

GRAPHONS AND CUT METRIC ON σ -FINITE MEASURE SPACES

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ABSTRACT. Borgs, Chayes, Cohn and Holden (2016+) recently extended the definition of graphons from probability spaces to arbitrary σ -finite measure spaces, in order to study limits of sparse graphs. They also extended the definition of the cut metric, and proved various results on the resulting metric space.

We continue this line of research and give various further results on graphons and the cut metric in this general setting, extending known results for the standard case of graphons on probability spaces. In particular, we characterize pairs of equivalent graphons, and we give new results on completeness and compactness.

1. INTRODUCTION

The theory of graph limits and graphons has become a successful tool to study large dense graphs. First, any sequence of graphs, with orders tending to infinity, has at least a subsequence that converges to a *graph limit*, which can be represented (non-uniquely) by a *graphon*, which in this context is a $[0, 1]$ -valued symmetric function defined on $S \times S$ where S is a probability space (that often is taken to be $[0, 1]$). Secondly, any such graphon W defines a sequence of random graphs $G(n, W)$, which gives a large family of dense random graphs with different properties. See e.g. Lovász and Szegedy [23], Borgs, Chayes, Lovász, Sós and Vesztergombi [7, 8], Austin [2], Diaconis and Janson [13] and Lovász [22].

There have been several partial extensions of the theory to sparse graphs, using more general graphons. Bollobás and Riordan [3] considered graphons that are bounded (but not necessarily $[0, 1]$ -valued), and this was extended by Borgs, Chayes, Cohn and Zhao [5] to unbounded graphons, assuming that the graphons are integrable (and usually in L^p for some $p > 1$). These papers also consider signed graphons (in connection with weighted graphs where the weights may be negative).

Another leap in increasing generality was taken by Veitch and Roy [26] and Borgs, Chayes, Cohn and Holden [4], with some special cases studied by Caron and Fox [9] and Herlau, Schmidt and Mørup [17]; the new idea is to let the graphons be defined on $S \times S$ for an arbitrary σ -finite measure space S (and not just a probability space, as earlier); it turns out that without loss of generality, the measure space S can be taken to be \mathbb{R}_+ with Lebesgue measure [4, Proposition 2.8]. (Only this case is considered in [26].) The graphons in [26] and [4] are mainly $[0, 1]$ -valued and generate random graphs

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by the construction described in Section 3.2 below; however, [4] considers also unbounded and signed graphons (that may occur as limits of weighted graphs). (The version of the construction in [26] also includes additional stars and isolated edges; we do not treat these parts in the present paper.)

Veitch and Roy [26] is focussed on properties of the resulting random graphs, and in particular the fact that, as a consequence of results by Kallenberg [19, 21], all random graphs that are exchangeable in a certain sense can be obtained in this way. Borgs, Chayes, Cohn and Holden [4] contains related results on exchangeable random graphs, and also many results on convergence of graphs and graphons in the cut metric δ_{\square} , as well as some results for the related metrics δ_1 and δ_p .

The present paper is mainly inspired by Borgs, Chayes, Cohn and Holden [4], and gives various further results on convergence in the cut metric for (unbounded, possibly signed) graphons defined on σ -finite measure spaces. We also give some related results for the metrics δ_1 and δ_p . The results should be compared to the corresponding results for standard graphons on probability spaces in [18].

Sections 2–4 contain definitions, some earlier results and other preliminaries.

Section 5 extends a result by Bollobás and Riordan [3] to the present generality and shows that for Borel spaces, the infimum in the definition of the cut distance is attained (Theorem 5.1). This leads to a characterisation (Theorem 5.3) of equivalent graphons on such spaces as having a pair of pullbacks that are a.e. equal, also extending a result by [3], and another, completely general, characterisation of equivalence (Theorem 5.6) as being generated by pull-backs and trivial extensions, extending [18, Theorem 8.3]. Several consequences of the latter characterisation are also given.

Section 6 gives results on completeness of sets for the cut metric (Theorem 6.6), after some preliminary results for the cut norm. Several counterexamples are also given, illustrating the conditions in the theorem; the set of all graphons is, unfortunately, not complete.

Sections 7–8 give results on (relative) compactness in the cut metric that extend and improve results in [4]. We give a complete characterisation of totally bounded sets (Theorem 7.3); however, since we do not have a complete characterisation of complete sets, we do not obtain a complete characterisation of (relatively) compact sets of graphons without adding extra conditions (for example Theorems 7.5–7.7).

Remark 1.1. The present paper thus studies the cut metric for graphons on σ -finite measure spaces. Since graphs may be represented by graphons, this includes results on convergence of graphs to graphons in this sense, see Section 3.1. Note, however, that the cut metric is only one of several conceivable metrics (or other ways of defining limits), see e.g. [7], [8], [3], [4], [27]. In the standard case of $[0, 1]$ -valued graphons on probability spaces, a number of different metrics and topologies are equivalent, basically because they define compact topologies that are comparable and thus equal, see e.g. [7; 8]. In extensions like the one treated here, compactness is lost, and there is no reason to expect various notions to be equivalent, although there are some partial results under extra assumptions, see e.g. [3]. On

the contrary, there are counter examples, see for example [4, Proposition 2.24(iv)], showing that different notions of convergence are not equivalent.

The cut metric has been hugely successful in the standard setting, but it is not at all clear that it is of equal importance in extensions like the one studied here. (For one thing, the fact that the metric is not complete on the space of graphons studied here, see Section 6, is a warning that the definitions may be not optimal. Moreover, [26] considers also some non-integrable graphons, although the definition of the cut metric requires integrability.) Nevertheless, the present paper considers exclusively the cut metric (and the related δ_1 and δ_p), hoping that this will inspire future studies of other metrics and modes of convergence for general graphons and (sparse) graphs.

2. DEFINITIONS AND NOTATION

We follow Borgs, Chayes, Cohn and Holden [4], with minor variations in the notation. For the readers convenience, and to set our notation, we repeat the basic definitions in this section. See [4] for further details and references, and see also [18] for further details in the (standard) special case of probability spaces.

For any topological space S , $\mathcal{B} = \mathcal{B}(S)$ denotes the Borel σ -algebra on S . λ denotes the Lebesgue measure on \mathbb{R} .

A *measure space* is, as usual, a triple (S, \mathcal{F}, μ) , where S is a set, \mathcal{F} a σ -algebra on S and μ a (non-negative) measure on (S, \mathcal{F}) . We shall often omit \mathcal{F} and μ from the notation when they are clear from the context and denote the measure space just by S . (In contrast to [4] which is more careful with the notation.) In particular, we let $\mathbb{R}_+ := [0, \infty)$ denote the measure space $(\mathbb{R}_+, \mathcal{B}, \lambda)$, and similarly for $[0, 1]$ and other intervals $[0, a]$ and $[0, a)$ with $0 < a \leq \infty$.

A *subspace* of a measure space (S, \mathcal{F}, μ) is a measure space $(A, \mathcal{F}_A, \mu_A)$, where A is a measurable subset of S , $\mathcal{F}_A = \{B \in \mathcal{F} : B \subseteq A\}$ and μ_A is the restriction of μ to \mathcal{F}_A .

If $f_1 : S_1 \rightarrow \mathbb{R}$ and $f_2 : S_2 \rightarrow \mathbb{R}$ are two functions, then $f_1 \otimes f_2 : S_1 \times S_2 \rightarrow \mathbb{R}$ is the function $f_1 \otimes f_2(x, y) := f_1(x)f_2(y)$.

2.1. Graphons. A *graphon* $W = (W, S) = (W, S, \mathcal{F}, \mu)$ is a symmetric integrable function $W : S \times S \rightarrow \mathbb{R}$, where $S = (S, \mathcal{F}, \mu)$ is a σ -finite measure space. The space S , its σ -algebra \mathcal{F} and its measure μ are important components of the graphon, but for convenience we often omit them from the notation. (Again, [4] is more careful.) We generally identify two graphons that are equal a.e.

Note that in the present paper, as in [4], in general, a graphon is neither required to be bounded nor non-negative. Note also that we assume our graphons to be integrable, as in [4] (with minor exceptions, see [4, Remarks 2.3 and 2.25]), while [26] allows for somewhat more general graphons, see [26, Theorem 4.9].

We repeat for emphasis that the essential feature of [4] and the present paper is that μ is allowed to be any σ -finite measure, and that the standard theory in e.g. [7], [22], [18] is the special case when μ is a probability measure.

A *trivial extension* of a graphon (W, S, μ) is a graphon $(\widetilde{W}, \widetilde{S}, \widetilde{\mu})$ such that the measure space (S, μ) is a subspace of $(\widetilde{S}, \widetilde{\mu})$ and

$$\widetilde{W}(x, y) = \begin{cases} W(x, y), & x, y \in S, \\ 0, & \text{otherwise.} \end{cases} \quad (2.1)$$

Remark 2.1. We assume, following [4], that the measure space where a graphon is defined is σ -finite. This is mainly because the standard construction of product measures such as $\mu \times \mu$ assumes μ to be σ -finite, since there are serious technical problems otherwise. (For example, Fubini's theorem may fail, see e.g. [10, Exercise 5.2.1].) Nevertheless, it is possible to consider more general measure spaces, provided we only consider W that vanish outside $S_1 \times S_1$ for some σ -finite subset S_1 (which is reasonable since W should be integrable); then W is a trivial extension of its restriction to S_1 . We shall not treat this rather trivial extension of the definition in general and leave it to the reader, but note that an example of a non- σ -finite measure space occurs in the proof of Theorem 5.1 below.

2.2. Cut Norm. If (S, \mathcal{F}, μ) is a σ -finite measure space and $F \in L^1(S \times S, \mu \times \mu)$, then the *cut norm* of F is defined by

$$\|F\|_{\square} := \sup_{T, U} \left| \int_{T \times U} F(x, y) \, d\mu(x) \, d\mu(y) \right|, \quad (2.2)$$

taking the supremum over all measurable $T, U \subseteq S$. We use also notations such as $\|F\|_{\square, S}$ or $\|F\|_{\square, S, \mu}$. Note that

$$\|F\|_{\square} \leq \|F\|_{L^1(S \times S)}. \quad (2.3)$$

It is easily verified that all properties in [18, Section 4 and Appendix E.1–E.2] hold also in the σ -finite case studied here. (This includes other, equivalent, versions of the cut norm.) In particular, for any $F \in L^1(S \times S)$,

$$\|F\|_{\square} = 0 \iff F = 0 \quad (\mu \times \mu)\text{-a.e.} \quad (2.4)$$

Moreover,

$$\|F\|_{\square, S, \mu} = \left| \sup_{g, h} \int_{S \times S} F(x, y) g(x) h(y) \, d\mu(x) \, d\mu(y) \right| \quad (2.5)$$

with the supremum taken over all measurable functions $g, h : S \rightarrow [0, 1]$. As a consequence, for any bounded $f_1, f_2 : S \rightarrow \mathbb{R}_+$, cf. [18, (4.5)],

$$\|f_1(x) f_2(y) F(x, y)\|_{\square} \leq \|f_1\|_{L^\infty} \|f_2\|_{L^\infty} \|F\|_{\square}. \quad (2.6)$$

2.3. Measure-preserving maps and couplings. If φ is a function $S_1 \rightarrow S_2$, we define for any functions f on S_2 and W on S_2^2 , the *pull-backs* $f^\varphi(x) := f(\varphi(x))$ and $W^\varphi(x, y) := W(\varphi(x), \varphi(y))$; these are functions on S_1 and S_1^2 , respectively.

Similarly, if $\varphi : S_1 \rightarrow S_2$ is measurable, for two measurable spaces (S_i, \mathcal{F}_i) , and μ is a measure on (S_1, \mathcal{F}_1) , then the *push-forward* of μ is the measure μ^φ on (S_2, \mathcal{F}_2) defined by $\mu^\varphi(A) := \mu(\varphi^{-1}(A))$. Note that $\int_{S_1} f^\varphi \, d\mu = \int_{S_2} f \, d\mu^\varphi$ for any measurable function f on S_2 and measure μ_1 on S_1 such

that one of the integrals is defined (finite or $+\infty$). Similarly, if $W \in L^1(S_2 \times S_2)$, then $\int_{S_1^2} W^\varphi d\mu^2 = \int_{S_2^2} W d(\mu^\varphi)^2$ and

$$\|W^\varphi\|_{\square, S_1, \mu} = \|W\|_{\square, S_2, \mu^\varphi}. \quad (2.7)$$

A map $\varphi : (S_1, \mathcal{F}_1, \mu_1) \rightarrow (S_2, \mathcal{F}_2, \mu_2)$ is *measure-preserving* if it is measurable and $\mu_1^\varphi = \mu_2$. Note that all properties in [18, Section 5] hold also in the σ -finite case studied here.

A *coupling* of two measure spaces $(S_1, \mathcal{F}_1, \mu_1)$ and $(S_2, \mathcal{F}_2, \mu_2)$ is a pair (φ_1, φ_2) of measure-preserving maps $\varphi_i : S \rightarrow S_i$ defined on a common measure space (S, \mathcal{F}, μ) . We consider in this paper only the σ -finite case. (Note that S automatically is σ -finite if S_1 or S_2 is.) An important special case is when $S = S_1 \times S_2$ and $\varphi_i = \pi_i$, the projection of $S_1 \times S_2$ onto S_i , $i = 1, 2$; we call such couplings *special*. In this case μ is thus a measure on $S_1 \times S_2$ such that $\mu^{\pi_i} = \mu_i$; we call such a measure μ a *coupling measure* of μ_1 and μ_2 .

If (φ_1, φ_2) is a general coupling of S_1 and S_2 with $\varphi_i : S \rightarrow S_i$, then $\varphi := (\varphi_1, \varphi_2)$ is a measurable map $S \rightarrow S_1 \times S_2$, and the push-forward measure μ^φ is a coupling measure of μ_1 and μ_2 . Using this, it is easy to see that it suffices to consider special couplings in, for example, (2.8), (2.11) and (2.13) below. (In fact, [4] consider only special couplings.)

Note that a coupling of S_1 and S_2 exists only if $\mu_1(S_1) = \mu_2(S_2)$; in that case there always exist coupling measures, see [4, Lemma 3.2].

2.4. The cut metric and equivalence. The cut metric $\delta_\square(W_1, W_2)$ for two graphons W_1, W_2 , possibly defined on different spaces, is defined by [4] in two steps:

- (i) If $\mu_1(S_1) = \mu_2(S_2)$, then (as in the standard case of probability spaces, see e.g. [7; 22; 18])

$$\delta_\square(W_1, W_2) := \inf_{(\varphi_1, \varphi_2)} \|W_1^{\varphi_1} - W_2^{\varphi_2}\|_\square, \quad (2.8)$$

taking the infimum over all couplings (φ_1, φ_2) of S_1 and S_2 (or, as in [4], only over special couplings).

- (ii) In general, take trivial extensions $(\widetilde{W}_i, \widetilde{S}_i, \widetilde{\mu}_i)$ of (W_i, S_i, μ_i) such that $\widetilde{\mu}_1(\widetilde{S}_1) = \widetilde{\mu}_2(\widetilde{S}_2)$ and define $\delta_\square(W_1, W_2) := \delta_\square(\widetilde{W}_1, \widetilde{W}_2)$.

It is shown in [4] that this is well-defined, and that the cut metric satisfies the triangle inequality and thus is a pseudo-metric.

Remark 2.2. By (2.5), for a special coupling with coupling measure μ we have explicitly

$$\|W_1^{\pi_1} - W_2^{\pi_2}\|_\square = \sup_{f, g} \left| \int_{(S_1 \times S_2)^2} (W_1(x_1, y_1) - W_2(x_2, y_2)) \times f(x_1, x_2) g(y_1, y_2) d\mu(x_1, x_2) d\mu(y_1, y_2) \right|, \quad (2.9)$$

taking the supremum over measurable $f, g : S_1 \times S_2 \rightarrow [0, 1]$.

Two graphons W_1 and W_2 are *equivalent* if $\delta_\square(W_1, W_2) = 0$; in this case we write $W_1 \cong W_2$. (This is sometimes called 'weakly equivalent'.) Since δ_\square is a pseudo-metric, \cong is an equivalence relation, and δ_\square is a metric

on the set of equivalence classes. When we talk about metric properties such as completeness and compactness for δ_\square , this should be interpreted as properties in the metric space of equivalence classes, but for convenience, we usually talk about graphons rather than equivalence classes.

Note that if \widetilde{W} is a pull-back W^φ or a trivial extension of a graphon W , then $\widetilde{W} \cong W$.

We shall repeatedly use the following propositions shown in [4]:

Proposition 2.3 ([4, Proposition 2.8]). *Every graphon is equivalent to a graphon defined on the space $[0, \infty)$.*

(Just as a graphon on a probability space is equivalent to a graphon on $[0, 1]$, see e.g. [18, Section 7].)

Proposition 2.4 ([4, Proposition 4.3(c)]). *If W_1 and W_2 are graphons defined on \mathbb{R}_+ , then*

$$\delta_\square(W_1, W_2) := \inf_{\varphi} \|W_1 - W_2^\varphi\|_\square, \quad (2.10)$$

taking the infimum over all measure-preserving bijections $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$.

(In other words, in this case, the infimum in (2.8) can be restricted to couplings with φ_1 the identity and φ_2 a bijection.)

Remark 2.5. We sometimes allow ourselves to talk about the set of all graphons, ignoring the technical set-theoretical fact that strictly speaking the graphons, as defined in this paper, form a class and not a set. This can when necessary be circumvented by the standard method of restricting the allowed measure spaces S to some sufficiently large set. In particular, note that by Proposition 2.3, the equivalence classes of graphons form a set.

2.5. The invariant L^1 and L^p metrics δ_1 and δ_p . The invariant L^1 -metric $\delta_1(W_1, W_2)$ is defined by [4] in the same way as the cut metric, replacing (2.8) in Case (i), i.e. when $\mu_1(S_1) = \mu_2(S_2)$, by

$$\delta_1(W_1, W_2) := \inf_{(\varphi_1, \varphi_2)} \|W_1^{\varphi_1} - W_2^{\varphi_2}\|_{L^1}, \quad (2.11)$$

and again using trivial extensions as in (ii) above for the general case. It is shown in [4] that this too is well-defined, and a quasi-metric. Note that (2.3) implies

$$\delta_\square(W_1, W_2) \leq \delta_1(W_1, W_2). \quad (2.12)$$

Moreover, [4] more generally defines the invariant L^p -metric $\delta_p(W_1, W_2)$, where $1 \leq p < \infty$, in the same way: when $\mu_1(S_1) = \mu_2(S_2)$,

$$\delta_p(W_1, W_2) := \inf_{(\varphi_1, \varphi_2)} \|W_1^{\varphi_1} - W_2^{\varphi_2}\|_{L^p}, \quad (2.13)$$

and in general trivial extensions are used as in (ii) above. However, for $p > 1$ we consider only graphons that satisfy

$$W_i \in L^1(\mu_i \times \mu_i) \cap L^p(\mu_i \times \mu_i) \quad \text{and} \quad W_i \geq 0; \quad (2.14)$$

for such graphons, [4] shows that δ_p is well-defined and a quasi-metric.

Remark 2.6. To understand the conditions (2.14), first recall that we, and [4], assume that a graphon is integrable, i.e., belongs to L^1 . (As said in Section 1, [26] allows somewhat more general graphons, see also [4, Remark 2.3], but it seems that the cut distance cannot be defined for them.) Secondly, taking, for example, $W_2 = 0$, (2.13) yields $\delta_p(W_1, 0) = \|W_1\|_{L^p}$, so we have to assume $W_1, W_2 \in L^p$ in order to have $\delta_p(W_1, W_2)$ finite in general; conversely, if $W_1, W_2 \in L^p$ then (2.13) yields $\delta_p(W_1, W_2) \leq \|W_1\|_{L^p} + \|W_2\|_{L^p} < \infty$, so δ_p is finite. The third condition, $W_i \geq 0$, is perhaps more surprising, but it is used in the proof in [4] that δ_p is invariant under trivial extensions, and it is, in fact, necessary for this when $p > 1$, see Example 2.7.

