TAIL BOUNDS FOR SUMS OF GEOMETRIC AND EXPONENTIAL VARIABLES

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Abstract. We give explicit bounds for the tail probabilities for sums of independent geometric or exponential variables, possibly with different parameters.

1. INTRODUCTION AND NOTATION

Let $X = \sum_{i=1}^{n} X_i$, where $n \geq 1$ and $X_i$, $i = 1, \ldots, n$, are independent geometric random variables with possibly different distributions: $X_i \sim \text{Ge}(p_i)$ with $0 < p_i \leq 1$, i.e.,

$$
P(X_i = k) = p_i (1 - p_i)^{k-1}, \quad k = 1, 2, \ldots (1.1)$$

Our goal is to estimate the tail probabilities $P(X \geq x)$. (Since $X$ is integer-valued, it suffices to consider integer $x$. However, it is convenient to allow arbitrary real $x$, and we do so.)

We define

$$
\mu := \mathbb{E} X = \sum_{i=1}^{n} \mathbb{E} X_i = \sum_{i=1}^{n} \frac{1}{p_i}, \quad (1.2)
$$

$$
p_* := \min_i p_i. \quad (1.3)
$$

We shall see that $p_*$ plays an important role in our estimates, which roughly speaking show that the tail probabilities of $X$ decrease at about the same rate as the tail probabilities of $\text{Ge}(p_*)$, i.e., as for the variable $X_i$ with smallest $p_i$ and thus fattest tail.

Recall the simple and well-known fact that (1.1) implies that, for any non-zero $z$ such that $|z|(1 - p_i) < 1$,

$$
\mathbb{E} z^{X_i} = \sum_{k=1}^{\infty} z^k P(X_i = k) = \frac{p_i z}{1 - (1 - p_i)z} = \frac{p_i}{z - 1 + p_i}. \quad (1.4)
$$

For future use, note that since $x \mapsto -\ln(1 - x)$ is convex on $(0, 1)$ and 0 for $x = 0$,

$$
-\ln(1 - x) \leq \frac{x}{y} \ln(1 - y), \quad 0 < x \leq y < 1. \quad (1.5)
$$
Remark 1.1. The theorems and corollaries below hold also, with the same proofs, for infinite sums \( X = \sum_{i=1}^{\infty} X_i \), provided \( E X = \sum_{i} p_i < \infty \).

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2. Upper bounds for the upper tail

We begin with a simple upper bound obtained by the classical method of estimating the moment generating function (or probability generating function) and using the standard inequality (an instance of Markov’s inequality)

\[
P(X \geq x) \leq z^{-x} E z^X, \quad z \geq 1,
\]

or equivalently

\[
P(X \geq x) \leq e^{-tx} E e^{tX}, \quad t \geq 0.
\]

(Cf. the related “Chernoff bounds” for the binomial distribution that are proved by this method, see e.g. [3, Theorem 2.1], and see e.g. [1] for other applications of this method. See also e.g. [2, Chapter 2] or [4, Chapter 27] for more general large deviation theory.)

Theorem 2.1. For any \( p_1, \ldots, p_n \in (0,1] \) and any \( \lambda \geq 1 \),

\[
P(X \geq \lambda \mu) \leq e^{-\mu \lambda (1 - \ln \lambda)}.
\]

Proof. If \( 0 \leq t < p_i \), then \( e^{-t} - 1 + p_i \geq p_i - t > 0 \), and thus by (1.4),

\[
E e^{tX_i} = \frac{p_i}{e^{-t} - 1 + p_i} \leq \frac{p_i}{p_i - t} = \left(1 - \frac{t}{p_i}\right)^{-1}.
\]

Hence, if \( 0 \leq t < p_* = \min_i p_i \), then

\[
\prod_{i=1}^n E e^{tX_i} \leq \prod_{i=1}^n \left(1 - \frac{t}{p_i}\right)^{-1}
\]

and, by (2.2),

\[
P(X \geq \lambda \mu) \leq e^{-t \lambda \mu} \prod E e^{tX_i} \leq \exp\left(-t \lambda \mu + \sum_{i=1}^{n} \ln \left(1 - \frac{t}{p_i}\right)\right).
\]

By (1.5) and \( 0 < p_*/p_i \leq 1 \), we have, for \( 0 \leq t < p_* \),

\[
- \ln \left(1 - \frac{t}{p_i}\right) \leq - \frac{p_*}{p_i} \ln \left(1 - \frac{t}{p_*}\right).
\]

Consequently, (2.6) yields

\[
P(X \geq \lambda \mu) \leq \exp\left(-t \lambda \mu - \sum_{i=1}^{n} \frac{p_*}{p_i} \ln \left(1 - \frac{t}{p_*}\right)\right)
\]

\[
= \exp\left(-t \lambda \mu - p_* \mu \ln \left(1 - \frac{t}{p_*}\right)\right).
\]

(2.8)
Choosing \( t = (1 - \lambda^{-1}) p_* \) (which is optimal in (2.8)), we obtain (2.3). \( \square \)

As a corollary we obtain a bound that is generally much cruder, but has the advantage of not depending on the \( p_* \)'s at all.

