positive probability of symmetry, we examine the probability of the  $n$ th node resulting in at least one cherry after making its choice of parent. To bound this probability below, we start by conditioning on the event that, after node  $n - 1$  has been added and its choice is made, there are at least  $Cn$  leaves, for an appropriately chosen constant  $C$ . This happens with high probability in both the uniform and preferential attachment model. Now, we split into three cases: there are no cherries, there is exactly one cherry, and there are at least two cherries.

In the case of no cherries, there are exactly as many parents of leaves as there are leaves, so that there are at least  $Cn$  leaf parents. In order to form a cherry, node  $n$  must choose a parent of a leaf, which happens with asymptotically positive probability: each leaf parent has degree exactly 2, so that the probability that  $n$  chooses such a node is at least

$$\frac{2Cn}{2(n-1)} \sim C$$

in the case of preferential attachment and

$$\frac{Cn}{n-1} \sim C$$

in the case of uniform attachment. In the case where there is exactly one cherry, the only way in which  $G_n$  can contain no cherries is by  $n$  choosing one of the leaves of the cherry. These leaves have total attractiveness 2 (in either model), so that the conditional probability that  $n$  destroys the cherry is at most  $\frac{2}{(n-1)}$ , which implies that  $G_n$  contains a cherry with conditional probability at least  $1 - \frac{2}{n-1}$ . In the final case, in which there are at least two cherries, the addition of node  $n$  cannot destroy more than one cherry, so that, with conditional probability 1, a cherry exists after  $n$  makes its choice. Putting everything together proves the positive probability claim.

In the uniform attachment case (i.e., for a random recursive tree), it follows from Example 3.2 of Aldous (1991), see also Theorem 1 of Feng and Mahmoud (2010), that the number of cherries with high probability is linear in  $n$ ; in particular, with high probability there is at least one cherry and thus a symmetry.

CASE  $m = 2$

We will show that, with positive probability, in both models, there is at least one *diamond* (i.e., a pair of nodes that choose the same parents and that are not chosen by any subsequent nodes) as shown in Figure 1(b). The details are technically more intricate than in the  $m = 1$  case, and the argument there does not work in this case, because node  $n$  must choose, from a set of size  $\Theta(n^2)$  (pairs of vertices), one of  $O(n)$  elements (previously chosen pairs). We thus rely on a birthday paradox-style argument to show that there is a positive probability of two vertices making the same choices, then condition on the lexicographically smallest such pair to complete the proof.

Let  $A(u, v)$  be the event that vertices  $u$  and  $v$  choose the same pair of parents, and let  $B(u, v)$  be the event that  $u$  and  $v$  are both unchosen. Now, define  $N(k)$  to be the number of pairs  $u, v$  of vertices such that  $k < u < v$  and  $A(u, v)$  and  $B(u, v)$  simultaneously hold. Define  $N_A(k)$  and  $N_B(k)$  analogously for pairs for which events  $A$  and  $B$  hold, respectively. Finally, denote by  $S_{>x}$  the set  $\{k \in [n] \mid k > x\}$ . We then aim to prove that  $\Pr[N(0) > 0] > C > 0$  for some constant  $C$  and  $n$  large enough. For any  $x$ , we have

$$\Pr[N(x) > 0] = \Pr[N(x) > 0 \cap N_A(x) > 0] = \Pr[N(x) > 0 \mid N_A(x) > 0] \cdot \Pr[N_A(x) > 0],$$

where the first equality is from the fact that  $[N(x) > 0] \subseteq [N(x)_A > 0]$ . The goal now is bound the remaining probabilities below by positive constants. We do this in the next two lemmas, which hold for both uniform and preferential attachment graphs. We will prove them in the uniform case, then explain the modifications needed to extend them to the preferential case.

**Lemma 6 (Probability of two vertices picking the same pair)** *There exists a positive constant  $C$  such that*

$$\Pr[N_A(n/2) > 0] > C$$

for all  $n$  sufficiently large.

**Proof:** To show this, we will instead compute  $\Pr[N_A(n/2) = 0]$  and bound it above by a constant less

than 1. The condition  $N_A(n/2) = 0$  means that all vertices  $> n/2$  choose distinct pairs. This is given by

$$\begin{aligned} \Pr[N_A(n/2) = 0] &= \prod_{k=1}^{n/2} \left(1 - \frac{k-1}{\left(\frac{n}{2} + k - 1\right)^2}\right) \\ &\leq \prod_{k=1}^{n/2} \left(1 - \frac{k-1}{n^2}\right) \leq \prod_{k=n/4+1}^{n/2} \left(1 - \frac{k-1}{n^2}\right) \\ &\leq \prod_{k=n/4+1}^{n/2} \left(1 - \frac{n}{4n^2}\right) = \left(1 - \frac{(1/4)}{n}\right)^{\frac{n}{4}} \xrightarrow{n \rightarrow \infty} e^{-\frac{1}{16}} < 1. \end{aligned}$$

In the preferential attachment case, the proof is similar, except that we apply the fact that the attractiveness of any vertex  $v < t$  at time  $t$  is at least  $\frac{m}{2m(t-1)} = \frac{1}{2(t-1)}$ .  $\square$

**Lemma 7 (Conditional probability of two vertices with the same neighborhood)** *There exists a positive constant  $C$  such that*

$$\Pr[N(n/2) > 0 | N_A(n/2) > 0] > C$$

for all  $n$  sufficiently large.