Example 2.7 (for signed graphons, δ_p is in general not invariant under trivial extensions). Let $W_1 = 1$ and $W_2 = -1$, on the one-point set $\mathcal{S} = \{1\}$ with measure $\mu\{1\} = 1$. Let \widetilde{W}_1 and \widetilde{W}_2 be the trivial extensions to $\widetilde{\mathcal{S}} = \{1, 2\}$, with $\widetilde{\mu}\{1\} = \widetilde{\mu}\{2\} = 1$. Then $\delta_p(W_1, W_2) = \|W_1 - W_2\|_p = 2$ but, letting $\sigma : \widetilde{\mathcal{S}} \rightarrow \widetilde{\mathcal{S}}$ denote the transposition $\sigma(1) = 2, \sigma(2) = 1$,

$$\delta_p(\widetilde{W}_1, \widetilde{W}_2) \leq \|\widetilde{W}_1 - \widetilde{W}_2^\sigma\|_{L^p} = 2^{1/p} < \delta_p(W_1, W_2). \quad (2.15)$$

(In fact, equality holds, since there are only two special couplings.) Hence, without the positivity condition in (2.14), δ_p is not preserved by trivial extensions.

As just said, δ_1 and δ_p (when defined) are quasi-metrics. Moreover, as will be shown in Theorems 5.6 and 5.7,

$$\delta_\square(W_1, W_2) = 0 \iff \delta_1(W_1, W_2) = 0 \iff \delta_p(W_1, W_2) = 0, \quad (2.16)$$

with the final equivalence assuming that $p > 1$ and (2.14) holds. Hence, the equivalence $W_1 \cong W_2$ is also characterised by $\delta_1(W_1, W_2) = 0$, and when (2.14) holds, by $\delta_p(W_1, W_2) = 0$. Consequently δ_1 is a metric on the set of equivalence classes of graphons, and δ_p is a metric on the set of equivalence classes of non-negative graphons in $L^p \cap L^1$.

Furthermore, by (2.12), convergence in δ_1 implies convergence in δ_\square . However, this fails for δ_p with $p > 1$, see Example 2.8.

Example 2.8 (convergence in δ_p with $p > 1$ does not imply convergence in cut norm). Let $W_n := n^{-2} \mathbf{1}_{[0, n] \times [0, n]}$ on \mathbb{R}_+ . Then, for any $p > 1$,

$$\delta_p(W_n, 0) \leq \|W_n\|_{L^p} = n^{-2(1-1/p)} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.17)$$

However, $\delta_\square(W_n, 0) = \inf_\varphi \|W_n - 0^\varphi\|_\square = \|W_n\|_\square = \|W_n\|_{L^1} = 1$ for every n . Thus convergence in δ_p does not imply convergence in cut norm for any $p > 1$. (This is in contrast to the case of graphons on probability spaces, where $\delta_p \geq \delta_1 \geq \delta_\square$ because $\|\cdot\|_{L^p} \geq \|\cdot\|_{L^1}$.)

2.6. Stretched graphons and the stretched metrics $\delta_\square^s, \delta_1^s, \delta_p^s$. Borgs, Chayes, Cohn and Holden [4] introduce also a new rescaling of graphons called *stretching*.

In general, given a graphon $W = (W, S, \mu)$ and $u > 0$, we define the rescaled graphon

$$\Upsilon_u^{(1)}W := (W, S, u^{1/2}\mu). \quad (2.18)$$

In other words, $\Upsilon_u^{(1)}W$ equals W as a function on S^2 , but we multiply the underlying measure by $u^{1/2}$.

In the special (and standard) case $(S, \mu) = (\mathbb{R}_+, \lambda)$, we can alternatively keep (S, μ) and define

$$\Upsilon_u^{(2)}W(x, y) := W(u^{-1/2}x, u^{-1/2}y). \quad (2.19)$$

It is easily seen that the two definitions are equivalent up to equivalence: $\Upsilon_u^{(1)}W \cong \Upsilon_u^{(2)}W$, since $\Upsilon_u^{(2)}W = (\Upsilon_u^{(1)}W)^\varphi$ where $\varphi : x \mapsto u^{-1/2}x$ is a measure-preserving map $(\mathbb{R}_+, \lambda) \rightarrow (\mathbb{R}_+, u^{1/2}\lambda)$. (It is the version (2.19) that motivates the name 'stretching'.) Therefore, the choice of version usually does not matter, and then we use the notation $\Upsilon_u W$ for any of $\Upsilon_u^{(1)}W$ and $\Upsilon_u^{(2)}W$ (when defined).

Note that (2.18) immediately implies

$$\|\Upsilon_u W\|_{L^1} = u\|W\|_{L^1}. \quad (2.20)$$

As a consequence, again following [4], we can normalize any non-zero graphon W to the *stretched graphon* W^s defined by

$$W^s := \Upsilon_{\|W\|_{L^1}^{-1}}(W) \quad (2.21)$$

with $\|W^s\|_{L^1} = 1$. For completeness, we also define $W^s = 0$ when $W = 0$ a.e.

Furthermore, [4] define the *stretched metric* δ_\square^s by

$$\delta_\square^s(W_1, W_2) := \delta_\square(W_1^s, W_2^s). \quad (2.22)$$

This is obviously a pseudo-metric on the set of all graphons, and thus a metric on the corresponding set of equivalence classes; moreover

$$\delta_\square^s(W_1, W_2) = 0 \iff W_2 \cong \Upsilon_u(W_1) \text{ for some } u > 0. \quad (2.23)$$

We can similarly define the pseudo-metrics $\delta_1^s(W_1, W_2) := \delta_1(W_1^s, W_2^s)$ and, for $p > 1$ and non-negative graphons in L^p , $\delta_p^s(W_1, W_2) := \delta_p(W_1^s, W_2^s)$. As a consequence of (2.16),

$$\delta_\square^s(W_1, W_2) = 0 \iff \delta_1^s(W_1, W_2) = 0 \iff \delta_p^s(W_1, W_2) = 0, \quad (2.24)$$

with the final equivalence holding for $p > 1$ and non-negative graphons in L^p .

3. GRAPHONS AND GRAPHS

Although the present paper is mainly about graphons, it should be remembered that the main motivation for studying graphons is the connection to (large) graphs. For the standard case of dense graphs and graphons on a probability space, see e.g. [23], [7], [8] and the book [22].

Two aspects of this connection are treated separately in the following subsections.

3.1. Graphons as limits of graphs. Given a finite graph G , the corresponding graphon W_G is defined by considering the vertex set $V(G)$ as a probability space, with the uniform measure, and defining W_G on $V(G)^2$ by $W_G(x, y) := \mathbf{1}\{x \sim y\}$. (This is just the adjacency matrix.) Alternatively, as is well-known, one can define an equivalent version of W_G on the standard space $[0, 1]$ by identifying the vertices of G with disjoint intervals of lengths $1/|V(G)|$, see e.g. [22], [18].

In the standard theory [22], one says that a sequence of graphs G_n converges to a graphon W if $|V(G_n)| \rightarrow \infty$ and

$$\delta_{\square}(W_{G_n}, W) \rightarrow 0. \quad (3.1)$$

In the case of sparse graphs, i.e., when the edge density $|E(G_n)|/|V(G_n)|^2 \rightarrow 0$, (3.1) just gives convergence to the graphon 0. In order to get interesting limits, Borgs, Chayes, Cohn and Holden [4] propose instead using stretched graphons (see Section 2.6 above) and thus the condition

$$\delta_{\square}^s(W_{G_n}, W) \rightarrow 0. \quad (3.2)$$

Given a graph G , we can also define a graphon \overline{W}_G by taking the same function W_G as above (i.e., the adjacency matrix) but consider it as a graphon defined on the measure space $V(G)$ with the counting measure (i.e., each point has measure 1). If G is a finite graph, then \overline{W}_G is a stretching of W_G , see (2.18), and thus by (2.23)

$$\delta_{\square}^s(W_G, \overline{W}_G) = 0. \quad (3.3)$$

Consequently, we can replace W_{G_n} by \overline{W}_{G_n} in (3.2). (We can also use any other stretching, for example $W_{G_n}^s$.) One technical advantage of \overline{W}_G is that it also is defined for countable infinite graphs G ; however, since we want our graphons to be integrable, we still have to assume that G has only a finite number of edges.

Remark 3.1. There is also another theory for sparse graphs due to Bollobás and Riordan [3] and further developed by Borgs, Chayes, Cohn and Zhao [5], where instead of stretching W_{G_n} , it is rescaled to $W_{G_n}/\|W_{G_n}\|_{L^1}$. As discussed in [4], it seems that the two theories have applications to different types of sparse graphs. We shall not consider the theory of [3] here.

3.2. Random graphs defined by graphons. In the standard theory for graphons on a probability space, there is a standard definition of a random graph $G(n, W)$ (with n vertices) for a given graphon W and any $n \geq 1$. Borgs, Chayes, Cohn and Holden [4] define a version of this for the present setting as follows. (Essentially the same construction is given by Veitch and Roy [26].)

Let W be a $[0, 1]$ -valued graphon on a measure space (S, μ) , and assume that W is not 0 a.e. Consider a Poisson point process Γ on $[0, \infty) \times S$, with intensity $\lambda \times \mu$. A realization of Γ is a countably infinite set of points $\{(t_i, x_i)\}$. Given such a realization, let $\tilde{G} = \tilde{G}(W)$ be the infinite graph with vertex set $\{(t_i, x_i)\}$, where two vertices (t_i, x_i) and (t_j, x_j) are connected by an edge with probability $W(x_i, x_j)$, independently of all other edges (conditionally on Γ).

Moreover, let $\tilde{G}_t = \tilde{G}_t(W)$ be the induced subgraph of \tilde{G} consisting of all vertices (t_i, x_i) with $t_i \leq t$. (It is useful to think of the parameter t_i as the time the vertex is born; then Γ_t is the subgraph existing at time t .)

Finally, we let $G_t = G_t(W)$ be the induced subgraph of \tilde{G}_t consisting of all non-isolated vertices. Note that the vertex set of \tilde{G}_t is a.s. infinite for every $t > 0$ if $\mu(S) = \infty$, but the expected number of edges is

$$\mathbb{E}|E(\tilde{G}_t)| = \mathbb{E}|E(G_t)| = \frac{1}{2}t^2 \iint W(x, y) d\mu(x) d\mu(y) < \infty, \quad (3.4)$$

so G_t is a.s. finite for every $t < \infty$.

Note also that the definition defines growing processes $(\tilde{G}_t)_{t \geq 0}$ and $(G_t)_{t \geq 0}$ of random graphs. (With $\tilde{G}_0 = G_0$ empty with no vertices.)

Remark 3.2. The graphs are usually regarded as unlabelled, so the identification (labelling) of the vertices by points in $[0, \infty) \times S$ is mainly for convenience. Some, equivalent, interesting alternative labellings are the following.

- (i) Since the measure μ is σ -finite, the coordinates t_i in the point process $\Gamma = \{(t_i, x_i)\}$ are a.s. distinct. Hence, we may just as well use t_i as the label, and let the vertex set of \tilde{G} be $\{t_i\}$. (With the edge probabilities still given by the x_i as above.) The random graph \tilde{G} then is exchangeable in the sense that its edge set is an exchangeable point process on \mathbb{R}_+^2 , see further Veitch and Roy [26], where this property is explored in depth.
- (ii) If the measure μ is atomless, then the coordinates x_i are also a.s. distinct, so we can use x_i as label and regard the vertex set of the random graphs defined above as (random) subsets of S . The vertex set of \tilde{G}_t then is a Poisson process on S with intensity $t\mu$. (If μ has atoms, then this vertex set generally has multiple points that have to be distinguished.)
- (iii) We may use an arbitrary measurable enumeration of the points in Γ or Γ_t as $\{(x_i, t_i)\}$, and then use i as the label; this means that the vertex set of \tilde{G} is \mathbb{N} . The vertex set of \tilde{G}_t is \mathbb{N} if $\mu(S) = \infty$ and a random finite set $\{1, \dots, N\}$ when $\mu(S) < \infty$, with $N \sim \text{Po}(t\mu(S))$.

Remark 3.3. Two stretched graphons define the same random graphs up to a change of time. In fact, if $u > 0$ then, by the definition above and (2.18), the random graphs $\tilde{G}_t(\Upsilon_u^{(1)}W)$ are constructed using a Poisson point process $\Gamma^{(u)}$ on $[0, \infty) \times S$ with intensity $\lambda \times u^{1/2}\mu$. The map $(t, x) \mapsto (u^{1/2}t, x)$ maps this to the Poisson process Γ with intensity $\lambda \times \mu$, and thus

$$\tilde{G}_t(\Upsilon_u W) \stackrel{d}{=} \tilde{G}_{u^{1/2}t}(W), \quad (3.5)$$

in the strong sense that both sides have the same distribution as processes on $\{t \geq 0\}$.

For the limit theory, we consider the corresponding graphons defined in Section 3.1. We have $\delta_{\square}^{\mathfrak{s}}(W_{G_t}, \overline{W}_{G_t}) = 0$ by (3.3). Moreover, since G_t is obtained from \tilde{G}_t by deleting isolated vertices, $\overline{W}_{\tilde{G}_t}$ is a trivial extension of \overline{W}_{G_t} , and thus $\delta_{\square}(\overline{W}_{G_t}, \overline{W}_{\tilde{G}_t}) = 0$, which implies $\delta_{\square}^{\mathfrak{s}}(\overline{W}_{G_t}, \overline{W}_{\tilde{G}_t}) = 0$, see (2.23). Hence, when using the stretched metric $\delta_{\square}^{\mathfrak{s}}$, it does not matter whether we use $\overline{W}_{\tilde{G}_t}$, \overline{W}_{G_t} or W_{G_t} .

Borgs, Chayes, Cohn and Holden [4] prove that the graphs $G_t(W)$ a.s. converge to W in the stretched metric $\delta_{\square}^{\mathfrak{s}}$, i.e., as $t \rightarrow \infty$,

$$\delta_{\square}^{\mathfrak{s}}(\overline{W}_{\tilde{G}_t}, W) = \delta_{\square}^{\mathfrak{s}}(\overline{W}_{G_t}, W) = \delta_{\square}^{\mathfrak{s}}(W_{G_t}, W) \rightarrow 0. \quad (3.6)$$

4. TOPOLOGICAL PRELIMINARIES

Although the definitions and main results are purely measure-theoretic and do not involve any topology, we shall use some topological notions in

some results and proofs. We use various standard results that can be found in several references; for convenience we give some specific references to [10].

A *Polish* space is a complete separable metric space. (Or, more generally, a topological space homeomorphic to such a space.)

A measurable space is *Borel* (also called *standard* [10] or *Lusin* [12]) if it is isomorphic to a Borel subset of a Polish space with its Borel σ -field. In fact, a Borel measurable space is either isomorphic to $[0, 1]$ (with the usual Borel σ -field) or countable (with every subset measurable). A measure space $(\Omega, \mathcal{F}, \mu)$ is *Borel* if (Ω, \mathcal{F}) is a Borel measurable space; equivalently, if the measure space is isomorphic to a Borel subset of a Polish space equipped with a Borel measure. See further [18, Appendix A.2].

Compact and *locally compact* spaces have the standard definitions. We consider only Hausdorff spaces; as said above, properties of the cut metric should be interpreted in the metric space of equivalence classes of graphons.

Second countable also has its standard definition, i.e., that the topology has a countable basis. Recall that a compact space is second countable if and only if it is metrizable [10, Proposition 7.1.12].

If K is a compact space, then $C(K)$ is the Banach space of continuous functions $K \rightarrow \mathbb{R}$. If X is a locally compact space, then $C_c(X)$ is the space of continuous functions $f : X \rightarrow \mathbb{R}$ with compact support

$$\text{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}. \quad (4.1)$$

Furthermore, we let $C_{[0,1]}(K)$ and $C_{c,[0,1]}(X)$ denote the subsets of functions with values in $[0, 1]$.

If X is locally compact and second countable, then a *Radon* measure on X is a Borel measure μ such that $\mu(K) < \infty$ for every compact $K \subseteq X$. A Radon measure μ defines a positive linear functional $f \mapsto \int_X f d\mu$ on $C_c(X)$, and by the Riesz representation theorem, this yields a one-to-one correspondence between Radon measures and positive linear functionals on $C_c(X)$ [10, Theorem 7.2.8]. (This extends to general locally compact spaces if one considers only measures that are *regular* [10, Section 7.2]; we only need this for second countable spaces, and then regularity is automatic [10, Proposition 7.2.3].)

Note that a locally compact second countable space is σ -compact. Hence a Radon measure on such a space is σ -finite.

We say that a sequence μ_n of Radon measures on a locally compact second countable space X converges *vaguely* to a Radon measure ν if $\int_X f d\mu_n \rightarrow \int_X f d\nu$ for every $f \in C_c(X)$; this is denoted by $\mu_n \xrightarrow{v} \nu$. (See [20, Theorem A2.3] for some properties of the vague topology.)

We shall use the following simple lemma. It is presumably well-known, but we have not found a reference so for completeness we include a proof. (We state it for one vaguely convergent sequence. The lemma and its proof generalize to two vaguely convergent sequences on two, possibly different, spaces X and Y ; this says that the product operation is vaguely continuous for Radon measures on locally compact second countable spaces.)

Lemma 4.1. *Let X be a locally compact second countable space, and let μ_n , $n = 1, 2, \dots, \infty$, be Radon measures on X such that $\mu_n \xrightarrow{v} \mu_\infty$ as $n \rightarrow \infty$. Then $\mu_n \times \mu_n \xrightarrow{v} \mu_\infty \times \mu_\infty$ on $X \times X$.*

Proof. Note that since the spaces are second countable, the Borel σ -algebra $\mathcal{B}(X \times X) = \mathcal{B}(X) \times \mathcal{B}(X)$.

Let $f \in C_c(X \times X)$. Then there exists a compact set $K \subseteq X$ such that $\text{supp}(f) \subseteq K \times K$. Let K_1 be a compact subset of X with $K \subseteq K_1^\circ$, the interior of K_1 . Then there exists a function $\psi \in C_{c,[0,1]}(X)$ such that $\text{supp}(\psi) \subseteq K_1$ and $\psi(x) = 1$ for $x \in K$.