**Corollary 2.2.** For any \( p_1, \ldots, p_n \in (0, 1] \) and any \( \lambda \geq 1 \),
\[
\Pr(X \geq \lambda \mu) \leq \lambda e^{1-\lambda} = e\lambda e^{-\lambda}.
\] (2.9)

**Proof.** Use \( \mu \geq 1/p_i \) for each \( i \), and thus \( \mu p_* \geq 1 \) in (2.3). (Alternatively, use \( t = (1 - \lambda^{-1})/\mu \) in (2.8).) \( \square \)

The bound in Theorem 2.1 is rather sharp in many cases. Also the cruder (2.9) is almost sharp for \( n = 1 \) (a single \( X_i \)) and small \( p_* = p_1 \); in this case \( \mu = 1/p_1 \) and
\[
\Pr(X \geq \lambda \mu) = (1 - p_1)^{\lceil \lambda \mu \rceil - 1} = \exp(\lambda + O(\lambda p_1)).
\] (2.10)

Nevertheless, we can improve (2.3) somewhat, in particular when \( p_* = \min_i p_i \) is not small, by using more careful estimates.

**Theorem 2.3.** For any \( p_1, \ldots, p_n \in (0, 1] \) and any \( \lambda \geq 1 \),
\[
\Pr(X \geq \lambda \mu) \leq \lambda(1 - p_*)^{\lambda - 1} (\lambda - 1) \mu.
\] (2.11)

The proof is given below. We note that Theorem 2.3 implies a minor improvement of Corollary 2.2:

**Corollary 2.4.** For any \( p_1, \ldots, p_n \in (0, 1] \) and any \( \lambda \geq 1 \),
\[
\Pr(X \geq \lambda \mu) \leq e^{1-\lambda}.
\] (2.12)

**Proof.** Use (2.11) and \( (1 - p_*)^{\mu} \leq e^{-\mu} \leq e^{-1} \). \( \square \)

We begin the proof of Theorem 2.3 with two lemmas yielding a minor improvement of (2.1) using the fact that the variables are geometric. (The lemmas actually use only that one of the variables is geometric.)

**Lemma 2.5.** (i) For any integers \( j \) and \( k \) with \( j \geq k \),
\[
\Pr(X \geq j) \geq (1 - p_*)^{j-k} \Pr(X \geq k).
\] (2.13)

(ii) For any real numbers \( x \) and \( y \) with \( x \geq y \),
\[
\Pr(X \geq x) \geq (1 - p_*)^{x-y+1} \Pr(X \geq y).
\] (2.14)

**Proof.** (i). We may without loss of generality assume that \( p_* = p_1 \). Then, for any integers \( i, j, k \) with \( j \geq k \),
\[
\Pr(X \geq j \mid X - X_1 = i) = \Pr(X_1 \geq j - i) = (1 - p_*)^{(j-i-1)_+},
\] (2.15)
and similarly for \( \Pr(X \geq k \mid X - X_1 = i) \). Since \( (j-i-1)_+ \leq j-k+(k-i-1)_+ \), it follows that
\[
\Pr(X \geq j \mid X - X_1 = i) \geq (1 - p_*)^{j-k} \Pr(X \geq k \mid X - X_1 = i)
\] (2.16)
for every \( i \), and thus (2.13) follows by taking the expectation.
(ii). For real $x$ and $y$ we obtain from (2.13)
\[
P(X \geq x) = P(X \geq \lceil x \rceil) \geq (1 - p_*)^{\lceil x \rceil - \lfloor y \rfloor} P(X \geq \lfloor y \rfloor)
\geq (1 - p_*)^{x - y + 1} P(X \geq y).
\]
(2.17)

Lemma 2.6. For any $x \geq 0$ and $z \geq 1$ with $z(1 - p_*) < 1$,
\[
P(X \geq x) \leq \frac{1 - z(1 - p_*)}{p_*} z^{-x} \mathbb{E} z^X.
\]
(2.18)