**Proof:** We condition on the lexicographically smallest pair  $X$  from  $S_{>n/2}$  such that  $A(X)$  holds. Let  $D(u, v)$  be the event that the pair  $(u, v)$  is the smallest pair from  $S_{>n/2}$  for which  $A$  holds. Then

$$\begin{aligned} \Pr[N(n/2) > 0 | N_A(n/2) > 0] &= \sum_{u < v \in S_{>n/2}} \Pr[N(n/2) > 0 | D(u, v)] \Pr[D(u, v) | N_A(n/2) > 0] \\ &\geq \sum_{u < v \in S_{>n/2}} \Pr[B(u, v) | D(u, v)] \Pr[D(u, v) | N_A(n/2) > 0] \\ &\geq C \sum_{u < v \in S_{>n/2}} \Pr[D(u, v) | N_A(n/2) > 0] = C. \end{aligned}$$

Here, the equalities are simply due to the law of total probability, and the first inequality is because the event  $B(u, v)$  is a subset of the event  $N(n/2) > 0$ . The second inequality is by direct computation. Note first that  $D(u, v)$  means that all lexicographically smaller pairs choose distinct pairs, and  $u$  and  $v$  choose the same pair (so that  $v$  cannot choose  $u$ ). So

$$\begin{aligned} \Pr[B(u, v) | D(u, v)] &= \prod_{j=u+1}^v \Pr[j \text{ avoids } u | D(u, v)] \prod_{j=v+1}^n \Pr[j \text{ avoids } v, u | D(u, v)] \\ &\geq \prod_{j=u+1}^{v-1} \left(\frac{(j-1)^2 - 2j - u}{(j-1)^2}\right) \cdot 1 \cdot \prod_{j=v+1}^n \left(\frac{(j-1)^2 - 4j - 2u}{(j-1)^2}\right) \\ &\geq \prod_{j=u+1}^{v-1} \left(1 - \frac{c}{n}\right) \prod_{j=v+1}^n \left(1 - \frac{c}{n}\right) \\ &\geq \left(1 - \frac{c}{n}\right)^n, \end{aligned}$$

where  $c > 0$  is some constant. Here, the first inequality results from bounding the numerators below by giving upper bounds for the number of pairs that vertex  $j$  must avoid in order to avoid  $u$  and  $v$  and for the number of pairs that  $j$  must avoid in order to pick a pair that is distinct from the choices of all vertices  $x$  such that  $(x, j)$  is lexicographically smaller than  $(u, v)$ . The 1 between the products is from the fact that  $v$  avoids  $u$  with probability 1, due to the conditioning by  $D(u, v)$ . The second inequality holds for all  $n$  sufficiently large, since  $j > \frac{n}{2}$ . The last inequality is because all factors are bounded above by 1. Finally, by taking  $n$  sufficiently large, the last value can be made arbitrarily close to  $e^{-c}$ . The proof in the preferential case is again similar, relying on the previously stated lower bound on vertex attractiveness.  $\square$

## References

- R. Albert and A.-L. Barabási. Statistical mechanics of complex networks. *Rev. Mod. Phys.*, 74:47–97, 2002.
- D. Aldous. Asymptotic fringe distributions for general families of random trees. *Annals of Applied Probability*, 1(2):228–266, 1991.
- B. Bollobás. The asymptotic number of unlabelled regular graphs. *Journal of the London Mathematical Society*, 26:201–206, 1982.
- B. Bollobás and O. Riordan. The diameter of a scale-free random graph. *Combinatorica*, 24(1):5–34, 2004. ISSN 0209-9683. doi: 10.1007/s00493-004-0002-2.
- Y. Choi and W. Szpankowski. Compression of graphical structures: Fundamental limits, algorithms, and experiments. *Information Theory, IEEE Transactions on*, 58(2):620–638, 2012. ISSN 0018-9448. doi: 10.1109/TIT.2011.2173710.
- R. Durrett. *Random Graph Dynamics (Cambridge Series in Statistical and Probabilistic Mathematics)*. Cambridge University Press, New York, NY, USA, 2006. ISBN 0521866561.
- P. Erdős and A. Rényi. Asymmetric graphs. *Acta Math. Acad. Sci. Hungar.*, 14:295–315, 1963.
- Q. Feng and H. M. Mahmoud. On the variety of shapes on the fringe of a random recursive tree. *Journal of Applied Probability*, 47:191–200, 2010. doi: 10.1239/jap/1269610825.
- J. H. Kim, B. Sudakov, and V. H. Vu. On the asymmetry of random regular graphs and random graphs. *Random Structures & Algorithms*, 21(3-4):216–224, 2002. ISSN 1098-2418. doi: 10.1002/rsa.10054.
- B. D. MacArthur and J. W. Anderson. Symmetry and Self-Organization in Complex Systems. *eprint arXiv:cond-mat/0609274*, Sept. 2006.
- N. Rashevsky. Life, information theory, and topology. *The bulletin of mathematical biophysics*, 17(3): 229–235, 1955. ISSN 0007-4985. doi: 10.1007/BF02477860.
- R. T. Smythe and H. M. Mahmoud. A survey of recursive trees. *Theory Prob. Math. Statist.*, 51:1–27, 1995.
- E. Trucco. A note on the information content of graphs. *The bulletin of mathematical biophysics*, 18(2): 129–135, 1956. ISSN 0007-4985. doi: 10.1007/BF02477836.