The set of linear combinations $\sum_1^N g_i(x)h_i(y)$ with $g_i, h_i \in C(K_1)$ is dense in $C(K_1 \times K_1)$, for example as a consequence of the Stone–Weierstrass theorem [11, V.8.1]. Hence, given any $\varepsilon > 0$, there exists such a linear combination $f_\varepsilon(x, y) = \sum_1^N g_i(x)h_i(y)$ with $\sup_{K_1 \times K_1} |f(x, y) - f_\varepsilon(x, y)| < \varepsilon$. Let $r_\varepsilon := f - f_\varepsilon$. Since $f(x, y) \neq 0$ implies $x, y \in K$ and thus $\psi(x) = \psi(y) = 1$, it follows that for $x, y \in K_1$,

$$\begin{aligned} f(x, y) &= \psi(x)\psi(y)f(x, y) = \psi(x)\psi(y)f_\varepsilon(x, y) + \psi(x)\psi(y)r_\varepsilon(x, y) \\ &= \sum_1^N \tilde{g}_i(x)\tilde{h}_i(y) + \tilde{r}_\varepsilon(x, y), \end{aligned} \quad (4.2)$$

where $\tilde{g}_i(x) := \psi(x)g_i(x)$ and similarly for $\tilde{h}_i(x)$ and $\tilde{r}_\varepsilon(x, y)$. We extend \tilde{g}_i and \tilde{h}_i to X by letting them be 0 outside K_1 ; since $\text{supp}(\psi) \subseteq K_1$, then $\tilde{g}_i, \tilde{h}_i \in C_c(X)$. We similarly define $\tilde{r}_\varepsilon(x, y) = 0$ when $(x, y) \notin K_1 \times K_1$. Then (4.2) holds for all $x, y \in X$, with $|\tilde{r}_\varepsilon(x, y)| < \varepsilon$. Since $\mu_n \xrightarrow{v} \mu_\infty$, we have, as $n \rightarrow \infty$, for each $i \leq N$,

$$\begin{aligned} \int_{X \times X} \tilde{g}_i(x)\tilde{h}_i(y) d\mu_n(x) d\mu_n(y) &= \int_X \tilde{g}_i(x) d\mu_n(x) \int_X \tilde{h}_i(y) d\mu_n(y) \\ &\rightarrow \int_X \tilde{g}_i(x) d\mu_\infty(x) \int_X \tilde{h}_i(y) d\mu_\infty(y) = \int_{X \times X} \tilde{g}_i(x)\tilde{h}_i(y) d\mu_\infty(x) d\mu_\infty(y) \end{aligned}$$

and hence by (4.2),

$$\begin{aligned} \int_{X \times X} f(x, y) d\mu_n(x) d\mu_n(y) &- \int_{X \times X} f(x, y) d\mu_\infty(x) d\mu_\infty(y) \\ &= \int_{X \times X} \tilde{r}_\varepsilon(x, y) d\mu_n(x) d\mu_n(y) - \int_{X \times X} \tilde{r}_\varepsilon(x, y) d\mu_\infty(x) d\mu_\infty(y) + o(1). \end{aligned} \quad (4.3)$$

Furthermore, choose a function $\psi_1 \in C_c(X)$ such that $\psi_1(x) = 1$ when $x \in K_1$. Then $|\tilde{r}_\varepsilon(x, y)| \leq \varepsilon\psi_1(x)\psi_1(y)$, and thus

$$\begin{aligned} \left| \int_{X \times X} \tilde{r}_\varepsilon(x, y) d\mu_n(x) d\mu_n(y) \right| &\leq \int_{X \times X} \varepsilon\psi_1(x)\psi_1(y) d\mu_n(x) d\mu_n(y) \\ &= \varepsilon \left(\int_X \psi_1 d\mu_n \right)^2, \quad n \leq \infty. \end{aligned} \quad (4.4)$$

Moreover, $\int_X \psi_1 d\mu_n \rightarrow \int_X \psi_1 d\mu_\infty < \infty$, and thus there exists a constant M (independent of ε) such that $\int_X \psi_1 d\mu_n \leq M$ for all $n \leq \infty$. As a result, if R_n is the right-hand side of (4.3), then $|R_n| \leq 2M^2\varepsilon + o(1)$ and thus $\limsup_{n \rightarrow \infty} |R_n| \leq 2M^2\varepsilon$. Since ε is arbitrary, this yields $R_n \rightarrow 0$. Consequently, (4.3) shows that, as $n \rightarrow \infty$,

$$\begin{aligned} \int_{X \times X} f \, d(\mu_n \times \mu_n) &= \int_{X \times X} f(x, y) \, d\mu_n(x) \, d\mu_n(y) \\ &\rightarrow \int_{X \times X} f(x, y) \, d\mu_\infty(x) \, d\mu_\infty(y) = \int_{X \times X} f \, d(\mu_\infty \times \mu_\infty). \end{aligned} \quad (4.5)$$

Since $f \in C_c(X)$ is arbitrary, this shows $\mu_n \times \mu_n \xrightarrow{v} \mu_\infty \times \mu_\infty$. \square

Lemma 4.2. *Let μ be a finite Borel measure on a compact metric space X . Then, for any $F \in L^1(K \times K)$,*

$$\|F\|_{\square, K, \mu} = \sup_{g, h \in C_{[0,1]}(K)} \left| \int_{K \times K} F(x, y) g(x) h(y) \, d\mu(x) \, d\mu(y) \right|. \quad (4.6)$$

Proof. Temporarily denote the right-hand side of (4.6) by $\|F\|_{\square}$. Then, by (2.5), $\|F\|_{\square} \leq \|F\|_C$, so it suffices to prove the opposite inequality.

Suppose first that F is bounded, say $|F(x, y)| \leq M$ for some M . Let $T, U \subset S$ be measurable, and let $\varepsilon > 0$. Since $C(K)$ is dense in $L^1(K, \mu)$, see [10, Proposition 7.4.2 (and 7.2.3)], there exists a function $g \in C(K)$ such that $\|\mathbf{1}_T - g\|_{L^1} < \varepsilon$. By replacing $g(x)$ by $\min(1, \max(0, g(x)))$, we may further assume that $g \in C_{[0,1]}(K)$. Similarly, there exists $h \in C_{[0,1]}(K)$ such that $\|\mathbf{1}_U - h\|_{L^1} < \varepsilon$. Then

$$|F(x, y)(\mathbf{1}_T(x)\mathbf{1}_U(y) - g(x)h(y))| \leq M(|\mathbf{1}_T(x) - g(x)| + |\mathbf{1}_U(y) - h(y)|)$$

and thus

$$\int_{K \times K} |F(x, y)(\mathbf{1}_T(x)\mathbf{1}_U(y) - g(x)h(y))| \, d\mu(x) \, d\mu(y) \leq 2M\mu(K)\varepsilon. \quad (4.7)$$

Hence,

$$\begin{aligned} \left| \int_{T \times U} F(x, y) \, d\mu(x) \, d\mu(y) \right| &= \left| \int_{K \times K} F(x, y) \mathbf{1}_T(x) \mathbf{1}_U(y) \, d\mu(x) \, d\mu(y) \right| \\ &\leq \left| \int_{K \times K} F(x, y) g(x) h(y) \, d\mu(x) \, d\mu(y) \right| + 2M\mu(K)\varepsilon \\ &\leq \|F\|_C + 2M\mu(K)\varepsilon. \end{aligned}$$

Taking the supremum over all T and U and letting $\varepsilon \rightarrow 0$, we obtain $\|F\|_{\square} \leq \|F\|_C$, which completes the proof for bounded F .

For a general F , consider the truncations

$$F_M(x, y) := F(x, y) \mathbf{1}\{|F(x, y)| \leq M\}. \quad (4.8)$$

By the first part of the proof, $\|F_M\|_{\square} = \|F_M\|_C$, and thus

$$\begin{aligned} \|F\|_{\square} &\leq \|F_M\|_{\square} + \|F - F_M\|_{\square} = \|F_M\|_C + \|F - F_M\|_{\square} \\ &\leq \|F\|_C + \|F - F_M\|_C + \|F - F_M\|_{\square} \leq \|F\|_C + 2\|F - F_M\|_{L^1}. \end{aligned}$$

Furthermore, as $M \rightarrow \infty$, $\|F - F_M\|_{L^1} \rightarrow 0$. Consequently, $\|F\|_{\square} \leq \|F\|_C$, which completes the proof. \square

5. EQUIVALENCE

We first extend a result by Bollobás and Riordan [3, Lemma 2.6] to the present setting of σ -finite measure spaces.

Theorem 5.1. *Let (W_1, S_1) and (W_2, S_2) be graphons where $S_i = (S_i, \mathcal{F}_i, \mu_i)$ are σ -finite Borel spaces, $i = 1, 2$. Then there exist trivial extensions $(\widetilde{W}_i, \widetilde{S}_i)$ of (W_i, S_i) and a coupling (φ_1, φ_2) of \widetilde{S}_1 and \widetilde{S}_2 such that*

$$\delta_{\square}(W_1, W_2) = \delta_{\square}(\widetilde{W}_1, \widetilde{W}_2) = \|\widetilde{W}_1^{\varphi_1} - \widetilde{W}_2^{\varphi_2}\|_{\square}. \quad (5.1)$$

The coupling may be assumed to be special.

In other words, the infimum in (2.8) is attained for \widetilde{W}_1 and \widetilde{W}_2 . We shall see in Example 5.11 that in general it is necessary to take trivial extensions \widetilde{W}_1 and \widetilde{W}_2 , even if $\mu_1(S_1) = \mu_2(S_2)$, unlike the corresponding result for graphons on probability spaces in [3] (see also [18, Theorem 6.16]). Note also that the result is not true for arbitrary measure spaces, not even in the standard probability space case, see [6] for a counter example.

Proof. First, we note that we may assume that S_1 and S_2 are atomless. In general, we let $\widehat{S}_i := S_i \times [0, 1]$ and $\widehat{W}_i := W_i^{\pi_i}$, the pull-back to \widehat{S}_i . The proof below applies to $(\widehat{W}_i, \widehat{S}_i)$ and shows that there exist trivial extensions \check{W}_i to $\check{S}_i := \widehat{S}_i \cup E_i$ and a coupling (φ_1, φ_2) of \check{S}_1 and \check{S}_2 such that $\|\check{W}_1^{\varphi_1} - \check{W}_2^{\varphi_2}\|_{\square} = \delta_{\square}(\widehat{W}_1, \widehat{W}_2) = \delta_{\square}(W_1, W_2)$. Here $\varphi_i : S \rightarrow \check{S}_i$ for some measure space (S, μ) , and $E_i = (E_i, \nu_i)$ are some measure spaces with E_i disjoint from \widehat{S}_i . We may assume that E_i also is disjoint from S_i and define $\check{S}_i := S_i \cup E_i$. Let \widetilde{W}_i be the trivial extension of W_i to \check{S}_i . Define $\psi_i : \check{S}_i \rightarrow \widetilde{S}_i$ by $\psi_i = \pi_i$ on $S_i \times [0, 1]$, and ψ_i the identity on E_i . Then ψ_i is measure-preserving, and $\check{W}_i = (\widetilde{W}_i)^{\psi_i}$. Hence, the pair of mappings $\varphi_i^* := \psi_i \circ \varphi_i : S \rightarrow \widetilde{S}_i$ give the desired coupling.

We may thus assume that S_1 and S_2 are atomless Borel spaces. In this case, by [4, Lemma 3.1], there exists a measure-preserving bijection of S_i onto $([0, \mu_i(S_i)), \mathcal{B}, \lambda)$. Hence we may without loss of generality assume that the measure space S_i is $([0, m_i), \mathcal{B}, \lambda)$ for some $m_i \in [0, \infty]$. Moreover, if $m_i < \infty$, we may make a trivial extension of W_i to $[0, \infty)$. Hence we may, and shall, assume that $S_1 = S_2 = \mathbb{R}_+$.

Let $\delta := \delta_{\square}(W_1, W_2)$. By the definition (2.8), there exists for every n a coupling measure $\hat{\mu}_n$ on \mathbb{R}_+^2 with both marginals equal to λ such that

$$\|W_1^{\pi_1} - W_2^{\pi_2}\|_{\square, \hat{\mu}_n} < \delta + 1/n. \quad (5.2)$$

Let $Z := [0, \infty) \times [0, \infty) \setminus \{(\infty, \infty)\}$. Then, Z is an open subset of the compact metric space $[0, \infty]^2$, so Z is a locally compact second countable space. Moreover, Z is a Polish space [10, Proposition 8.1.1] and $\mathbb{R}_+^2 \subset Z \subset [0, \infty]^2$. The measure $\hat{\mu}_n$ is defined on \mathbb{R}_+^2 , and we can regard it as a measure on Z .

Let, for $N \in \mathbb{N}$, $K_N := ([0, N] \times [0, \infty]) \cup ([0, \infty) \times [0, N])$. Then each K_N is a compact subset of Z and $Z = \bigcup_{N \geq 1} K_N$; moreover, every compact subset of Z is a subset of some K_N .

For each n , since $\hat{\mu}_n$ has marginals λ ,

$$\begin{aligned}\hat{\mu}_n(K_N) &\leq \hat{\mu}_n([0, N] \times [0, \infty]) + \hat{\mu}_n([0, \infty] \times [0, N]) \\ &= \lambda([0, N]) + \lambda([0, N]) = 2N.\end{aligned}\tag{5.3}$$

Hence, $\hat{\mu}_n(K) < \infty$ for every compact $K \subset Z$, so $\hat{\mu}_n$ is a Radon measure on Z . Moreover, (5.3) implies that the sequence $\hat{\mu}_n(K)$ is bounded for each compact $K \subset Z$, which means that the sequence $\hat{\mu}_n$ is relatively compact in the vague topology, see [20, Theorem A2.3(ii)]. Furthermore, by [20, Theorem A2.3(i)], the set of Radon measures on Z with the vague topology is metrizable. (In fact, a Polish space.) Consequently, there exists a subsequence $(\tilde{\mu}_n)$ of $(\hat{\mu}_n)$ that converges vaguely to some Radon measure ν on Z , i.e.,

$$\lim_{n \rightarrow \infty} \int_Z f d\tilde{\mu}_n = \int_Z f d\nu, \quad f \in C_c(Z).\tag{5.4}$$

Since $Z \subset [0, \infty]^2$, we may also regard ν as a measure on $[0, \infty]^2$. Since Z is σ -compact, ν is σ -finite on Z and thus on $[0, \infty]^2$. (But note that ν is an infinite measure and thus not a Radon measure on the compact space $[0, \infty]^2$.)

We next consider the marginals ν^{π_i} of ν ; these are measures on $[0, \infty]$. If $g \in C_c(\mathbb{R}_+)$, then the function $g^{\pi_1}(x, y) = g(x)$ has compact support $\text{supp}(g) \times [0, \infty]$ in Z , so $g^{\pi_1} \in C_c(Z)$ and (5.4) yields, recalling that each $\tilde{\mu}_n$ has marginals $(\tilde{\mu}_n)^{\pi_1} = \lambda$,

$$\int_{\mathbb{R}_+} g d\nu^{\pi_1} = \int_Z g^{\pi_1} d\nu = \lim_{n \rightarrow \infty} \int_Z g^{\pi_1} d\tilde{\mu}_n = \lim_{n \rightarrow \infty} \int_{\mathbb{R}_+} g d(\tilde{\mu}_n)^{\pi_1} = \int_{\mathbb{R}_+} g d\lambda.\tag{5.5}$$

Consequently, the marginal ν^{π_1} of ν equals λ on $\mathbb{R}_+ = [0, \infty)$. By symmetry, the same holds for ν^{π_2} . However, note that each marginal also may have a point mass at $\{\infty\}$; this point mass may even be infinite, in which case the marginal is not σ -finite. (We shall see that this causes no serious problem.)

Let $\varepsilon > 0$. Since $W_i \in L^1(\mathbb{R}_+^2)$ and $C_c(\mathbb{R}_+^2)$ is dense in $L^1(\mathbb{R}_+^2)$, there exist $\overline{W}_i \in C_c(\mathbb{R}_+^2)$ such that $\|W_i - \overline{W}_i\|_{L^1(\mathbb{R}_+^2)} < \varepsilon$, $i = 1, 2$. Then

$$\|W_i^{\pi_i} - \overline{W}_i^{\pi_i}\|_{\square, \tilde{\mu}_n} \leq \|W_i^{\pi_i} - \overline{W}_i^{\pi_i}\|_{L^1(\tilde{\mu}_n^2)} = \|W_i - \overline{W}_i\|_{L^1(\lambda^2)} < \varepsilon,\tag{5.6}$$

and thus by (5.2) and the triangle inequality, recalling that $\tilde{\mu}_n = \hat{\mu}_{k_n}$ for some $k_n \geq n$,

$$\|\overline{W}_1^{\pi_1} - \overline{W}_2^{\pi_2}\|_{\square, \tilde{\mu}_n} < \delta + 1/n + 2\varepsilon.\tag{5.7}$$

Extend each W_i and \overline{W}_i trivially (i.e., by 0) to $[0, \infty]^2$.

Let N be such that $\text{supp}(\overline{W}_i) \subseteq [0, N] \times [0, N]$ for $i = 1, 2$. Then $\text{supp}(\overline{W}_i^{\pi_i}) \subseteq K_N \times K_N$, and thus $\overline{W}_i^{\pi_i} \in C_c(Z^2)$.

Let $f, g \in C_{[0,1]}(K_N)$. We can extend f and g to functions in $C_{c,[0,1]}(Z)$; moreover, there exist sequences f_m and g_m in $C_{c,[0,1]}(Z)$ such that

$$f_m(x_1, x_2) \rightarrow \begin{cases} f(x_1, x_2), & (x_1, x_2) \in K_N, \\ 0, & \text{otherwise,} \end{cases}\tag{5.8}$$

as $m \rightarrow \infty$, and similarly for g_m . Then, by (5.7) and (2.9), for any m and n ,

$$\left| \int_{Z^2} (\overline{W}_1(x_1, y_1) - \overline{W}_2(x_2, y_2)) f_m(x_1, x_2) g_m(y_1, y_2) d\tilde{\mu}_n(x_1, x_2) d\tilde{\mu}_n(y_1, y_2) \right| \leq \delta + 2\varepsilon + 1/n. \quad (5.9)$$

The integrand in (5.9) is a continuous function with compact support in Z^2 , and by Lemma 4.1, $\tilde{\mu}_n \times \tilde{\mu}_n \xrightarrow{\nu} \nu \times \nu$. Hence, we can take the limit as $n \rightarrow \infty$ in (5.9) and obtain, with $z_i = (x_i, y_i)$,

$$\left| \int_{Z^2} (\overline{W}_1^{\pi_1}(z_1, z_2) - \overline{W}_2^{\pi_2}(z_1, z_2)) f_m(z_1) g_m(z_2) d\nu(z_1) d\nu(z_2) \right| \leq \delta + 2\varepsilon. \quad (5.10)$$

Now let $m \rightarrow \infty$; by (5.8) and dominated convergence (noting that $\overline{W}_1^{\pi_1} - \overline{W}_2^{\pi_2} \in L^1(\nu \times \nu)$), the integral in (5.10) converges to

$$\int_{K_N \times K_N} (\overline{W}_1^{\pi_1}(z_1, z_2) - \overline{W}_2^{\pi_2}(z_1, z_2)) f(z_1) g(z_2) d\nu(z_1) d\nu(z_2). \quad (5.11)$$

Since $f, g \in C_{[0,1]}(K_N)$ are arbitrary, (5.10) and Lemma 4.2 thus yield

$$\|\overline{W}_1^{\pi_1} - \overline{W}_2^{\pi_2}\|_{\square, K_N, \nu} \leq \delta + 2\varepsilon. \quad (5.12)$$

Furthermore, $\overline{W}_1^{\pi_1}$ and $\overline{W}_2^{\pi_2}$ vanish off $K_N \times K_N$, and thus

$$\|\overline{W}_1^{\pi_1} - \overline{W}_2^{\pi_2}\|_{\square, [0, \infty]^2, \nu} = \|\overline{W}_1^{\pi_1} - \overline{W}_2^{\pi_2}\|_{\square, K_N, \nu} \leq \delta + 2\varepsilon. \quad (5.13)$$

Consequently, on $[0, \infty]^2$, using the analogue of (5.6) for ν ,

$$\|W_1^{\pi_1} - W_2^{\pi_2}\|_{\square, \nu} \leq \|\overline{W}_1^{\pi_1} - \overline{W}_2^{\pi_2}\|_{\square, \nu} + \|W_1^{\pi_1} - \overline{W}_1^{\pi_1}\|_{\square, \nu} + \|W_2^{\pi_2} - \overline{W}_2^{\pi_2}\|_{\square, \nu} \leq \delta + 4\varepsilon. \quad (5.14)$$

Since $\varepsilon > 0$ is arbitrary, this yields

$$\|W_1^{\pi_1} - W_2^{\pi_2}\|_{\square, [0, \infty]^2, \nu} \leq \delta = \delta_{\square}(W_1, W_2). \quad (5.15)$$

If $\nu(\{\infty\} \times [0, \infty))$ and $\nu([0, \infty) \times \{\infty\})$ are finite, then the projections ν^{π_1} and ν^{π_2} are σ -finite measures on $[0, \infty]$. In this case, (5.15) shows that if we take the trivial extensions of W_1 and W_2 to $[0, \infty]^2$, then ν is a coupling measure such that the special coupling (π_1, π_2) yields equality in (5.1). (Recall that $\delta_{\square}(W_1, W_2) \leq \|W_1^{\pi_1} - W_2^{\pi_2}\|_{\square, \nu}$ by the definition (2.8).)