Proof. Since $z \geq 1$, (2.13) implies that for every $k \geq 1$, 
\[
\mathbb{E} z^X \geq \mathbb{E}(z^X \cdot 1\{X \geq k\}) = \mathbb{E}\left(\left(\frac{z^k + (z - 1) \sum_{j=k}^{X-1} z^j}{1} \right) 1\{X \geq k\}\right)
\geq z^k \mathbb{P}(X \geq k) \left(1 + (z - 1) \sum_{j=k}^{\infty} z^j (1 - p_*)^{j+1-k}\right)
\geq z^k \mathbb{P}(X \geq k) \frac{p_*}{1 - z(1 - p_*)}.
\]
(2.19)

The result (2.18) follows when $x = k$ is a positive integer. The general case follows by taking $k = \max(\lceil x \rceil, 1)$ since then $\mathbb{P}(X \geq x) = \mathbb{P}(X \geq k)$. □

Proof of Theorem 2.3. We may assume that $p_* < 1$. (Otherwise every $p_i = 1$ and $X_i = 1$ a.s., so $X = n = \mu$ a.s. and the result is trivial.) We then choose
\[
z := \frac{\lambda - p_*}{\lambda(1 - p_*)},
\]
(2.20)

i.e.,
\[
z^{-1} = \frac{\lambda(1 - p_*)}{\lambda - p_*} = 1 - \frac{(\lambda - 1)p_*}{\lambda - p_*};
\]
(2.21)

note that $z^{-1} \leq 1$ so $z \geq 1$ and $z^{-1} > 1 - p_* \geq 1 - p_i$ for every $i$. Thus, by (1.4),
\[
\mathbb{E} z^X = \prod_{i=1}^{n} \mathbb{E} z^{X_i} = \prod_{i=1}^{n} \frac{p_i}{z^{-1} - 1 + p_i} = \prod_{i=1}^{n} \frac{1}{1 - (1 - z^{-1})/p_i}.
\]
(2.22)
By (2.22), (2.7) (with $t = 1 - z^{-1} < p_*$) and (2.21),

$$
\ln E z^X = -\sum_{i=1}^n \ln \left(1 - \frac{1 - z^{-1}}{p_i} \right) \leq -\sum_{i=1}^n \frac{p_*}{p_i} \ln \left(1 - \frac{1 - z^{-1}}{p_*} \right)
$$

$$
= -\sum_{i=1}^n \frac{p_*}{p_i} \ln \left(1 - \frac{\lambda - 1}{\lambda - p_*} \right) = -\mu p_* \ln \frac{1 - p_*}{\lambda - p_*} = \mu p_* \ln \frac{\lambda - p_*}{1 - p_*},
$$

(2.23)

Furthermore, by (2.20),

$$
\frac{1 - z(1 - p_*)}{p_*} = 1 - \frac{(\lambda - 1)p_*}{\lambda(1 - p_*)} = \frac{1}{\lambda}.
$$

(2.24)

Hence, Lemma 2.6, (2.20) and (2.23) yield

$$
\ln \mathbb{P}(X \geq \lambda \mu) \leq -\ln \lambda - \lambda \mu \ln z + \ln E z^X
$$

$$
\leq -\ln \lambda - \lambda \mu \ln \frac{\lambda - p_*}{\lambda(1 - p_*)} + \mu p_* \ln \frac{\lambda - p_*}{1 - p_*}
$$

$$
= -\ln \lambda + \lambda \mu \ln(1 - p_*) + \mu f(\lambda),
$$

(2.25)

where

$$
f(\lambda) := -\frac{\lambda}{\lambda - p_*} + \frac{\lambda}{1 - p_*}
$$

$$
= -(\lambda - p_*) \ln(\lambda - p_*) + \lambda \ln \lambda - p_* \ln(1 - p_*).
$$

(2.26)

We have $f(1) = -\ln(1 - p_*)$ and, for $\lambda \geq 1$, using (1.5),

$$
f'(\lambda) = -\ln(\lambda - p_*) + \ln \lambda = -\ln \left(1 - \frac{p_*}{\lambda} \right) \leq -\frac{1}{\lambda} \ln(1 - p_*).
$$

(2.27)

Consequently, by integrating (2.27), for all $\lambda \geq 1$,

$$
f(\lambda) \leq -\ln(1 - p_*) - \ln \lambda \cdot (1 - p_*),
$$

(2.28)

and the result (2.11) follows by (2.25).