In general, the projections are not σ -finite, since they may have infinite atoms at ∞ , but we may easily modify the construction. Let $\tilde{S} := \mathbb{R}_+ \cup \mathbb{R}'_+$, where \mathbb{R}_+ and \mathbb{R}'_+ are two disjoint copies of $[0, \infty)$, with variables denoted x and x' , respectively. Define a map $\varphi : Z \rightarrow \tilde{S} \times \tilde{S}$ by

$$\varphi(x, y) = \begin{cases} (x, y), & x, y < \infty, \\ (x, x'), & y = \infty, \\ (y', y), & x = \infty. \end{cases} \quad (5.16)$$

This means that the lines $\mathbb{R}_+ \times \{\infty\}$ and $\{\infty\} \times \mathbb{R}_+$ are mapped to the diagonals in $\mathbb{R}_+ \times \mathbb{R}'_+$ and $\mathbb{R}'_+ \times \mathbb{R}_+$, respectively. Let $\tilde{\nu} := \nu^{\varphi}$. Then there is no problem with the projections: $\tilde{\nu} := \nu^{\varphi}$ is a σ -finite measure on $\tilde{S} \times \tilde{S}$, and the marginals $\tilde{\nu}_i := \tilde{\nu}^{\pi_i}$ are σ -finite measures on \tilde{S} ; moreover, $\tilde{\nu}_i = \lambda$

on $\mathbb{R}_+ \subset \tilde{S}$. Hence, we can define $\tilde{S}_i := (\tilde{S}, \tilde{\nu}_i)$ and let \widetilde{W}_i be the trivial extension of W_i to \tilde{S}_i ; then $\tilde{\nu}$ is a coupling measure. Moreover,

$$\|\widetilde{W}_1^{\pi_1} - \widetilde{W}_2^{\pi_2}\|_{\square, \tilde{S} \times \tilde{S}, \tilde{\nu}} = \|W_1^{\pi_1} - W_2^{\pi_2}\|_{\square, [0, \infty]^2, \nu}. \quad (5.17)$$

One way to see (5.17) is to define $\psi : \tilde{S} \rightarrow [0, \infty]$ by $\psi(x) = x$ and $\psi(x') = \infty$, and let $\psi^{\otimes 2} := \psi \otimes \psi : \tilde{S}^2 \rightarrow [0, \infty]^2$. Then it follows from (5.16) that $\psi^{\otimes 2} \circ \varphi$ is the identity embedding $Z \rightarrow [0, \infty]^2$. Hence, $\psi^{\otimes 2}$ is measure-preserving $(\tilde{S} \times \tilde{S}, \tilde{\nu}) \rightarrow ([0, \infty] \times [0, \infty], \nu)$. Furthermore, $\widetilde{W}_i = W_i^\psi$ and thus $\widetilde{W}_i^{\pi_i} = (W_i^\psi)^{\pi_i} = W_i^{\psi \circ \pi_i} = W_i^{\pi_i \circ \psi^{\otimes 2}}$, and thus

$$\|\widetilde{W}_1^{\pi_1} - \widetilde{W}_2^{\pi_2}\|_{\square, \tilde{S} \times \tilde{S}, \tilde{\nu}} = \|\widetilde{W}_1^{\pi_1 \circ \psi^{\otimes 2}} - \widetilde{W}_2^{\pi_2 \circ \psi^{\otimes 2}}\|_{\square, \tilde{S} \times \tilde{S}, \tilde{\nu}} = \|W_1^{\pi_1} - W_2^{\pi_2}\|_{\square, [0, \infty]^2, \nu},$$

showing (5.17).

Finally, (5.17) and (5.15) show that

$$\|\widetilde{W}_1^{\pi_1} - \widetilde{W}_2^{\pi_2}\|_{\square, \tilde{S} \times \tilde{S}, \tilde{\nu}} = \|W_1^{\pi_1} - W_2^{\pi_2}\|_{\square, [0, \infty]^2, \nu} \leq \delta_{\square}(W_1, W_2), \quad (5.18)$$

and thus equality holds by the definition (2.8). \square

Remark 5.2. The analogue of Theorem 5.1 for δ_1 holds too. The proof is essentially the same, with the difference that we do not need f, g, f_m, g_m ; we proceed directly from the L^1 version of (5.7) to the L^1 version of (5.13) using Lemma 4.1 and the fact that $|\overline{W}_1^{\pi_1} - \overline{W}_2^{\pi_2}| \in C_c(Z^2)$.

Similarly, Theorem 5.1 holds for δ_p too, for any $p > 1$, provided $W_1, W_2 \in L^p$ and $W_1, W_2 \geq 0$.

As a special case, we obtain the following characterisation of equivalent graphons on Borel spaces, for example \mathbb{R}_+ ; again this extends a result by Bollobás and Riordan [3, Corollary 2.7].

Theorem 5.3. *Let (W_1, S_1) and (W_2, S_2) be graphons with $\delta_{\square}(W_1, W_2) = 0$, and assume that $S_i = (S_i, \mathcal{F}_i, \mu_i)$ are σ -finite Borel spaces, $i = 1, 2$. Then there exist trivial extensions $(\widetilde{W}_i, \tilde{S}_i)$ of (W_i, S_i) and a coupling (φ_1, φ_2) of \tilde{S}_1 and \tilde{S}_2 such that $\widetilde{W}_1^{\varphi_1} = \widetilde{W}_2^{\varphi_2}$ a.e. The coupling may be assumed to be special.*

Proof. An immediate consequence of Theorem 5.1 and (2.4). \square

Remark 5.4. In the standard case of graphons on probability spaces, Borgs, Chayes and Lovász [6] proved a similar result using pull-backs in the opposite direction, viz. that $W_1 \cong W_2$ if and only if there exists a graphon W on some probability space S and measure preserving maps $\varphi_i : S_i \rightarrow S$ such that $W_i = W^{\varphi_i}$ a.e. See also [18, Theorem 8.4] with an alternative proof. (For this result, no assumption on the probability spaces S_1 and S_2 is needed, so it yields a general characterisation of equivalence.)

We conjecture that this result too extends (in some form) to the σ -finite case, but we leave this as an open problem.

We can elaborate Theorem 5.3 as follows, cf. the corresponding result [18, Theorem 8.3] for graphons on probability spaces.

Definition 5.5. Two graphons (W_1, S_1) and (W_2, S_2) are *elementarily equivalent* if one of the following conditions holds:

- (i) $W_1 = W_2^\varphi$ a.e. for some measure-preserving map $\varphi : S_1 \rightarrow S_2$, or conversely.
- (ii) W_1 is a trivial extension of W_2 , or conversely.

Note that (i) includes the case of two a.e. equal graphons on the same measure space.

Theorem 5.6. *Let W and W' be two graphons. Then the following are equivalent:*

- (i) $W \cong W'$.
- (ii) $\delta_\square(W, W') = 0$.
- (iii) $\delta_1(W, W') = 0$.
- (iv) *There exists a finite sequence of graphons $W = W_0, \dots, W_N = W'$ such that W_{i-1} and W_i are elementarily equivalent for $i = 1, \dots, N$.*

The equivalence (ii) \iff (iii) is [4, Proposition 2.6].

Proof. (i) \iff (ii) holds by definition.

(ii) \implies (iv): By [4, Lemma 4.1], $W = W_1^{\varphi_1}$ and $W' = W_2^{\varphi_2}$ for some graphons W_1 and W_2 on Borel measure spaces S_1 and S_2 . Then $\delta_\square(W_1, W_2) = \delta_\square(W, W') = 0$. Hence Theorem 5.3 shows the existence of trivial extensions \widetilde{W}_i of W_i and a coupling (ψ_1, ψ_2) such that $\widetilde{W}_1^{\psi_1} = \widetilde{W}_2^{\psi_2}$ a.e. Consequently, $W, W_1, \widetilde{W}_1, \widetilde{W}_1^{\psi_1}, \widetilde{W}_2^{\psi_2}, \widetilde{W}_2, W_2, W'$ is a sequence where every graphon is elementarily equivalent to the next.

(iv) \implies (iii): It is clear by the definitions in Section 2 that $\delta_1(W, W') = 0$ whenever W and W' satisfy one of the two cases in Definition 5.5. The result follows by the triangle inequality.

(iii) \implies (ii): Immediate by (2.12). \square

Theorem 5.7. *Let $p > 1$ and let W and W' be two non-negative graphons in L^p . Then the conditions (i)–(iv) in Theorem 5.6 are also equivalent to*

- (v) $\delta_p(W, W') = 0$.

Proof. (iv) \implies (v): Immediate, as the proof of (iv) \implies (iii) above.

(v) \implies (iv): As the proof of (ii) \implies (iv) above, now using Remark 5.2.

Alternatively, we can prove (v) \implies (iii) directly as follows. Using Proposition 2.3, we may assume that W_1 and W_2 are defined on \mathbb{R}_+ .

Let $\varepsilon > 0$. Since $\delta_p(W, W') = 0$, there exists by [4, Proposition 4.3(c) and Remark 4.4] a measure-preserving bijection $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\|W - (W')^\varphi\|_{L^p} < \varepsilon$.

Also, let $M > 0$. Let $I_M := [0, M]$ and $A := I_M \cup \varphi^{-1}(I_M)$. Then, $\lambda(A) \leq \lambda(I_M) + \lambda(\varphi^{-1}(I_M)) = 2M$ and thus, by Hölder's inequality,

$$\|(W - (W')^\varphi)\mathbf{1}_{A \times A}\|_{L^1} \leq \|W - (W')^\varphi\|_{L^p} (\lambda^2(A \times A))^{1-1/p} < \varepsilon (2M)^{2-2/p}. \quad (5.19)$$

Moreover,

$$\|W\mathbf{1}_{(A \times A)^c}\|_{L^1} \leq \|W\mathbf{1}_{(I_M \times I_M)^c}\|_{L^1} \quad (5.20)$$

and

$$\begin{aligned} \|(W')^\varphi\mathbf{1}_{(A \times A)^c}\|_{L^1} &\leq \|(W')^\varphi\mathbf{1}_{(\varphi^{-1}(I_M) \times \varphi^{-1}(I_M))^c}\|_{L^1} = \|(W'\mathbf{1}_{(I_M \times I_M)^c})^\varphi\|_{L^1} \\ &= \|W'\mathbf{1}_{(I_M \times I_M)^c}\|_{L^1}. \end{aligned} \quad (5.21)$$

Consequently, by the triangle inequality and (5.19)–(5.21),

$$\begin{aligned} \delta_1(W, W') &\leq \|W - (W')^\varphi\|_{L^1} \\ &< \varepsilon(2M)^{2-2/p} + \|W\mathbf{1}_{(I_M \times I_M)^c}\|_{L^1} + \|W'\mathbf{1}_{(I_M \times I_M)^c}\|_{L^1}. \end{aligned} \quad (5.22)$$

Letting first $\varepsilon \rightarrow 0$ and then $M \rightarrow \infty$, we obtain $\delta_1(W, W') = 0$. \square

Theorem 5.6 has an important consequence. In order to state it generally, let a *property* of a graphon be anything that is determined uniquely by the graphon; i.e., any function Φ from the set of all graphons to some arbitrary set X .

Theorem 5.8. *Let Φ be a property of graphons such that $\Phi(W_1) = \Phi(W_2)$ whenever W_1 and W_2 are elementarily equivalent graphons. Then $\Phi(W) = \Phi(W')$ whenever W and W' are two equivalent graphons, and consequently, Φ is well-defined on the set of equivalence classes of graphons.*

Proof. Obvious by Theorem 5.6(i) \implies (iv). \square

Note that it does not matter whether the property depends continuously on the graphon or not.

Example 5.9. We give some examples of applications of Theorem 5.8. In all of them, it is immediately verified that two elementarily equivalent graphons give the same result. In the first examples, the property Φ is binary (true or false); the last examples are real parameters. Another example, with a more complicated Φ , follows in the proof of Theorem 5.10.

- (i) If the graphon W is a.e. non-negative, then so is every graphon equivalent to W .
- (ii) If the graphon W is a.e. $[0, 1]$ -valued, then so is every graphon equivalent to W .
- (iii) For any $p > 0$, the L^p -norm $\|W\|_{L^p} \in [0, \infty]$ is the same for any two equivalent graphons, and is thus well-defined for every equivalence class of graphons.
- (iv) Let $h(p) := -p \log p - (1 - p) \log(1 - p)$ for $p \in [0, 1]$, and define the *entropy* of a $[0, 1]$ -valued graphon W by

$$\mathcal{E}(W) := \int_{S \times S} h(W(x, y)) \, d\mu(x) \, d\mu(y) \in [0, \infty]. \quad (5.23)$$

Then $\mathcal{E}(W)$ is the same for any two equivalent $[0, 1]$ -valued graphons, and is thus well-defined for every equivalence class of such graphons.

For the entropy (5.23) in the standard case of graphons on probability spaces, see e.g. [1], [18, Appendix D.2], [16] and the further references given there. We leave it as an open problem whether results in these references can be extended to the present setting, at least under suitable conditions.

As another application, we obtain a new (simpler) proof of [4, Theorem 2.21]. (The statement is slightly expanded here.)

Theorem 5.10 ([4]). *Let W_1 and W_2 be two graphons. Then the following are equivalent.*

- (i) $W_1 \cong W_2$.

- (ii) *The random graph processes $(\tilde{G}_t(W_1))_{t \geq 0}$ and $(\tilde{G}_t(W_2))_{t \geq 0}$ have the same distribution, up to vertices that stay isolated for all t .*
- (iii) *The random graph processes $(G_t(W_1))_{t \geq 0}$ and $(G_t(W_2))_{t \geq 0}$ have the same distribution.*
- (iv) *The random graphs $\tilde{G}_t(W_1)$ and $\tilde{G}_t(W_2)$ have the same distribution, up to isolated vertices, for every $t > 0$.*
- (v) *The random graphs $G_t(W_1)$ and $G_t(W_2)$ have the same distribution, for every $t > 0$.*

Note that a trivial extension of a graphon in general adds permanently isolated vertices to \tilde{G}_t , so the result would not be true without the provisions for them in (ii) and (iv).

Proof. (i) \implies (ii). By Theorem 5.8 (taking $\Phi(W)$ to be the distribution of the process $(\tilde{G}_t(W))_t$), it suffices to consider the case of two elementarily equivalent graphons, which is obvious for both cases in Definition 5.5.

(ii) \implies (iii) \implies (v) and (ii) \implies (iv) \implies (v) are trivial.

(v) \implies (i). We can couple the random graphs for each fixed t such that $G_t(W_1) = G_t(W_2)$ a.s., and thus $W_{G_t(W_1)} = W_{G_t(W_2)}$ a.s. Moreover, (3.6) shows that $\delta_{\square}^s(W_{G_t(W_i)}, W_i) \rightarrow 0$ in probability as $t \rightarrow \infty$, for $i = 1, 2$. Consequently,

$$\delta_{\square}^s(W_1, W_2) = 0. \quad (5.24)$$

This implies by (2.23) that $W_2 \cong \Upsilon_u(W_1)$ for some $u > 0$. Using (3.5) and the assumption (v), this in turn implies

$$G_t(W_1) \stackrel{d}{=} G_t(W_2) \stackrel{d}{=} G_t(\Upsilon_u(W_1)) \stackrel{d}{=} G_{u^{1/2}t}(W_1). \quad (5.25)$$

However, this equality implies (except in the trivial case $W_1 = 0$) that $u = 1$, for example by considering the expected number of edges, see (3.4). Hence $W_2 \cong \Upsilon_1(W_1) = W_1$. \square

Example 5.11 (trivial extensions are necessary in Theorems 5.1 and 5.3). Define $W(x, y) := e^{-x-y} \mathbf{1}\{x \geq 0\} \mathbf{1}\{y \geq 0\}$ for real x, y . Let $a \in (0, \infty]$ and let $S_1 := [-a, \infty)$, $S_2 := [0, \infty)$, both with Lebesgue measure. Define the graphons (W_1, S_1) and (W_2, S_2) by $W_1 = W$ and $W_2 = W$. Then W_1 is a trivial extension of W_2 , and thus $\delta_{\square}(W_1, W_2) = 0$. Nevertheless, there exists no coupling $(\varphi_1, \varphi_2) : S \rightarrow (S_1, S_2)$ such that $\|W_1^{\varphi_1} - W_2^{\varphi_2}\|_{\square} = 0$ and thus, by (2.4), $W_1^{\varphi_1} = W_2^{\varphi_2}$ a.e. In fact, suppose that such a coupling exists, and let $D_i(x) := \int_{S_i} W_i(x, y) d\lambda(y)$, $x \in S_i$. Then, for a.e. $x \in S$, since $\mu^{\varphi_i} = \lambda$,

$$\begin{aligned} D_1^{\varphi_1}(x) &= \int_{S_1} W_1(\varphi_1(x), y) d\lambda(y) = \int_S W_1(\varphi_1(x), \varphi_1(z)) d\mu(z) \\ &= \int_S W_1^{\varphi_1}(x, z) d\mu(z) = \int_S W_2^{\varphi_2}(x, z) d\mu(z) = D_2^{\varphi_2}(x), \end{aligned} \quad (5.26)$$

where the final equality follows by symmetry. However,

$$\mu\{x \in S : D_1^{\varphi_1}(x) = 0\} = \lambda\{y \in S_1 : D_1(y) = 0\} = a, \quad (5.27)$$

while, similarly, $\mu\{x \in S : D_2^{\varphi_2}(x) = 0\} = 0$. This is a contradiction.

6. COMPLETENESS

Borgs, Chayes, Cohn and Holden [4, Corollary 2.13] show that the set of (equivalence classes of) $[0, 1]$ -valued graphons is complete for the δ_{\square} metric. Their proof is based on their characterisation of compactness for ($[0, 1]$ -valued) graphons, see Section 7. We discuss completeness further in this section, and give a new, more direct, proof of their result; we also extend the result somewhat. (Whether our proof is simpler or not is a matter of taste and background.) We also investigate in more detail why a restriction such as $[0, 1]$ -valued is needed; the set of all graphons is not complete and we give several examples that illustrate that.

6.1. Completeness for the cut norm. We begin by studying convergence in the cut norm $\|\cdot\|_{\square}$ for graphons on a fixed σ -finite measure space.

We say that a set \mathcal{A} of integrable functions (possibly defined on different measure spaces) is *uniformly integrable* if it satisfies

- (UI1) $\sup_{F \in \mathcal{A}} \int_S |F| \, d\mu < \infty$, and
- (UI2) $\sup_{F \in \mathcal{A}} \int_{|F| > B} |F| \, d\mu \rightarrow 0$ as $B \rightarrow \infty$.

This is a standard notion for probability spaces, where furthermore (UI2) implies (UI1). (For infinite measure spaces, the notion is much less used and when used, the definitions vary between different authors.) Note also that assuming (UI1), (UI2) is equivalent to

$$(UI2') \sup_{E \subseteq S: \mu(E) \leq \delta} \sup_{F \in \mathcal{A}} \int_E |F| \, d\mu \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

We say that a set \mathcal{A} is *semiuniformly integrable* if it satisfies (UI2). (For a finite measure space, semiuniformly integrable is thus equivalent to uniformly integrable, but in general it is weaker.)

Recall that a subset of a Banach space is relatively weakly compact if it is a subset of a weakly compact set. Recall also that a set is relatively weakly compact if and only if it is sequentially weakly compact, i.e., every sequence in the set has a convergent subsequence. (The Eberlein–Šmulian theorem [15, Theorem V.6.1].) Moreover, let (S, μ) be a σ -finite measure space, and let S_n be an increasing sequence of subsets of S with finite measures such that $S = \bigcup_n S_n$. Then a subset \mathcal{A} of $L^1(S, \mu)$ is relatively weakly compact if and only if (UI1)–(UI2) hold together with

$$(WC3) \sup_{F \in \mathcal{A}} \int_{S \setminus S_n} |F| \, d\mu \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This is (one form of) the Dunford–Pettis theorem; see [14, Theorem 3.2.6] and [15, Theorem IV.8.9] for two slightly different formulations, and note [15, Corollary IV.8.10].

The Dunford–Pettis theorem is perhaps best known in the case of a probability space. In that case, and more generally for any finite measure space (S, μ) , (WC3) is trivial; thus, the theorem then says that a subset of $L^1(S, \mu)$ is relatively weakly compact if and only if it is uniformly integrable.

We say that a set \mathcal{A} in a metric space is *relatively complete* if every Cauchy sequence in \mathcal{A} converges to some limit (which does not have to belong to \mathcal{A}). It is easy to see that \mathcal{A} is relatively complete if and only if $\overline{\mathcal{A}}$ is complete.

We now give our main result on completeness for the cut norm. Although we only are interested in symmetric functions, the theorem and its proof

hold for general (integrable) functions on a product $S_1 \times S_2$; such functions appear for example in the study of bipartite graphs.

Theorem 6.1. *Let (S, μ) be a σ -finite measure space. Any uniformly integrable set of graphons in $L^1(S^2, \mu^2)$ is relatively complete for the cut norm.*

Proof. Step 1: $\mu(S) < \infty$. First, consider the case when (S, μ) is a finite measure space. Let (W_n) be a Cauchy sequence for the cut norm, with $\{W_n\}$ uniformly integrable. By the comments before the theorem, the set $\{W_n\}$ is relatively weakly compact in $L^1(S \times S)$, and thus sequentially weakly compact; hence, there exists a subsequence $W'_n = W_{k_n}$ and some $V \in L^1(S \times S)$ such that $W'_n \rightarrow V$ weakly in $L^1(S \times S)$ as $n \rightarrow \infty$. Clearly, V is symmetric and thus a graphon.

In particular, whenever $T, U \subseteq S$,

$$\int_{T \times U} W'_n \, d\mu^2 \rightarrow \int_{T \times U} V \, d\mu^2. \quad (6.1)$$

Hence, for every m , recalling the definition (2.2),

$$\left| \int_{T \times U} (W'_m - V) \, d\mu^2 \right| = \lim_{n \rightarrow \infty} \left| \int_{T \times U} (W'_m - W'_n) \, d\mu^2 \right| \leq \limsup_{n \rightarrow \infty} \|W'_m - W'_n\|_{\square}. \quad (6.2)$$

Taking the supremum over all measurable subsets T and U , we obtain

$$\|W'_m - V\|_{\square} \leq \limsup_{n \rightarrow \infty} \|W'_m - W'_n\|_{\square}. \quad (6.3)$$

Since W'_n is a Cauchy sequence, the right-hand side of (6.3) tends to 0 as $m \rightarrow \infty$, and thus $\|W'_m - V\|_{\square} \rightarrow 0$.

We have shown that the original sequence has a subsequence that converges to V for the cut norm. Since the sequence is Cauchy, the full sequence (W_n) converges to the same limit, i.e., $\|W_n - V\|_{\square} \rightarrow 0$ as $n \rightarrow \infty$.

Moreover, note that since $W'_n \rightarrow V$ weakly in L^1 ,

$$\|V\|_{L^1} \leq \sup_n \|W'_n\|_{L^1} \leq \sup_n \|W_n\|_{L^1}. \quad (6.4)$$

Step 2: $\mu(S) = \infty$. In general, let $S = \bigcup_N S_N$, where S_N is an increasing sequence of subsets of S with finite measure. Let again (W_n) be a uniformly integrable Cauchy sequence for the cut norm. Consider the restrictions $W_n^{(N)} := W_n|_{S_N \times S_N}$. Then

$$\|W_m^{(N)} - W_n^{(N)}\|_{\square, S_N} = \|W_m - W_n\|_{\square, S_N} \leq \|W_m - W_n\|_{\square}, \quad (6.5)$$

and thus, for each N , $(W_n^{(N)})_n$ is a Cauchy sequence for the cut norm on $S_N \times S_N$. Furthermore, these restrictions are uniformly integrable, since the graphons W_n are.

Hence, the first part applies to S_N , and shows that for each N there exists some graphon $V^{(N)}$ on S_N such that

$$\|W_n - V^{(N)}\|_{\square, S_N} = \|W_n^{(N)} - V^{(N)}\|_{\square, S_N} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (6.6)$$

If $M < N$, then (6.6) implies $\|W_n - V^{(N)}\|_{\square, S_M} \rightarrow 0$ as $n \rightarrow \infty$, and thus $\|V^{(M)} - V^{(N)}\|_{\square, S_M} = 0$, so $V^{(M)} = V^{(N)}|_{S_M \times S_M}$ a.e. Consequently,

there exists a symmetric measurable function V on $S \times S$ such that $V^{(N)} = V|_{S_N \times S_N}$ a.e. for every N . Moreover, (6.4) implies that

$$\int_{S_N^2} |V| = \int_{S_N^2} |V^{(N)}| \leq \sup_n \|W_n^{(N)}\|_{L^1} \leq \sup_n \|W_n\|_{L^1}, \quad (6.7)$$

which is finite by (UI1). Consequently, by monotone convergence,

$$\int_{S^2} |V| \leq \sup_n \|W_n\|_{L^1} < \infty, \quad (6.8)$$

and thus V is integrable and thus a graphon.

It remains to show that $\|W_n - V\|_{\square} \rightarrow 0$. Let $T, U \subseteq S$, and let $T_N := T \cap S_N$, $U_N := U \cap S_N$. It follows from (6.6) that for any fixed N ,

$$\begin{aligned} \left| \int_{T_N \times U_N} (W_m - V) \right| &= \left| \int_{T_N \times U_N} (W_m - V^{(N)}) \right| = \lim_{n \rightarrow \infty} \left| \int_{T_N \times U_N} (W_m - W_n) \right| \\ &\leq \limsup_{n \rightarrow \infty} \|W_m - W_n\|_{\square}. \end{aligned} \quad (6.9)$$

Letting $N \rightarrow \infty$, we see, by dominated convergence, that

$$\left| \int_{T \times U} (W_m - V) \right| \leq \limsup_{n \rightarrow \infty} \|W_m - W_n\|_{\square}, \quad (6.10)$$

and taking the supremum over all T and U we obtain

$$\|W_m - V\|_{\square} \leq \limsup_{n \rightarrow \infty} \|W_m - W_n\|_{\square}, \quad (6.11)$$

which tends to 0 as $m \rightarrow \infty$. \square

Remark 6.2. In particular, a relatively weakly compact set in $L^1(S^2, \mu^2)$ is relatively complete for the cut norm. Moreover, a weakly compact set in $L^1(S^2, \mu^2)$ is complete for the cut norm $\|\cdot\|_{\square}$, since the argument in the first part of the proof shows that a Cauchy sequence for the cut norm in a weakly compact set converges to an element of that set. Note, however, that the identity map is, in general, *not* continuous $(L^1(S^2), \text{weak}) \rightarrow (L^1(S^2), \|\cdot\|_{\square})$, and that a weakly compact set does not have to be compact for the cut norm; see Example 6.14.

Theorem 6.1 is related to Borgs, Chayes, Cohn and Zhao [5, Theorem C.7], which shows that a uniformly integrable set of graphons on $[0, 1]$ is relatively compact (and thus relatively complete) for the cut metric, see Corollary 7.16 below.

We assumed in Theorem 6.1 uniform integrability, i.e., (UI1) and (UI2). None of these conditions suffices alone; Example 6.11 shows that (UI1) is not enough and Example 6.12 shows that (UI2) is not enough for relative completeness in the cut norm. However, if we consider only non-negative graphons, then (UI2) suffices.

Theorem 6.3. *Let (S, μ) be a σ -finite measure space. A semiuniformly integrable set of non-negative graphons in $L^1(S^2, \mu^2)$ is relatively complete for the cut norm.*

Proof. If (W_n) is a Cauchy sequence for the cut norm on S , then $\int_{S \times S} W_n$ converges; hence, if every $W_n \geq 0$, then

$$C_1 := \sup_n \|W_n\|_{L^1} = \sup_n \int_{S \times S} W_n < \infty, \quad (6.12)$$

so $\{W_n\}$ satisfies (UI1). Consequently, if $\{W_n\}$ also is semiuniformly integrable, then it is uniformly integrable and Theorem 6.1 implies that the sequence converges. \square

In particular, these results yield results for L^p -bounded sets of graphons. (This too is related to results in [5, in particular Theorem 2.13] for the cut metric in the case $S = [0, 1]$.)

On a finite measure space, for example a probability space, it is well-known, and easy to see by Hölder's inequality, that a set that is bounded in L^p for some $p > 1$ is uniformly integrable, and thus Theorem 6.1 applies; thus the set is relatively complete for the cut norm. In particular, a uniformly bounded set of graphons on a finite measure space is relatively complete for the cut norm. This fails for infinite measure spaces, see Examples 6.12 and Example 6.13. Nevertheless, an L^p -bounded set, for $1 < p \leq \infty$, is semiuniformly integrable, which leads to the following results. (We shall see in Example 6.11 that the results do not hold for $p = 1$.)

Theorem 6.4. *Let (S, μ) be a σ -finite measure space. Let $1 < p \leq \infty$ and $C < \infty$ and let $\mathcal{W}(p, C)$ be the set of graphons W on (S, μ) with $\|W\|_{L^p} \leq C$.*

- (i) *The set of non-negative graphons in $\mathcal{W}(p, C)$ is complete for the cut norm. Hence, a set of non-negative graphons on S that is bounded in L^p is relatively complete for the cut norm.*

In particular, the set of $[0, 1]$ -valued graphons on S is complete for the cut norm.

- (ii) *If $C_1 < \infty$, then the set of graphons W in $\mathcal{W}(p, C)$ such that also $\|W\|_{L^1} \leq C_1$ is complete for the cut norm. Hence, a set of graphons on S that is bounded in both L^1 and L^p is relatively complete for the cut norm.*
- (iii) *If μ is a finite measure, then $\mathcal{W}(p, C)$ is complete for the cut norm. Hence, a set of graphons on a finite measure space that is bounded in L^p is relatively complete for the cut norm.*

Before giving the proof, we give a simple lemma. It is certainly known, but we have not found an explicit reference so for completeness, we give a proof. (Part (ii) follows in the case $S = [0, 1]$ from the more advanced [5, Theorem 2.13], and the proof uses similar ideas as there.)

Lemma 6.5. *Let (S, μ) be a σ -finite measure space and let W_n and W be graphons on S such that $\|W_n - W\|_{\square} \rightarrow 0$ as $n \rightarrow \infty$.*

- (i) *If each $W_n \geq 0$, then $W \geq 0$ a.e.*
- (ii) *If $1 \leq p \leq \infty$ and each $W_n \in L^p(S, \mu)$, then $\|W\|_{L^p} \leq \sup_n \|W_n\|_{L^p}$.*

In other words, the set of non-negative graphons on S and, for any $p \geq 1$ and C , the set of graphons W on S with $\|W\|_{L^p} \leq C$ are closed for the cut norm.

Proof. First, consider the case when (S, μ) is a probability space. There exists a sequence $(A_i)_{i=1}^\infty$ of measurable subsets of S such that if \mathcal{F}_∞ is the σ -field generated by $\{A_i\}$, then W is $(\mathcal{F} \times \mathcal{F})$ -measurable; see e.g. the proof of [18, Lemma 7.3].

Let \mathcal{F}_N be the sub- σ -field generated by $\{A_1, \dots, A_N\}$. Then $\mathcal{F}_N \times \mathcal{F}_N$ is an increasing sequence of σ -fields on $S \times S$ and their union generates $\mathcal{F}_\infty \times \mathcal{F}_\infty$, so the martingale limit theorem yields

$$\mathbb{E}(W \mid \mathcal{F}_N \times \mathcal{F}_N) \xrightarrow{\text{a.s.}} \mathbb{E}(W \mid \mathcal{F}_\infty \times \mathcal{F}_\infty) = W \quad \text{as } N \rightarrow \infty. \quad (6.13)$$

Furthermore, each \mathcal{F}_N is finite, and generated by some partition $\mathcal{P}_N := \{B_{Nj} : 1 \leq j \leq m_N\}$ of S . The conditional expectation $\mathbb{E}(W \mid \mathcal{F}_N \times \mathcal{F}_N)$ is constant on each ‘‘rectangle’’ $B_{Ni} \times B_{Nj}$, and equals there, provided the rectangle has positive measure, the average $(\mu(B_{Ni})\mu(B_{Nj}))^{-1} \int_{B_{Ni} \times B_{Nj}} W \, d\mu^2$. The same holds for each W_n , and consequently, the assumption $\|W_n - W\|_\square \rightarrow 0$ implies that

$$\mathbb{E}(W_n \mid \mathcal{F}_N \times \mathcal{F}_N) \xrightarrow{\text{a.s.}} \mathbb{E}(W \mid \mathcal{F}_N \times \mathcal{F}_N) \quad \text{as } n \rightarrow \infty, \quad (6.14)$$

for each fixed N .

For (i), we note that if $W_n \geq 0$, then $\mathbb{E}(W_n \mid \mathcal{F}_N \times \mathcal{F}_N) \geq 0$ and thus (6.14) and (6.13) yield $W \geq 0$ a.s. Similarly, for (ii), if $C := \sup_n \|W_n\|_{L^p}$, then $\|\mathbb{E}(W_n \mid \mathcal{F}_N \times \mathcal{F}_N)\|_{L^p} \leq C$ and (6.14) and (6.13) yield $\|W\|_{L^p} \leq C$, using Fatou’s lemma twice if $p < \infty$ (and directly if $p = \infty$).

This completes the proof if $\mu(S) = 1$. If $\mu(S) < \infty$, we replace μ by $\mu/\mu(S)$ and the result follows from the case just treated.

In general, $S = \bigcup_m S_m$, where S_m is an increasing sequence of subsets with finite measure. Since $\|W_n - W\|_\square \rightarrow 0$ implies $\|W_n - W\|_{\square, S_m} \rightarrow 0$, it follows from the finite measure case that for every m , in case (i), $W \geq 0$ a.e. on $S_m \times S_m$ and in case (ii), $\|W\|_{L^p(S_m)} \leq \sup_n \|W_n\|_{L^p(S_m)} \leq \sup_n \|W_n\|_{L^p}$. The conclusions follow, for (ii) using monotone convergence when $p < \infty$. \square

Proof of Theorem 6.4. Note first that, as said above, $\mathcal{W}(p, C)$ is semiuniformly integrable. In fact, if $1 < p < \infty$ and $W \in \mathcal{W}(p, C)$, then we have $\int_{|W| > B} |W| \, d\mu \leq B^{1-p} C^p$ and (UI2) follows; if $p = \infty$, take $B = C$ in (UI2).

(i): The set of non-negative graphons in $\mathcal{W}(p, C)$ is relatively complete by Theorem 6.3 and closed by Lemma 6.5. Hence the set is complete.

For the final sentence, note that the set of $[0, 1]$ -valued graphons equals the set of non-negative graphons in $\mathcal{W}(\infty, 1)$.

(ii): The set $\mathcal{W}(p, C) \cap \mathcal{W}(1, C_1)$ of graphons W on S such that $\|W\|_{L^p} \leq C$ and $\|W\|_{L^1} \leq C_1$ is uniformly integrable, since (UI1) is assumed and (UI2) follows from the L^p -bound as seen above. Hence, $\mathcal{W}(p, C) \cap \mathcal{W}(1, C_1)$ is relatively complete for the cut norm by Theorem 6.1. Furthermore, $\mathcal{W}(p, C)$ and $\mathcal{W}(1, C_1)$ are closed by Lemma 6.5(ii). Hence, $\mathcal{W}(p, C) \cap \mathcal{W}(1, C_1)$ is complete.

(iii): If μ is a finite measure, then $\mathcal{W}(p, C)$ is L^1 -bounded by Hölder’s inequality, and thus, as said before the theorem, $\mathcal{W}(p, C)$ is uniformly integrable. Hence the result follows by Theorem 6.1 and Lemma 6.5, or from (ii). (Alternatively, it follows that $\mathcal{W}(p, C)$ is weakly compact in L^1 ; hence the result follows by Remark 6.2.) \square

6.2. Completeness for the cut metric. It is time to turn to our main interest, the cut metric δ_{\square} defined in Section 2.4. We repeat that part (iv) of the following theorem was originally proved in [4, Corollary 2.13]. Furthermore, for the special case of graphons on $[0, 1]$, (i) follows from Borgs, Chayes, Cohn and Zhao [5, Theorem C.7], and (iii) and (v) follow from [5, Theorem 2.13]. (These theorems in [5] show much stronger results on compactness, see Corollaries 7.16–7.17 below.)

Theorem 6.6. (i) *Any uniformly integrable set of graphons is relatively complete for the cut metric.*

(ii) *A semiuniformly integrable set of non-negative graphons is relatively complete for the cut metric.*

(iii) *If $1 < p \leq \infty$ and $C < \infty$, then the set of non-negative graphons W with $\|W\|_{L^p} \leq C$ is complete for the cut metric. Hence, a set of non-negative graphons that is bounded in L^p is relatively complete for the cut metric.*

(iv) *The set of $[0, 1]$ -valued graphons is complete for the cut metric.*

(v) *If $1 < p \leq \infty$ and $C, C_1 < \infty$, then the set of graphons W with $\|W\|_{L^p} \leq C$ and $\|W\|_{L^1} \leq C_1$ is complete for the cut metric. Hence, a set of graphons that is bounded in both L^1 and L^p is relatively complete for the cut metric.*

We first prove a general lemma that will enable us to reduce various parts to the corresponding claims for the cut norm.

Lemma 6.7. *Suppose that (W_n) is a sequence of graphons that is a Cauchy sequence for the cut metric. Then there exists a sequence (W'_n) of graphons on \mathbb{R}_+ with $W_n \cong W'_n$ such that (W'_n) is a Cauchy sequence for the cut norm $\|\cdot\|_{\square, \mathbb{R}_+}$.*

Proof. By Proposition 2.3, we may replace the graphons by equivalent graphons on (\mathbb{R}_+, λ) , and we may thus assume that every W_n is defined on \mathbb{R}_+ .

First, suppose that $\delta_{\square}(W_n, W_{n+1}) < 2^{-n}$ for all n . By Proposition 2.4, this implies the existence of measure-preserving bijections $\varphi_n : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\|W_n - W_{n+1}^{\varphi_n}\|_{\square} < 2^{-n}. \quad (6.15)$$

Let $\psi_n := \varphi_{n-1} \circ \dots \circ \varphi_1$ (with ψ_1 the identity); this is a measure-preserving bijection $\mathbb{R}_+ \rightarrow \mathbb{R}_+$. Then $\psi_{n+1} = \varphi_n \circ \psi_n$, and thus $W_{n+1}^{\psi_{n+1}} = W_{n+1}^{\varphi_n \circ \psi_n} = (W_{n+1}^{\varphi_n})^{\psi_n}$. Hence, (6.15) implies

$$\begin{aligned} \|W_n^{\psi_n} - W_{n+1}^{\psi_{n+1}}\|_{\square} &= \|W_n^{\psi_n} - (W_{n+1}^{\varphi_n})^{\psi_n}\|_{\square} = \|(W_n - W_{n+1}^{\varphi_n})^{\psi_n}\|_{\square} \\ &= \|W_n - W_{n+1}^{\varphi_n}\|_{\square} < 2^{-n}. \end{aligned} \quad (6.16)$$

Consequently, the sequence $(W_n^{\psi_n})$ is a Cauchy sequence for the cut norm $\|\cdot\|_{\square}$ on \mathbb{R}_+ , so we may take $W'_n := W_n^{\psi_n}$.

In general, we can select a subsequence n_k such that

$$\delta_{\square}(W_{n_k}, W_m) < 2^{-k} \quad \text{for all } k \geq 1 \text{ and } m \geq n_k. \quad (6.17)$$

In particular, $\delta_{\square}(W_{n_k}, W_{n_{k+1}}) < 2^{-k}$, so the case just treated applies to the subsequence (W_{n_k}) and shows the existence of graphons $W'_{n_k} \cong W_{n_k}$

defined on \mathbb{R}_+ such that $\|W'_{n_k} - W'_{n_{k+1}}\|_{\square} < 2^{-k}$. Moreover, for any $m \in (n_k, n_{k+1})$, (6.17) implies $\delta_{\square}(W'_{n_k}, W_m) = \delta_{\square}(W_{n_k}, W_m) < 2^{-k}$, and thus Proposition 2.4 shows that there exists a measure-preserving bijection φ_m such that $\|W'_{n_k} - W_m^{\varphi_m}\|_{\square} < 2^{-k}$; let $W'_m := W_m^{\varphi_m}$. It follows that (W'_n) is a Cauchy sequence for the cut norm. (Actually, for our purposes it would be enough to consider the subsequence W'_{n_k} ; the full statement and the last part of the proof is only for completeness.) \square

We also need a version of Lemma 6.5 for the cut metric.