\[\square\]

Remark 2.7. Note that for large $\lambda$, the exponents above are roughly linear in $\lambda$, while for $\lambda = 1 + o(1)$ we have $\lambda - 1 - \ln \lambda \sim \frac{1}{2}(\lambda - 1)^2$ so the exponents are quadratic in $\lambda - 1$. The latter is to be expected from the central limit theorem. However, if $\lambda = 1 + \varepsilon$ with $\varepsilon$ very small and the central limit theorem is applicable, then $\mathbb{P}(X \geq (1 + \varepsilon)\mu)$ is roughly $\exp(-\varepsilon^2 \mu^2 / (2\sigma^2))$, where $\sigma^2 = \text{Var} \ X = \sum_{i=1}^n \text{Var} X_i = \sum_{i=1}^n \frac{1 - p_i}{\mu^2}$. Hence, in this case the exponents in (2.3) and (2.11) are asymptotically too small by a factor of roughly, for small $p_i$,

$$
\frac{p_* \mu}{\mu^2 / \sigma^2} \approx \frac{p_* \sum_{i=1}^n p_i^{-2}}{\sum_{i=1}^n p_i^{-1}},
$$

(2.29)

which may be much smaller than 1. (For example if $p_2 = \cdots = p_n$ and $p_1 = p_2/n^{1/3}$.)
3. Upper bounds for the lower tail

We can similarly bound the probability $\mathbb{P}(X \leq \lambda \mu)$ for $\lambda \leq 1$. We give only a simple bound corresponding to Theorem 2.1. (Note that $\lambda - 1 - \ln \lambda > 0$ for both $\lambda \in (0, 1)$ and $\lambda \in (1, \infty)$.)

**Theorem 3.1.** For any $p_1, \ldots, p_n \in (0, 1]$ and any $\lambda \leq 1$,

$$
\mathbb{P}(X \leq \lambda \mu) \leq e^{-p_s \mu (\lambda - 1 - \ln \lambda)}.
$$

(3.1)

**Proof.** We follow closely the proof of Theorem 2.1. If $t \geq 0$, then by (1.4),

$$
\mathbb{E} e^{-tX_i} = \frac{p_i}{e^t - 1 + p_i} \leq \frac{p_i}{t + p_i} = \left(1 + \frac{t}{p_i}\right)^{-1}.
$$

(3.2)

Hence

$$
\mathbb{E} e^{-tX} = \prod_{i=1}^{n} \mathbb{E} e^{-tX_i} \leq \prod_{i=1}^{n} \left(1 + \frac{t}{p_i}\right)^{-1}
$$

(3.3)

and, in analogy to (2.2),

$$
\mathbb{P}(X \leq \lambda \mu) \leq e^{t \lambda \mu} \mathbb{E} e^{-tX} \leq \exp\left(t \lambda \mu - \sum_{i=1}^{n} \ln\left(1 + \frac{t}{p_i}\right)\right).
$$

(3.4)

In analogy with (2.7), still by the convexity of $-\ln x$,

$$
-\ln\left(1 + \frac{t}{p_i}\right) \leq -\frac{p_s}{p_i} \ln\left(1 + \frac{t}{p_s}\right),
$$

(3.5)

and (3.4) yields

$$
\mathbb{P}(X \leq \lambda \mu) \leq \exp\left(t \lambda \mu - \ln\left(1 + \frac{t}{p_s}\right) \sum_{i=1}^{n} \frac{p_s}{p_i}\right)
$$

$$
= \exp\left(t \lambda \mu - p_s \mu \ln\left(1 + \frac{t}{p_s}\right)\right).
$$

(3.6)

Choosing $t = (\lambda - 1)p_s$, we obtain (3.1). \qed

4. A lower bound

We show also a general lower bound for the upper tail probabilities, which shows that for constant $\lambda > 1$, the exponents in Theorems 2.1 and 2.3 are at most a constant factor away from best possible.