Lemma 6.8. *Let W_n and W be graphons such that $\delta_{\square}(W_n, W) \rightarrow 0$ as $n \rightarrow \infty$.*

- (i) *If each $W_n \geq 0$, then $W \geq 0$ a.e.*
- (ii) *If each $W_n \in L^p(S, \mu)$, then $\|W\|_{L^p} \leq \sup_n \|W_n\|_{L^p}$.*

In other words, the set of non-negative graphons and, for any C , the set of graphons W with $\|W\|_{L^p} \leq C$ are closed for the cut metric.

Proof. As above, by Proposition 2.3, we may replace the graphons by equivalent graphons on (\mathbb{R}_+, λ) , and assume that W and every W_n is defined on \mathbb{R}_+ , using also Example 5.9(i)(iii). By Proposition 2.4, this implies the existence of pull-backs $W'_n := W_n^{\varphi_n} \cong W_n$ such that $\|W - W'_n\|_{\square} < \delta_{\square}(W, W_n) + 1/n \rightarrow 0$ as $n \rightarrow \infty$. Now apply Lemma 6.5. \square

Remark 6.9. By considering suitable subsequences, it follows that the conclusion in Lemma 6.8(ii) can be improved to $\|W\|_{L^p} \leq \liminf_n \|W_n\|_{L^p}$.

Proof of Theorem 6.6. (i): Suppose that (W_n) is a sequence of graphons (possibly defined on different measure spaces) that is uniformly integrable and a Cauchy sequence for the cut metric.

By Lemma 6.7, there exist $W'_n \cong W_n$ such that W'_n are defined on \mathbb{R}_+ and the sequence (W'_n) is a Cauchy sequence for the cut norm $\|\cdot\|_{\square}$ on \mathbb{R}_+ . Note that replacing W_n by the equivalent W'_n preserves uniform integrability; this follows from Theorem 5.8 applied to (UI1) and (UI2) with explicit bounds.

Hence, Theorem 6.1 shows that there exists a graphon W on \mathbb{R}_+ such that W'_n converges to W in cut norm as $n \rightarrow \infty$. This implies

$$\delta_{\square}(W_n, W) \leq \|W'_n - W\|_{\square} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (6.18)$$

and thus $W_n \rightarrow W$ in the cut metric.

(ii): Argue as in the proof of (i), now using Theorem 6.3.

(iii): Argue as in the proof of (i), now using Theorem 6.4(i) and Lemma 6.8. (Alternatively, note that (6.12) holds in this case too, and use (v).)

(iv): This is just a special case of (iii).

(v): The set is uniformly integrable, as seen in the proof of Theorem 6.4, so it is relatively complete for the cut metric by (i). Moreover, the set is closed by Lemma 6.8. (Alternatively, one could use Theorem 6.4(ii).) \square

Theorem 6.6 is our main result about completeness. Note, however, that the conditions there are not necessary. In particular, as shown in Example 6.15, (semi)uniform integrability is not necessary for convergence and completeness in cut metric.

On the other hand, some conditions are needed, and we give a number of (counter) examples to illustrate that. In particular, Example 6.11 shows

that the set of all non-negative graphons is not complete, and also that Theorem 6.6(iii) does not hold for $p = 1$; furthermore, Example 6.12 shows that Theorem 6.6(ii) and (iii) do not hold without the assumption that the graphons are non-negative, even if we assume uniform boundedness (i.e., $p = \infty$).

Remark 6.10. For a given measure space (S, μ) , the cut norm is equivalent to the injective tensor product norm in $L^1 \check{\otimes} L^1$, which is given by taking the supremum over all $g, h : S \rightarrow [-1, 1]$ in (2.5), see e.g. [25, Chapter 3] or [18, Remark 4.2]. Hence every Cauchy sequence for the cut norm converges in the completed injective tensor product $L^1(S) \check{\otimes} L^1(S)$. However, the limit may lie outside $L^1(S \times S)$. In fact, $L^1(S \times S)$ is a dense subspace of $L^1(S) \check{\otimes} L^1(S)$, but typically (e.g. for $S = [0, 1]$ or \mathbb{R}_+) the norms are not equivalent, as is witnessed e.g. by V_n in Example 6.12 below, and thus $L^1(S \times S) \subsetneq L^1(S) \check{\otimes} L^1(S)$. Hence there exists Cauchy sequences for the cut norm (and thus also for the cut metric) with limits not in $L^1(S \times S)$. It is also easy to see that such sequences can be made symmetric, i.e., graphons.

On an abstract level, the failure of completeness in general for the cut norm is thus almost obvious. The examples below give some simple concrete examples. (In these examples, the Cauchy sequences thus have limits in $L^1(S) \check{\otimes} L^1(S) \setminus L^1(S \times S)$.)

See further e.g. [25] and note, in contrast, that the completed projective tensor product $L^1(S) \hat{\otimes} L^1(S)$ equals $L^1(S \times S)$.

One might be tempted to extend the definition of graphons to include all symmetric elements of $L^1(S) \check{\otimes} L^1(S)$. However, we doubt that this is useful. In particular, we do not see any way to define random graphs generalizing the construction in Section 3.2 unless W is a function.

We say that a graphon W on $S = [0, 1]$ or \mathbb{R}_+ is a *step* graphon if there is a partition of S into a finite number of intervals I_i (the steps) such that W is constant on each $I_i \times I_j$. (Note that the definition in [18] is more general.)

Example 6.11 (An L^1 -bounded set of non-negative graphons on $[0, 1]$ that is not relatively complete). This example is essentially the same as Borgs, Chayes, Cohn and Zhao [5, Proposition 2.12b] but stated differently; for completeness we give full details.

Note that for each $n \geq 1$ there exists a step graphon U_n on $[0, 1]$ with steps of equal measure, values in $\{0, 1\}$, and

$$\int U_n = \frac{1}{2}, \quad \|U_n - \frac{1}{2}\|_{\square} < 4^{-n}. \quad (6.19)$$

One way to see this is to consider the Erdős–Rényi random graph $G(N, M)$ with $M = N^2/4$ for large even N . Then, as $N \rightarrow \infty$, the graphon $W_{G(N, M)}$ converges in probability to the constant graphon $\frac{1}{2}$. Consequently, we may take $U_n = W_G$ where G is a realization of $G(N, N^2/4)$ for some large $N = N(n)$. (If N is chosen large enough, most realizations will do.) Alternatively we may take W_G where G is a realization of $G(N, \frac{1}{2})$, or a sufficiently large deterministic quasi-random graph such as a Paley graph, adjusted (arbitrarily) to have exactly $N^2/4$ edges.

We construct inductively a sequence of step graphons W_n on $[0, 1]$, with values in $\{0, 2^{-n}\}$. Let $W_0 = 1$. Given W_n , denote its steps by $I_{n,i}$. On each

rectangle (actually square) $I_{n,i} \times I_{n,j}$ where W_n is non-zero, and thus equal to 2^n , let W_{n+1} be a scaled copy of $2^{n+1}U_{n+1}$; on the other rectangles, where $W_n = 0$, let $W_{n+1} = 0$. In other words: Let U_n have m_n steps of length m_n^{-1} , and let $M_n := \prod_{i=1}^n m_i$. Then, let $\tilde{U}_n(x, y) := U_n(\{M_{n-1}x\}, \{M_{n-1}y\})$ and $W_n := 2^n \prod_{j=1}^n \tilde{U}_j$.

It is easily seen that $\|W_n - W_{n+1}\|_{\square} < 2^{-n}$, so (W_n) is a Cauchy sequence for the cut norm, and thus for the cut metric. However, W_n does not converge in the cut metric (and thus also not in the cut norm). To see this, suppose that $\delta_{\square}(W_n, W) \rightarrow 0$ for some graphon W . Then $W \geq 0$ a.e. We do not assume that W is defined on $[0, 1]$, but we may and shall assume that is defined on \mathbb{R}_+ , see Proposition 2.3. We also extend each W_n trivially to \mathbb{R}_+ , adding another step $I_{n,0} = (1, \infty)$. Then, by Proposition 2.4, there exist measure-preserving bijections $\varphi_n : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\|W_n - W^{\varphi_n}\|_{\square} < \delta_{\square}(W_n, W) + 1/n \rightarrow 0$.

For any $N \geq 1$, on each rectangle $I_{N,i} \times I_{N,j}$ where $W_N = 0$, we have $W_n = 0$ for all $n \geq N$. Call such rectangles *good* (for N). Then, on each good rectangle $Q = I_{N,i} \times I_{N,j}$, for $n \geq N$,

$$\int_Q W^{\varphi_n} = \int_Q (W^{\varphi_n} - W_n) \leq \|W^{\varphi_n} - W_n\|_{\square} \rightarrow 0. \quad (6.20)$$

Let A_N be the union of all good rectangles for a given N , and let $B_N := \mathbb{R}_+^2 \setminus A_N$; note that $\lambda^2(B_N) = 2^{-N}$ since $1 = \int W_N = 2^N \lambda^2(B_N)$. Given any $\varepsilon > 0$, we can find N such that if B is a set with $\lambda^2(B) < 2^{-N}$, then $\int_B W < \varepsilon$. This implies that $\int_{B_N} W^{\varphi_n} = \int_{\varphi_n \otimes \varphi_n(B_N)} W < \varepsilon$ for every n . Furthermore, (6.20) implies that $\int_{A_N} W^{\varphi_n} \rightarrow 0$ as $n \rightarrow \infty$. Since

$$\int_{\mathbb{R}_+^2} W = \int_{\mathbb{R}_+^2} W^{\varphi_n} = \int_{A_N} W^{\varphi_n} + \int_{B_N} W^{\varphi_n}, \quad (6.21)$$

it follows by letting $n \rightarrow \infty$ that $\int_{\mathbb{R}_+^2} W \leq \varepsilon$. Since ε is arbitrary, this implies $\int W = 0$ (and thus $W = 0$ a.e.).

On the other hand, we have $\int W_n = 1$ for each n , and it follows from $\delta_{\square}(W_n, W) \rightarrow 0$ that $\int W = 1$, a contradiction.

This shows that (W_n) is a Cauchy sequence for the cut metric that does not converge.

Note that $\|W_n\|_{L^1} = 1$, so this example shows that we cannot take $p = 1$ in Theorem 6.4(i)–(iii) or Theorem 6.6(iii),(v).

Example 6.12 (the set of $[-1, 1]$ -valued graphons on \mathbb{R}_+ is not complete). Let $S = \mathbb{R}_+$. For $n \geq 1$, let V_n be a graphon on $[0, 1]$ with values in $\{\pm 1\}$, and thus $\|V_n\|_{L^1} = 1$, such that $\|V_n\|_{\square} < 2^{-n}$. For example, we can take $V_n := 2U_n - 1$ with U_n as in Example 6.11.

Let $\tilde{V}_n(x, y) := V(x - n + 1, y - n + 1)$ when $(x, y) \in (n - 1, n]^2$ and 0 otherwise, i.e., \tilde{V}_n is V_n translated to $(n - 1, n]^2$. Finally, let $W_n := \sum_{k=1}^n \tilde{V}_k$. Then

$$\|W_n - W_{n-1}\|_{\square} = \|\tilde{V}_n\|_{\square} = \|V_n\|_{\square} < 2^{-n}, \quad (6.22)$$

and thus (W_n) is a Cauchy sequence for the cut norm.

However, there is no graphon W such that $\|W_n - W\|_{\square} \rightarrow 0$. In fact, suppose that this holds for some W . Then, for any integer N ,

$$\|W_n - W\|_{\square, [0, N]} = \|(W_n - W)\mathbf{1}_{[0, N]^2}\|_{\square} \leq \|W_n - W\|_{\square} \rightarrow 0 \quad (6.23)$$

as $n \rightarrow \infty$. On the other hand, for $N \geq n$, $W_n = W_N$ on $[0, N]^2$. Hence, $\|W_N - W\|_{\square, [0, N]} = 0$, and thus $W = W_N$ a.e. on $[0, N]^2$. Consequently,

$$\int_{[0, N]^2} |W| = \int_{[0, N]^2} |W_N| = \sum_{k=1}^N \int_{[0, N]^2} |\tilde{V}_k| = N. \quad (6.24)$$

Letting $N \rightarrow \infty$, we see that $\int |W| = \infty$, which contradicts our assumption that W is a graphon and thus integrable. Consequently, the uniformly bounded Cauchy sequence (W_n) does not converge in the cut norm.

We have so far considered the cut norm; we now show that the same properties hold for the cut metric. It follows from (6.22) that (W_n) also is a Cauchy sequence for the cut metric.

Suppose that $\delta_{\square}(W_n, W) \rightarrow 0$ for some graphon W . We may suppose that W too is defined on \mathbb{R}_+ . Then, by Proposition 2.4, there exist measure-preserving bijections $\varphi_n : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\|W_n - W^{\varphi_n}\|_{\square} < \delta_{\square}(W_n, W) + 1/n \rightarrow 0 \quad (6.25)$$

as $n \rightarrow \infty$. Taking the restrictions to $[0, N] \times [0, N]$, we see that for any $n \geq N$,

$$\|W_n - W^{\varphi_n}|_{[0, N] \times [0, N]}\|_{\square} = \|(W_n - W^{\varphi_n})|_{[0, N] \times [0, N]}\|_{\square} \leq \|W_n - W^{\varphi_n}\|_{\square} \rightarrow 0 \quad (6.26)$$

and thus, by Lemma 6.5(ii), for every $N \geq 1$,

$$\|W_N\|_{L^1} \leq \sup_n \|W^{\varphi_n}|_{[0, N] \times [0, N]}\|_{L^1} \leq \sup_n \|W^{\varphi_n}\|_{L^1} = \|W\|_{L^1}. \quad (6.27)$$

However, $\|W_N\|_{L^1} = N$ by (6.24), a contradiction.

Consequently, (W_n) is also for the cut metric a Cauchy sequence that does not converge.

Example 6.13 (An L^p -bounded set that is not relatively complete). Let V_n be as in Example 6.12 and let $V_n^* := 2^{-2n}\Upsilon_{2^{2n}}^{(2)}(V_n)$, see (2.19). Define now $\tilde{V}_n(x, y) := V_n^*(x - 2^n, y - 2^n)$ on $(2^n, 2^{n+1}]^2$ and 0 elsewhere, and let again $W_n := \sum_{k=1}^n \tilde{V}_k$. We have, as in (6.22),

$$\|\tilde{V}_n\|_{\square} = \|V_n^*\|_{\square} = 2^{-2n}\|\Upsilon_{2^{2n}}^{(2)}(V_n)\|_{\square} = \|V_n\|_{\square} < 2^{-n}. \quad (6.28)$$

Furthermore, for any $p \geq 1$,

$$\int |\tilde{V}_n|^p = \int |V_n^*|^p = 2^{-2pn}\lambda([0, 2^n]^2) = 2^{2(1-p)n}; \quad (6.29)$$

in particular, $\|\tilde{V}_n\|_{L^1} = 1$, and the same argument as in Example 6.12 (now using $[0, 2^{N+1}]^2$) shows that (W_n) is a Cauchy sequence in cut norm and cut metric that does not converge to any graphon.

Moreover, for any $p > 1$, by (6.29),

$$\int |W_n|^p = \sum_{k=1}^n \int |\tilde{V}_k|^p \leq \sum_{k=1}^{\infty} 2^{-2(p-1)k} < \infty, \quad (6.30)$$

and thus the sequence (W_n) is L^p -bounded for every $p > 1$.

Example 6.14 (weak convergence does not imply convergence in cut norm or cut metric). Let $h_n := \text{sgn}(\sin(2^n \pi x))$ on $S = [0, 1]$ (the Rademacher functions). Define

$$W_n(x, y) := \begin{cases} h_n(x), & 0 < x < \frac{1}{2} < y < 1, \\ h_n(y), & 0 < y < \frac{1}{2} < x < 1, \\ 0, & \text{otherwise.} \end{cases} \quad (6.31)$$

Then $h_n \rightarrow 0$ weakly in $L^1([0, 1])$ as $n \rightarrow \infty$, and it follows easily that $W_n \rightarrow 0$ weakly in $L^1(S^2)$. However, if $T_n := \{x \in (0, \frac{1}{2}) : h_n(x) > 0\}$, then

$$\|W_n - 0\|_{\square} = \|W_n\|_{\square} \geq \int_{T_n \times (\frac{1}{2}, 1)} W_n = \frac{1}{8}. \quad (6.32)$$

Hence $W_n \not\rightarrow 0$ in the cut norm.

Moreover, it is easily seen that for $n \geq 2$, W_n is the pull-back $W_2^{\varphi_n}$ of W_2 by the measure-preserving map φ_n defined by $\varphi_n(x) = \frac{1}{2}\{2^{n-1}x\}$ for $x \leq \frac{1}{2}$, $\varphi_n(x) = x$ for $x > \frac{1}{2}$. Hence, $W_n \cong W_2$ and, trivially, $W_n \rightarrow W_2 \neq 0$ for the metric δ_{\square} . Consequently, $W_n \not\rightarrow 0$ also for the cut metric δ_{\square} .

It follows that the set $\{W_n\}_{n \geq 1} \cup \{0\}$ is weakly compact, but discrete and thus not compact for both the cut norm and the cut metric.

We can get a similar example with $[0, 1]$ -valued graphons by considering $\widetilde{W}_n := (W_n + 1)/2$.

Example 6.15 (uniform integrability is not necessary I). Similarly to the construction of U_n in Example 6.11, we can for each n find a step graphon V_n on $[0, 1]$ with steps of equal measure, values in $\{0, 1\}$, and

$$\|V_n - \frac{1}{n}\|_{\square} < 4^{-n}. \quad (6.33)$$

For example, we can take V_n as W_G for a realization of $G(N, 1/n)$ for a sufficiently large $N = N(n)$.

If we further define $W_n := nV_n$, then by (6.33), W_n is $\{0, n\}$ -valued and

$$\|W_n - 1\|_{\square} < n4^{-n} < 2^{-n}. \quad (6.34)$$

Hence, $W_n \rightarrow 1$ as $n \rightarrow \infty$ for the cut norm, and thus also for the cut metric. However, since W_n is $\{0, n\}$ -valued, for any B and all $n > B$,

$$\int_{W_n > B} W_n = \int_{[0, 1]^2} W_n \rightarrow 1, \quad (6.35)$$

where we also used (6.34). Hence, (UI2) does not hold for $\{W_n\}$, so the set is not uniformly integrable, and not even semiuniformly integrable.

Consequently, the set $\{W_n\}_{n \geq 1} \cup \{1\}$ is compact and complete for both the cut norm on $[0, 1]$ and the cut metric, but not uniformly integrable.

Note that in this example, the graphons are all non-negative and defined on $[0, 1]$, but unbounded.

Example 6.16 (uniform integrability is not necessary II). We obtain a related example, where the graphons are $[-1, 1]$ -valued but defined on \mathbb{R}_+ , by letting V_n be as in Example 6.12 and taking the stretched graphons $\overline{W}_n := \Upsilon_n^{(2)}V_n$. Then, cf. (2.20), $\|\overline{W}_n\|_{L^1} = n$ and $\|\overline{W}_n\|_{\square} < n2^{-n}$, so

$\overline{W}_n \rightarrow 0$ in cut norm and thus in cut metric, but \overline{W}_n is not uniformly integrable, since (UI1) does not hold.

Note that in this example, in contrast to Example 6.15, the graphons are uniformly bounded, but take negative values and are defined on an infinite measure space.

Remark 6.17. We cannot combine the assertions of Examples 6.15 and 6.16. First, a uniformly bounded set of graphons defined on $[0, 1]$ is automatically uniformly integrable. Secondly, a sequence of non-negative graphons that converges in the cut norm or cut metric is bounded in L^1 ; hence, if it also is uniformly bounded, then it is uniformly integrable.

6.3. Completeness for δ_1 and δ_p . Our main interest is the cut metric δ_{\square} , but we also give a simple corresponding result for δ_1 and δ_p .

Theorem 6.18. (i) *The set of all graphons is complete for the metric δ_1 .*
(ii) *For any $p > 1$, and any $C < \infty$, the set of non-negative graphons W in L^p such that $\|W\|_{L^1} \leq C$ is complete for the metric δ_p .*

Proof. Both parts follow by the same argument as in the proof of Theorem 6.6, using [4, Remark 4.4] to see that a δ_p -version of Lemma 6.7 holds for $p \geq 1$, and (instead of Theorem 6.1) the fact that $L^1(\mathbb{R}_+^2)$ and $L^p(\mathbb{R}_+^2)$ are complete; for (ii) also the fact that if $W_n \rightarrow W$ in L^p and $\|W_n\|_{L^1} \leq C$ for all n , then $\|W\|_{L^1} \leq C$. \square

The L^1 -boundedness in (ii) is necessary in general, as is seen by the following example.

Example 6.19 (the set of non-negative graphons is not complete for δ_p , $p > 1$). Let $f(x) := 1/(x+1)$ and define the graphon W on \mathbb{R}_+ by $W(x, y) := f(x)f(y)$. Let further $W_n := W\mathbf{1}_{[0, n]^2}$. Then, for every $p > 1$, $f \in L^p(\mathbb{R}_+)$ and thus $W \in L^p(\mathbb{R}_+^2)$ and $W_n \rightarrow W$ in L^p , and thus in δ_p . Hence (W_n) is a Cauchy sequence of graphons for δ_p . However, $W \notin L^1$, so W is not a graphon. Furthermore, W_n cannot have any other limit V that is a non-negative graphon. To see this, suppose that V is a non-negative graphon and that $\delta_p(W_n, V) \rightarrow 0$. We may, as usual, assume that V is defined on \mathbb{R}_+ . By [4, Proposition 4.3(c) and Remark 4.4], there exist measure-preserving bijections $\varphi_n : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\|W_n - V^{\varphi_n}\|_{L^p} \rightarrow 0$. Consequently,

$$\|W - V^{\varphi_n}\|_{L^p} \leq \|W - W_n\|_{L^p} + \|W_n - V^{\varphi_n}\|_{L^p} \rightarrow 0 \quad (6.36)$$

as $n \rightarrow \infty$. Hence $V^{\varphi_n} \rightarrow W$ in L^p , but this implies

$$\|W\|_{L^1} \leq \sup_n \|V^{\varphi_n}\|_{L^1} = \|V\|_{L^1} < \infty, \quad (6.37)$$

a contradiction.

Remark 6.20. If we only consider non-negative L^p graphons on probability spaces as in [5], then the set of all of them is complete for δ_p , for any $p > 1$; this follows since a Cauchy sequence (W_n) has

$$\sup_n \|W_n\|_{L^1} \leq \sup_n \|W_n\|_{L^p} = \sup_n \delta_p(W_n, 0) < \infty \quad (6.38)$$

and thus W_n converges by Theorem 6.18(ii).

7. COMPACTNESS

Borgs, Chayes, Cohn and Holden [4, Theorem 2.12] prove a partial characterisation of relatively compact sets for the cut metric. Their theorem is stated in terms on convergent (sub)sequences, but it implies immediately a statement on relative compactness, viz. the case of uniformly bounded graphons in Theorem 7.7 below. (In fact, [4, Theorem 2.12] is equivalent to this compactness result together with the completeness result in Theorem 6.6(iv).)

We give a new proof of their result and extend it in several ways; in particular, we give extensions from uniformly bounded to uniformly integrable graphons, and from non-negative to signed graphons. (Recall, however, that uniform integrability is not needed for convergence, and thus not for compactness, not even for graphons on $[0, 1]$, see Example 6.15.)

We begin with some definitions.

We say, as in [4, Definition 2.11] that a set \mathcal{W} of graphons has *uniformly regular tails* if for every $\varepsilon > 0$, there exists $M < \infty$ such that for every graphon $(W, S, \mu) \in \mathcal{W}$, there exists a set $U \subseteq S$ such that $\mu(U) \leq M$ and

$$\|W - W\mathbf{1}_{U \times U}\|_{L^1} < \varepsilon. \quad (7.1)$$

Similarly, we say that a set \mathcal{W} of graphons has *uniformly cut regular tails* if for every $\varepsilon > 0$, there exists $M < \infty$ such that for every graphon $(W, S, \mu) \in \mathcal{W}$, there exists a set $U \subseteq S$ such that $\mu(U) \leq M$ and

$$\|W - W\mathbf{1}_{U \times U}\|_{\square} < \varepsilon. \quad (7.2)$$

I.e., we relax the L^1 -norm in (7.1) to the cut norm. For non-negative graphons the two notions are equivalent, as shown in the following simple lemma. (Example 7.11 shows that the notions differ in general.)

Lemma 7.1. (i) *Any set of graphons with uniformly regular tails has uniformly cut regular tails.*

(ii) *A set of non-negative graphons has uniformly regular tails if and only if it has uniformly cut regular tails.*

Proof. (i): By (2.3).

(ii): If $W \geq 0$, then also $W - W\mathbf{1}_{U \times U} \geq 0$ and thus $\|W - W\mathbf{1}_{U \times U}\|_{\square} = \|W - W\mathbf{1}_{U \times U}\|_{L^1}$. \square

We say that a set \mathcal{W} of graphons is *upper cut regular* if for every $\varepsilon > 0$, there exists $B < \infty$ such that for every graphon $(W, S, \mu) \in \mathcal{W}$, there exists a graphon V on S such that $|V| \leq B$ and

$$\|W - V\|_{\square} < \varepsilon. \quad (7.3)$$

Note that the corresponding notion with $\|\cdot\|_{L^1}$ in (7.3) is equivalent to (UI2). Hence a semiuniformly integrable set is upper cut regular.

Remark 7.2. Borgs, Chayes, Cohn and Zhao [5, Definition C.2] give, for graphons on $[0, 1]$, a definition of what they call *uniformly upper regular* sequences of graphons and use this for (essentially) a characterisation of sequential compactness in [5, Theorems C.13 and C.5]. Our definition of upper cut regular is quite different, but it is used for a similar purpose. It

seems interesting to investigate the relation between these notions, but we have not yet done so, and leave it as an open problem.

Recall that if A is a set in a metric space (X, d) , then an ε -net for A is a subset $Y \subseteq X$ such that for every $x \in A$ there exists $y \in Y$ with $d(x, y) < \varepsilon$. The set A is *totally bounded* if for every $\varepsilon > 0$ there exists a finite ε -net for A . (We may here further assume that the ε -net is a subset of A , but we find it more convenient not to do so.) Recall also that A is compact if and only if it is complete and totally bounded; it follows that A is relatively compact if and only if it is relatively complete and totally bounded.

We can now characterise totally bounded sets of graphons. Note that condition in (iii) below is a combination of the two conditions in (ii).

Theorem 7.3. *Let \mathcal{W} be a set of graphons. Then the following are equivalent.*

- (i) \mathcal{W} is totally bounded for the cut metric.
- (ii) \mathcal{W} is upper cut regular and has uniformly cut regular tails.
- (iii) For every $\varepsilon > 0$ there exists $M < \infty$ and $B < \infty$ such that for every graphon $(W, S, \mu) \in \mathcal{W}$ there exists a set $U \subseteq S$ with $\mu(U) \leq M$ and a graphon V on S such that $|V| \leq B \mathbf{1}_{U \times U}$ and $\|W - V\|_{\square} < \varepsilon$.

We postpone the proof to the next section, and give first some consequences of the theorem. We consider two different simplifying assumptions in the following two subsections.

7.1. Compactness, the semiuniformly integrable case. In this subsection, we consider for simplicity only sets of graphons that are semiuniformly integrable, for example sets that are uniformly integrable or uniformly bounded. Such sets are always upper cut regular, as said above, since we may take $V := W \mathbf{1}_{|W| \leq B}$ in (7.3) for some large B . Hence, Theorem 7.3 implies the following.

Corollary 7.4. *A semiuniformly integrable set \mathcal{W} of graphons is totally bounded for the cut metric if and only if it has uniformly cut regular tails. \square*

We combine Corollary 7.4 with results on completeness in Section 6 to obtain results on compactness.

Theorem 7.5. *A uniformly integrable set of graphons is relatively compact for the cut metric if and only if it has uniformly cut regular tails.*

Proof. By Theorem 6.6(i) and Corollary 7.4. \square

Theorem 7.6. *A semiuniformly integrable set of non-negative graphons is relatively compact for the cut metric if and only if it has uniformly cut regular tails.*

Proof. By Theorem 6.6(ii) and Corollary 7.4. \square

Borgs, Chayes, Cohn and Holden [4, Theorem 2.12] give a similar result (for uniformly bounded graphons) using uniformly regular tails instead of uniformly cut regular tails; we obtain a (reformulation of) their result as a corollary, where we furthermore extend their result from uniformly bounded graphons to semiuniformly integrable graphons.

Theorem 7.7 (Mainly [4, Theorem 2.12]). *Let \mathcal{W} be a semiuniformly integrable set of graphons. (For example a uniformly bounded set, or a set bounded in L^p for some $p > 1$.)*

- (i) *If \mathcal{W} has uniformly regular tails, then \mathcal{W} is relatively compact for the cut metric.*
- (ii) *The converse holds if all graphons in \mathcal{W} are non-negative.*

We prove first a simple lemma.

Lemma 7.8. *A set of graphons that is semiuniformly integrable and has uniformly regular tails is uniformly integrable.*

Proof. Denote the set by \mathcal{W} . By the definition (UI2) of semiuniformly integrable, there exists $B < \infty$ such that $\int_{|W|>B} |W| < 1$ for every $W \in \mathcal{W}$. Furthermore, take $\varepsilon = 1$ in the definition of uniformly regular tails. Thus, there exists $M < \infty$ such that if $W \in \mathcal{W}$ is defined on (S, μ) , then there exists $U \subseteq S$ with $\mu(U) \leq M$ and $\|W - W\mathbf{1}_{U \times U}\|_{L^1} < 1$. Consequently,

$$\begin{aligned} \|W\|_{L^1} &= \|W - W\mathbf{1}_{U \times U}\|_{L^1} + \int_{U \times U} |W| \\ &\leq 1 + \int_{|W|>B} |W| + \int_{U \times U} B \leq 1 + 1 + M^2 B. \end{aligned} \quad (7.4)$$

Hence, (UI1) holds, so \mathcal{W} is uniformly integrable. \square

Proof of Theorem 7.7. (i): By Lemma 7.8, \mathcal{W} is uniformly integrable, and \mathcal{W} has uniformly cut regular tails, so \mathcal{W} is relatively compact by Theorem 7.5.

(ii): If \mathcal{W} is relatively compact, then it is totally bounded, and Theorem 7.3 shows that it has uniformly cut regular tails, and the result follows by Lemma 7.1(ii). \square

Example 7.9 below shows that a semiuniformly integrable set of graphons with uniformly cut regular tails does not have to be relatively compact. Note that Theorems 7.5, 7.6 and 7.7(i) strengthen the assumption in three different ways (uniformly integrable, non-negative and uniformly regular tails, respectively), and that we thus need these stronger assumptions.

Furthermore, Example 7.10 shows that uniform integrability is not enough to imply relative compactness, even for $[0, 1]$ -valued graphons; hence the first condition in Theorems 7.5–7.7 does not suffice to imply the other conditions.

Example 7.11 shows that a compact set of signed graphons does not have to have uniformly regular tails, so Theorem 7.7(ii) does not hold without assuming non-negativity.

Example 7.9 (A semiuniformly integrable set with uniformly cut regular tails that is not relatively compact). The sequence (W_n) in Example 6.12 is a Cauchy sequence for the cut metric, and thus it is totally bounded; hence it has uniformly cut regular tails by Theorem 7.3. Furthermore, the sequence is uniformly bounded and is thus semiuniformly integrable. Nevertheless, the sequence does not converge, so there can be no convergent subsequence and thus the $\{W_n\}$ is not relatively compact.

Thus, we cannot replace uniformly integrable by semiuniformly integrable in Theorem 7.5, non-negative by arbitrary (signed) in Theorem 7.6, or uniformly regular tails by uniformly cut regular tails in Theorem 7.7.

Example 7.10 (A uniformly integrable sequence that is not relatively compact). Let W_n be the graphon on \mathbb{R}_+ given by $W_n := n^{-2}\mathbf{1}_{[0,n]\times[0,n]}$. The graphons W_n are $[0,1]$ -valued and thus uniformly bounded; furthermore, $\|W_n\|_{L^1} = 1$ so the set $\{W_n\}$ is also L^1 -bounded and thus uniformly integrable.

However, $\|W_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$, and it follows by Remark 6.9 that if $\delta_\square(W_n, W) \rightarrow 0$ for some subsequence, then $\|W\|_\infty = 0$, so $W = 0$ a.e.; however, this is impossible since $\delta_\square(W_n, 0) = \int W_n = 1$. Hence, no subsequence converges, and thus $\{W_n\}$ is not relatively compact.

By Theorem 7.3, $\{W_n\}$ cannot have uniformly cut regular tails.

Example 7.11 (convergence does not imply uniformly regular tails without non-negativity). Let V_n be as in Example 6.12 and both stretch and rescale them to $W_n := n^{-1}\Upsilon_n(V_n)$. Then, see (2.20),

$$\|W_n\|_{L^1} = n^{-1}\|\Upsilon_n(V_n)\|_{L^1} = \|V_n\|_{L^1} = 1, \quad (7.5)$$

while

$$\|W_n\|_\square = n^{-1}\|\Upsilon_n(V_n)\|_\square = \|V_n\|_\square < 2^{-n}. \quad (7.6)$$

By (7.6), $W_n \rightarrow 0$ in cut norm and thus in cut metric; hence the set $\{W_n\}$ is relatively compact for the cut metric, and $\{W_n\} \cup \{0\}$ is compact.

It follows from Theorem 7.3 (or directly from the definition) that the graphons W_n have uniformly cut regular tails.

However, W_n do not have uniformly regular tails. In fact, $|W_n| \leq n^{-1}$, and thus for any M and any set U with $\lambda(U) \leq M$,

$$\|W_n \mathbf{1}_{U \times U}\|_{L^1} \leq n^{-1}\lambda^2(U \times U) \leq M^2/n; \quad (7.7)$$

hence, by (7.5),

$$\|W_n - W_n \mathbf{1}_{U \times U}\|_{L^1} \geq 1 - M^2/n. \quad (7.8)$$

Since $1 - M^2/n \geq \frac{1}{2}$ for all large n , (7.8) shows that the graphons do not have uniformly regular tails.

Hence, uniformly regular tails is not necessary for compactness.

Note also that $|W_n| \leq n^{-1} \leq 1$, and thus the graphons W_n are all uniformly bounded. By this and (7.5), they are also uniformly integrable.

7.2. Compactness, the standard case of probability spaces. Finally, we consider the standard setting of graphons defined on probability spaces. (Or, equivalently, graphons defined on $[0,1]$.) In this setting, it is well-known, and of fundamental importance, that the set of all $[0,1]$ -valued graphons is compact, as proved by Lovász and Szegedy [24]. This was extended to L^p -bounded and uniformly integrable sets of graphons on probability spaces by Borgs, Chayes, Cohn and Zhao [5, Theorems 2.13 and C.7]. We recover these results as corollaries.

Note first that a graphon W defined on a probability space obviously is also equivalent to graphons defined on other spaces; one example is a trivial extension of W , and the following lemma, the proof of which is postponed to the next section, shows that this is essentially the only possibility.

Lemma 7.12. *Let $W = (W, S, \mu)$ be a graphon. Then the following are equivalent.*

- (i) $W \cong W'$ for some graphon W' defined on a probability space.
- (ii) W is a.e. equal to a trivial extension of a graphon defined on a measure space (S', μ') with $\mu'(S') \leq 1$.
- (iii) There exists a set $U \subseteq S$ with $\mu(U) \leq 1$ such that $W(x, y) = 0$ a.e. on $(S \times S) \setminus (U \times U)$.
- (iv) There exists a function $f : S \rightarrow [0, 1]$ with $\int f \, d\mu \leq 1$ such that $f(x)f(y)W(x, y) = W(x, y)$ a.e.

Furthermore, for two graphons W_1 and W_2 defined on probability spaces, the definition by [4], see Section 2.4, of the cut distance $\delta_{\square}(W_1, W_2)$ is the same as the usual definition for probability spaces in e.g. [3; 5; 7; 18; 22]. Moreover, the next lemma shows that when considering limits of sequences of graphons on probability spaces, it does not matter whether we require also the limit to be defined on a probability space or allow it to be defined on an arbitrary σ -finite measure space. In particular, completeness and compactness properties of a set \mathcal{W} of graphons on probability spaces do not depend on whether we consider \mathcal{W} as a subset of the set of all such graphons, or of all graphons on σ -finite measure spaces (We are more careful than usually in the statement and talk explicitly about equivalence classes, since as just noted, a graphon on a probability space is equivalent to graphons on other measure spaces.)

Lemma 7.13. *If W_n are graphons defined on probability spaces, and W is a graphon such that $W_n \rightarrow W$ in the cut metric, then there exists an equivalent graphon $W' \cong W$ that is defined on a probability space. In other words, for the cut metric, the set of equivalence classes of graphons defined on probability spaces is a closed subset of the set of equivalence classes of all graphons defined on σ -finite measure spaces.*

We postpone the proof of this lemma too to next section.

We record also a trivial fact.

Lemma 7.14. *Any set of graphons defined on probability spaces has uniformly regular tails, and thus uniformly cut regular tails.*

Proof. Take $M = 1$ and $U = S$ in the definition. □

We return to compactness properties.

Theorem 7.15. *Let \mathcal{W} be a set of graphons defined on probability spaces. Then \mathcal{W} is totally bounded for the cut metric if and only if it is upper cut regular.*

Hence, \mathcal{W} is relatively compact if and only if it is upper cut regular and relatively complete.

Proof. By Lemma 7.14 and Theorem 7.3. □

Corollary 7.16 ([5, Theorem C.7]). *A uniformly integrable set of graphons defined on probability spaces is relatively compact for the cut metric.*

Proof. A uniformly integrable set is upper cut regular, as said in Section 7.1, so the result follows by Theorems 7.15 and 6.1. (Or by Lemma 7.14 and Theorem 7.5.) □

Corollary 7.17 ([5, Theorem 2.13]). *Let $1 < p < \infty$ and $C < \infty$. Then the set of all graphons W defined on probability spaces such that $\|W\|_{L^p} \leq C$ is compact for the cut metric.*

Proof. Denote this set by \mathcal{W} . As remarked in Section 6, the set \mathcal{W} is uniformly integrable, and thus it is relatively compact by Corollary 7.16, so it remains only to show that \mathcal{W} is closed, i.e., that if W_n is a sequence of graphons in \mathcal{W} and $W_n \rightarrow W$ in cut metric, then $W \in \mathcal{W}$ for some $W' \cong W$. This follows by Lemmas 6.8 and 7.13. \square

Remark 7.18. Borgs, Chayes, Cohn and Zhao [5, Section 2.5] give a definition of L^p upper regular sequences of graphons (defined on $[0, 1]$), similar to their definition of uniformly upper regular sequences mentioned in Remark 7.2 above. They use this in criteria for sequential compactness with a limit in L^p . As in Remark 7.2, we leave it as an open problem to investigate the relation with our notions and results.

8. PROOFS OF THEOREM 7.3 AND LEMMAS 7.12–7.13

We first prove a couple of technical lemmas. The first yields alternative (but equivalent) characterisations of the properties uniformly [cut] regular tails.

Lemma 8.1. (i) *A set \mathcal{W} of graphons has uniformly regular tails if for every $\varepsilon > 0$, there exists $M < \infty$ such that for every graphon $(W, S, \mu) \in \mathcal{W}$, there exists a measurable function $f : S \rightarrow [0, 1]$ such that $\int f \, d\mu \leq M$ and*

$$\|W - (f \otimes f)W\|_{L^1} < \varepsilon. \quad (8.1)$$

(ii) *A set \mathcal{W} of graphons has uniformly cut regular tails if for every $\varepsilon > 0$, there exists $M < \infty$ such that for every graphon $(W, S, \mu) \in \mathcal{W}$, there exists a measurable function $f : S \rightarrow [0, 1]$ such that $\int f \, d\mu \leq M$ and*

$$\|W - (f \otimes f)W\|_{\square} < \varepsilon. \quad (8.2)$$

Proof. We prove (ii); the same proof works for (i).

If (7.2) holds, then take $f := \mathbf{1}_U$.