**Theorem 4.1.** For any $p_1, \ldots, p_n \in (0, 1]$ and any $\lambda \geq 1$,

$$
\mathbb{P}(X \geq \lambda \mu) \geq \frac{(1 - p_s)^{1/p_s}}{2p_s \mu} (1 - p_s)^{(\lambda - 1)\mu}.
$$

(4.1)

**Lemma 4.2.** If $A \geq 1$ and $0 \leq x \leq 1/A$, then

$$
A(x + \ln(1 - x)) \leq \ln(1 - Ax^2/2).
$$

(4.2)
Proof. Let \( f(x) := A(x + \ln(1 - x)) - \ln(1 - Ax^2/2) \). Then \( f(0) = 0 \) and
\[
f'(x) = A \left(1 - \frac{1}{1 - x} \right) + \frac{Ax}{1 - Ax^2/2} = -\frac{Ax}{1 - x} + \frac{Ax}{1 - Ax^2/2} \leq 0 \tag{4.3}
\]
for \( 0 \leq x < 1/A \leq 1 \), since then \( 0 < 1 - x \leq 1 - Ax^2/2 \). Hence \( f(x) \leq 0 \) for \( 0 \leq x \leq 1/A \). □

Proof of Theorem 4.1. Let \( \varepsilon := 1/(p_*\mu) \). By Theorem 3.1 (with \( \lambda = 1 - \varepsilon \)) and Lemma 4.2 (with \( A = p_*\mu \geq 1 \)),
\[
\mathbb{P}(X \leq (1 - \varepsilon)\mu) \leq \exp(-p_*\mu(-\varepsilon - \ln(1 - \varepsilon))) \leq 1 - \frac{1}{2p_*\mu}. \tag{4.4}
\]
Hence, \( \mathbb{P}(X \geq (1 - \varepsilon)\mu) \geq 1/(2p_*\mu) \), and by Lemma 2.5(ii),
\[
\mathbb{P}(X \geq \lambda\mu) \geq (1 - p_*)^{(\lambda - 1 + \varepsilon)\mu + 1} \mathbb{P}(X \geq (1 - \varepsilon)\mu) \geq (1 - p_*)^{(\lambda - 1 + \varepsilon)\mu + 1} \frac{1}{2p_*\mu},
\]
which completes the proof since \( \varepsilon\mu = 1/p_* \). □

5. Exponential distributions

In this section we assume that \( X = \sum_{i=1}^{n} X_i \) where \( X_i, i = 1, \ldots, n \), are independent random variables with exponential distributions: \( X_i \sim \text{Exp}(a_i) \), with density function \( a_i x e^{-a_i x}, x > 0 \), and expectation \( \mathbb{E} X_i = 1/a_i \). (Thus \( a_i \) can be interpreted as a rate.) The exponential distribution is the continuous analogue of the geometric distributions, and the results above have (simpler) analogues for exponential distributions. We now define
\[
\mu := \mathbb{E} X = \sum_{i=1}^{n} \mathbb{E} X_i = \sum_{i=1}^{n} \frac{1}{a_i}, \tag{5.1}
\]
\[
a_* := \min_{i} a_i. \tag{5.2}
\]

Theorem 5.1. Let \( X = \sum_{i=1}^{n} X_i \) with \( X_i \sim \text{Exp}(a_i) \) independent.

(i) For any \( \lambda \geq 1 \),
\[
\mathbb{P}(X \geq \lambda\mu) \leq \lambda^{-1} e^{-a_*\mu(\lambda - 1 - \ln \lambda)}. \tag{5.3}
\]

(ii) For any \( \lambda \geq 1 \), we have also the simpler but weaker
\[
\mathbb{P}(X \geq \lambda\mu) \leq e^{1 - \lambda}. \tag{5.4}
\]

(iii) For any \( \lambda \leq 1 \),
\[
\mathbb{P}(X \leq \lambda\mu) \leq e^{-a_*\mu(\lambda - 1 - \ln \lambda)}. \tag{5.5}
\]

(iv) For any \( \lambda \geq 1 \),
\[
\mathbb{P}(X \geq \lambda\mu) \geq \frac{1}{2e a_* \mu} e^{-a_*\mu(\lambda - 1)}. \tag{5.6}
\]
Proof. Let $X_i^{(N)} \sim \text{Ge}(a_i/N)$ be independent (for $N > \max_i a_i$). Then $X_i^{(N)}/N \xrightarrow{d} X_i$, where $\xrightarrow{d}$ denotes convergence in distribution, and thus $X^{(N)}/N \xrightarrow{d} X$, where $X^{(N)} := \sum_{i=1}^n X_i^{(N)}$. Furthermore, $\mu^{(N)} := \mathbb{E} X^{(N)} = M \nu$ and $p_* := \min_i (a_i/N) = a_*/N$. The results follow by taking the limit as $N \to \infty$ in (2.11), (2.12), (3.1) and (4.1). (Alternatively, we may imitate the proofs above, using $\mathbb{E} e^{t X_i} = a_i/(a_i - t)$ for $t < a_i$.)$\square$

References