Conversely, suppose that (8.2) holds. Define $U := \{x : f(x) > \frac{1}{2}\}$. Then $\mu(U) \leq 2\|f\|_{L^1} \leq 2M$. Moreover,

$$\mathbf{1}_{U^c}(x)W(x, y) = \sum_{k=0}^{\infty} \mathbf{1}_{U^c}(x)(f(x)f(y))^k(1 - f(x)f(y))W(x, y) \quad (8.3)$$

and thus, using (2.6) and (8.2),

$$\begin{aligned} \|\mathbf{1}_{U^c}(x)W(x, y)\|_{\square} &\leq \sum_{k=0}^{\infty} \left\| \mathbf{1}_{U^c}(x)(f(x)f(y))^k(1 - f(x)f(y))W(x, y) \right\|_{\square} \\ &\leq \sum_{k=0}^{\infty} \sup_x (\mathbf{1}_{U^c}(x)f(x)^k) \sup_y (f(y)^k) \left\| (1 - f(x)f(y))W(x, y) \right\|_{\square} \\ &< \sum_{k=0}^{\infty} 2^{-k}\varepsilon = 2\varepsilon. \end{aligned} \quad (8.4)$$

By symmetry also $\|\mathbf{1}_{U^c}(y)W(x, y)\|_{\square} < 2\varepsilon$, and thus

$$\begin{aligned} \|W - \mathbf{1}_{U \times U}W\|_{\square} &\leq \|\mathbf{1}_{U^c}(x)W(x, y)\|_{\square} + \|\mathbf{1}_U(x)\mathbf{1}_{U^c}(y)W(x, y)\|_{\square} \\ &\leq \|\mathbf{1}_{U^c}(x)W(x, y)\|_{\square} + \|\mathbf{1}_{U^c}(y)W(x, y)\|_{\square} \\ &< 4\varepsilon. \end{aligned} \quad (8.5)$$

Consequently, (7.2) holds, with (M, ε) replaced by $(2M, 4\varepsilon)$. \square

We say that two sets of graphons are equivalent if every graphon in one of the sets is equivalent to some graphon in the other set.

Lemma 8.2. *Let \mathcal{W} and \mathcal{W}' be two equivalent sets of graphons.*

- (i) *If \mathcal{W} has uniformly regular tails, then so has \mathcal{W}' , and conversely.*
- (ii) *If \mathcal{W} has uniformly cut regular tails, then so has \mathcal{W}' , and conversely.*

A special case of (i) is given in [4, Lemma 6.1].

Proof. Again, the same proof works for both parts; we choose (ii).

We show that for every ε and M , if W and W' are two equivalent graphons and one of them satisfies (8.2) for some f as in Lemma 8.1, then so does the other. The result then follows by Lemma 8.1.

By Theorem 5.6, it suffices to consider the case when W and W' are elementarily equivalent. The case of a trivial extension is trivial, and thus it suffices to consider the case of a pull-back $W' = W^\varphi$ for some measure-preserving map $\varphi : (S_1, \mathcal{F}_1, \mu_1) \rightarrow (S_2, \mathcal{F}_2, \mu_2)$.

First, if (8.2) holds, then using (2.7),

$$\|W^\varphi - (f^\varphi \otimes f^\varphi)W^\varphi\|_{\square, \mu_1} = \|W - (f \otimes f)W\|_{\square, \mu_2} < \varepsilon, \quad (8.6)$$

and $f^\varphi : S_1 \rightarrow [0, 1]$ with $\int f^\varphi d\mu_1 = \int f d\mu_2 \leq M$.

Conversely, suppose that $\|W^\varphi - (f \otimes f)W^\varphi\|_{\square, \mu_1} < \varepsilon$ for some $f : S_1 \rightarrow [0, 1]$ with $\int_{S_1} f \leq M$. Let $\mathcal{F}' := \varphi^{-1}(\mathcal{F}_2) := \{\varphi^{-1}(A) : A \in \mathcal{F}_2\}$, and let $f' := \mathbb{E}(f | \mathcal{F}')$. (Although conditional expectations usually are defined for probability spaces only, there is no problem to extend the definition to σ -finite measure spaces, for example by considering a partition $S = \bigcup_k S_k$ into subsets $S_k \in \mathcal{F}'$ with finite measure.) Since f' is measurable for $\mathcal{F}' = \varphi^{-1}(\mathcal{F}_2)$, $f' = g^\varphi$ for some $g : S_2 \rightarrow [0, 1]$ with

$$\int_{S_2} g d\mu_2 = \int_{S_1} g^\varphi d\mu_1 = \int_{S_1} f' d\mu_1 = \int_{S_1} f d\mu_1 \leq M. \quad (8.7)$$

Furthermore, W^φ is $(\mathcal{F}' \times \mathcal{F}')$ -measurable, and thus $\mathbb{E}((f \otimes f)W^\varphi | \mathcal{F}' \times \mathcal{F}') = (f' \otimes f')W^\varphi$. Consequently,

$$\begin{aligned} \|W - (g \otimes g)W\|_{\square, \mu_2} &= \|(W - (g \otimes g)W)^\varphi\|_{\square, \mu_1} \\ &= \|W^\varphi - (f' \otimes f')W^\varphi\|_{\square, \mu_1} \\ &= \|\mathbb{E}(W^\varphi - (f \otimes f)W^\varphi | \mathcal{F}' \times \mathcal{F}')\|_{\square, \mu_1} \\ &\leq \|W^\varphi - (f \otimes f)W^\varphi\|_{\square, \mu_1} < \varepsilon \end{aligned} \quad (8.8)$$

where the last but one inequality easily follows from (2.5). \square

Proof of Theorem 7.3. (i) \implies (ii). First, note that Lemma 8.2 shows that if we replace \mathcal{W} by an equivalent set of graphons, then (ii) is preserved. Thus, using Proposition 2.3 we may assume that every graphon W in \mathcal{W} is defined on \mathbb{R}_+ .

Let $\varepsilon > 0$. By assumption, there exists a finite ε -net $\{W_i\}_{i=1}^N$ for \mathcal{W} ; we may assume that also every W_i is defined on \mathbb{R}_+ . Thus, if $W \in \mathcal{W}$, then there exists W_i such that $\delta_{\square}(W, W_i) < \varepsilon$, and thus by Proposition 2.4 a measure-preserving bijection $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\|W - W_i^{\varphi}\|_{\square} < \varepsilon. \quad (8.9)$$

Next, every W_i is integrable, and thus we may find a set U_i of finite measure such that $\|W_i - W_i \mathbf{1}_{U_i \times U_i}\|_{\square} \leq \|W_i - W_i \mathbf{1}_{U_i \times U_i}\|_{L^1} < \varepsilon$ and a number B_i such that if $\overline{W}_i := W_i \mathbf{1}_{|W_i| \leq B_i}$, then $\|W_i - \overline{W}_i\|_{\square} \leq \|W_i - \overline{W}_i\|_{L^1} < \varepsilon$. Let $M := \max_{i \leq N} \mu(U_i)$ and $B := \max_{i \leq N} B_i$.

If (8.9) holds, then let $U := \varphi^{-1}(U_i)$. Then $W_i^{\varphi} \mathbf{1}_{U \times U} = W_i^{\varphi} \mathbf{1}_{U_i \times U_i}^{\varphi} = (W_i \mathbf{1}_{U_i \times U_i})^{\varphi}$ and thus

$$\begin{aligned} \|W - W \mathbf{1}_{U \times U}\|_{\square} & \leq \|W - W_i^{\varphi}\|_{\square} + \|(W_i - W_i \mathbf{1}_{U_i \times U_i})^{\varphi}\|_{\square} + \|(W_i^{\varphi} - W) \mathbf{1}_{U \times U}\|_{\square} \\ & < \varepsilon + \|W_i - W_i \mathbf{1}_{U_i \times U_i}\|_{\square} + \varepsilon < 3\varepsilon. \end{aligned} \quad (8.10)$$

Similarly,

$$\|W - \overline{W}_i^{\varphi}\|_{\square} \leq \|W - W_i^{\varphi}\|_{\square} + \|(W_i - \overline{W}_i)^{\varphi}\|_{\square} < \varepsilon + \|W_i - \overline{W}_i\|_{\square} < 2\varepsilon. \quad (8.11)$$

Since $\lambda(U) = \lambda(U_i) \leq M$ and $|\overline{W}_i^{\varphi}| \leq B_i \leq B$, it follows that \mathcal{W} has uniformly cut regular tails and is upper cut regular.

(ii) \implies (iii). Given $\varepsilon > 0$, let M and B be as in the definitions of uniformly cut regular tails and upper cut regular; thus if $W = (W, S, \mu) \in \mathcal{W}$, there exist $U \subseteq S$ with $\mu(U) \leq M$ such that (7.2) holds and a graphon V on S with $|V| \leq B$ such that (7.3) holds. Let $V' := V \mathbf{1}_{U \times U}$. Then $|V'| \leq B \mathbf{1}_{U \times U}$ and

$$\begin{aligned} \|W - V'\|_{\square} & \leq \|W - W \mathbf{1}_{U \times U}\|_{\square} + \|(W - V) \mathbf{1}_{U \times U}\|_{\square} \\ & \leq \|W - W \mathbf{1}_{U \times U}\|_{\square} + \|W - V\|_{\square} < 2\varepsilon. \end{aligned} \quad (8.12)$$

Thus (iii) follows.

(iii) \implies (i). Let $\varepsilon > 0$ and let M and B be as in (iii). Let $\mathcal{W}(M, B)$ be the set of all graphons (V, S, μ) with $|V| \leq B$ and $\mu(S) = M$. If W, U and V are as in (iii), then V is a trivial extension of the restriction $V_U := V|_{U \times U}$, which is defined on U with $\mu(U) \leq M$, and thus there exists a trivial extension V'_U of V_U to a measure space (S', μ') with $\mu'(S') = M$. Then $V'_U \in \mathcal{W}(M, B)$; furthermore,

$$\delta_{\square}(W, V'_U) = \delta_{\square}(W, V_U) = \delta_{\square}(W, V) \leq \|W - V\|_{\square} < \varepsilon. \quad (8.13)$$

Next, the mapping $\Psi : V \mapsto (\Upsilon_{M^{-2}}^{(1)} V + B)/(2B)$ maps $\mathcal{W}(M, B)$ bijectively onto the set of $[0, 1]$ -valued graphons defined on probability spaces, see (2.18), and Ψ is a homeomorphism for the cut metric. (In fact, $\delta_{\square}(V, V') = 2BM^2 \delta_{\square}(\Psi(V), \Psi(V'))$ for all $V, V' \in \mathcal{W}(M, B)$.) As is well-known from the standard theory of graphons on probability spaces (or on $[0, 1]$), the latter set is compact for the cut metric [24]. Consequently, $\mathcal{W}(M, B)$ is compact

for the cut metric and thus totally bounded. Hence, there exists a finite ε -net $\{V_i\}_{i=1}^N$ for $\mathcal{W}(M, B)$.

We have shown in (8.13) that if $W \in \mathcal{W}$, then there exists a graphon $V'_U \in \mathcal{W}(M, B)$ such that $\delta_{\square}(W, V'_U) < \varepsilon$; furthermore, since $\{V_i\}$ is an ε -net for $\mathcal{W}(M, B)$, there exists V_i such that $\delta_{\square}(V'_U, V_i) < \varepsilon$. Hence, $\delta_{\square}(W, V_i) < 2\varepsilon$, and it follows that $\{V_i\}_1^N$ is a finite 2ε -net for \mathcal{W} . Since ε is arbitrary, \mathcal{W} is totally bounded. \square

Proof of Lemma 7.12. (ii) \iff (iii). Clear by the definition of trival extension.

(iii) \implies (iv). Take $f := \mathbf{1}_U$.

(iv) \implies (iii). Take $U := \{x : f(x) = 1\}$.

(ii) \implies (i). If W is a trival extension of (W', S', μ') with $\mu'(S') \leq 1$, then W' has a trival extension W'' defined on a probability space, and $W \cong W' \cong W''$.

(i) \implies (iv). This follows by the same proof as for Lemma 8.2, with $M = 1$ and ' $< \varepsilon$ ' replaced by ' $= 0$ ' (or, equivalently, for all ε simultaneously), recalling (2.4). \square

Proof of Lemma 7.13. As usual, we may by Proposition 2.3 replace the graphons by equivalent graphons on (\mathbb{R}_+, λ) and assume that W and every W_n is defined on \mathbb{R}_+ . By Proposition 2.4, this implies the existence of pull-backs $W'_n := W_n \circ \varphi_n \cong W_n$ such that $\|W - W'_n\|_{\square} < \delta_{\square}(W, W_n) + 1/n \rightarrow 0$ as $n \rightarrow \infty$.

By Lemma 7.12, there exist sets $U_n \subset \mathbb{R}_+$ with $\lambda(U_n) \leq 1$ such that $W'_n = W'_n \mathbf{1}_{U_n \times U_n}$ a.e. Hence,

$$\|W - W \mathbf{1}_{U_n \times U_n}\|_{\square} \leq \|W - W'_n\|_{\square} + \|\mathbf{1}_{U_n \times U_n}(W'_n - W)\|_{\square} \leq 2\|W - W'_n\|_{\square} \rightarrow 0, \quad (8.14)$$

as $n \rightarrow \infty$.

The unit ball of $L^{\infty}(\mathbb{R}_+) = L^1(\mathbb{R}_+)^*$ is weak-* compact and metrizable (since $L^1(\mathbb{R}_+)$ is separable) [15, Theorems V.4.2 and V.5.1]. Hence, by considering a subsequence, we may assume that $\mathbf{1}_{U_n} \xrightarrow{w^*} f$ for some $f \in L^{\infty}(\mathbb{R}_+)$ with $|f| \leq 1$; furthermore, this implies $f : \mathbb{R}_+ \rightarrow [0, 1]$. For any $T \subset \mathbb{R}_+$ with $\lambda(T) < \infty$, $\int_T f = \int_{\mathbb{R}_+} \mathbf{1}_T f = \lim \int_{\mathbb{R}_+} \mathbf{1}_T \mathbf{1}_{U_n} \leq 1$; hence, by monotone convergence, $\int_{\mathbb{R}_+} f \leq 1$.

Moreover, it follows, since $L^1(\mathbb{R}_+) \otimes L^1(\mathbb{R}_+)$ is dense in $L^1(\mathbb{R}_+^2)$, that $\mathbf{1}_{U_n \times U_n} = \mathbf{1}_{U_n} \otimes \mathbf{1}_{U_n} \xrightarrow{w^*} f \otimes f$ in $L^{\infty}(\mathbb{R}_+^2)$. Hence, for any measurable sets $T, U \subseteq \mathbb{R}_+$,

$$\begin{aligned} \int_{T \times U} W \mathbf{1}_{U_n \times U_n} &= \int_{\mathbb{R}_+^2} W \mathbf{1}_{T \times U} \mathbf{1}_{U_n \times U_n} \\ &\rightarrow \int_{\mathbb{R}_+^2} W \mathbf{1}_{T \times U} (f \otimes f) = \int_{T \times U} (f \otimes f) W. \end{aligned} \quad (8.15)$$

On the other hand, (8.14) implies

$$\int_{T \times U} (W \mathbf{1}_{U_n \times U_n} - W) \rightarrow 0. \quad (8.16)$$

Consequently, $\int_{T \times U} (f \otimes f)W = \int_{T \times U} W$ for all $T, U \subseteq S$, and thus, by (2.2), $\|(f \otimes f)W - W\|_{\square} = 0$, so by (2.4), $(f \otimes f)W = W$ a.e. The result follows by Lemma 7.12(iv) \implies (i). \square

This completes the proof of the results in Section 7.

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REFERENCES

- [1] David J. Aldous, Exchangeability and related topics. *École d'été de probabilités de Saint-Flour, XIII – 1983*, Lecture Notes in Math. 1117, 1–198. Springer, Berlin, 1985.
- [2] Tim Austin, On exchangeable random variables and the statistics of large graphs and hypergraphs. *Probability Surveys* **5** (2008), 80–145.
- [3] Béla Bollobás & Oliver Riordan, Metrics for sparse graphs. *Surveys in Combinatorics 2009*, London Math. Soc. Lecture Note Ser. 365, 211–287. Cambridge Univ. Press, Cambridge, 2009.
- [4] Christian Borgs, Jennifer T. Chayes, Henry Cohn & Nina Holden, Sparse exchangeable graphs and their limits via graphon processes. Preprint, 2016. [arXiv:1601.07134v1](https://arxiv.org/abs/1601.07134)
- [5] Christian Borgs, Jennifer T. Chayes, Henry Cohn & Yufei Zhao, An L^p theory of sparse graph convergence I: limits, sparse random graph models, and power law distributions. Preprint, 2014. [arXiv:1401.2906](https://arxiv.org/abs/1401.2906)
- [6] Christian Borgs, Jennifer Chayes & László Lovász. Moments of two-variable functions and the uniqueness of graph limits. *Geom. Funct. Anal.* **19** (2010), no. 6, 1597–1619.
- [7] Christian Borgs, Jennifer T. Chayes, László Lovász, Vera T. Sós & Katalin Vesztegombi, Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing, *Advances in Math.* **219** (2008), 1801–1851.
- [8] Christian Borgs, Jennifer T. Chayes, László Lovász, Vera T. Sós & Katalin Vesztegombi, Convergent sequences of dense graphs II. Multiway cuts and statistical physics. *Ann. of Math. (2)* **176** (2012), no. 1, 151–219.
- [9] François Caron & Emily B. Fox, Sparse graphs using exchangeable random measures. Preprint, 2014. [arXiv:1401.1137v3](https://arxiv.org/abs/1401.1137v3)
- [10] Donald L. Cohn, *Measure Theory*, Birkhäuser, Boston, 1980.
- [11] John B. Conway, *A Course in Functional Analysis*. 2nd ed., Springer-Verlag, New York, 1990.

- [12] Claude Dellacherie & Paul-André Meyer, *Probabilités et potentiel*. Édition entièrement refondue, Hermann, Paris, 1975; English transl.: *Probabilities and Potential*. North-Holland, Amsterdam, 1978.
- [13] Persi Diaconis & Svante Janson, Graph limits and exchangeable random graphs. *Rend. Mat. Appl. (7)* **28** (2008), 33–61.
- [14] Nelson Dunford & B. J. Pettis, Linear operations on summable functions. *Trans. Amer. Math. Soc.* **47** (1940), 323–392.
- [15] Nelson Dunford & Jacob T. Schwartz, *Linear Operators. I. General Theory*. Interscience Publishers, New York, 1958.
- [16] Hamed Hatami, Svante Janson & Balázs Szegedy, Graph properties, graph limits and entropy. Preprint, 2013. [arXiv:1312.5626](https://arxiv.org/abs/1312.5626)
- [17] Tue Herlau, Mikkel N. Schmidt & Morten Mørup, Completely random measures for modelling block-structured networks. Preprint, 2015. [arXiv:1507.02925v3](https://arxiv.org/abs/1507.02925v3)
- [18] Svante Janson, Graphons, cut norm and distance, rearrangements and coupling. *New York J. Math. Monographs*, **4**, 2013.
- [19] Olav Kallenberg, Exchangeable random measures in the plane. *J. Theoret. Probab.* **3** (1990), no. 1, 81–136.
- [20] Olav Kallenberg, *Foundations of Modern Probability*. 2nd ed., Springer, New York, 2002.
- [21] Olav Kallenberg, *Probabilistic Symmetries and Invariance Principles*. Springer, New York, 2005.
- [22] László Lovász, *Large Networks and Graph Limits*. American Mathematical Society, Providence, RI, 2012.
- [23] László Lovász & Balázs Szegedy, Limits of dense graph sequences. *J. Comb. Theory Ser. B* **96** (2006), 933–957.
- [24] László Lovász & Balázs Szegedy, Szemerédi’s lemma for the analyst. *Geom. Funct. Anal.* **17** (2007), no. 1, 252–270.
- [25] Raymond A. Ryan, *Introduction to Tensor Products of Banach Spaces*. Springer-Verlag, London, 2002.
- [26] Victor Veitch & Daniel M. Roy, The class of random graphs arising from exchangeable random measures. Preprint, 2015. [arXiv:1512.03099](https://arxiv.org/abs/1512.03099)
- [27] Victor Veitch & Daniel M. Roy, Sampling and estimation for Kallenberg exchangeable graphs. In preparation.

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