A graphon counter example

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Abstract. We give an example of a graphon such that there is no equivalent graphon with a degree function that is (weakly) increasing.

1. Introduction

A central fact in the theory of dense graph limits (see e.g. the book by Lovász [7]) is that each graph limit can be represented by a graphon, but this representation is not unique. We say that two graphons are equivalent (also called weakly isomorphic) if they define the same graph limit; thus there is a bijection between graph limits and equivalence classes of graphons. (Recall that equivalence of graphons can be described by the homomorphism densities being the same; furthermore, it is equivalent to the cut distance being 0; see [7] for details.)

Recall that graphons are symmetric measurable functions \( W : \Omega \times \Omega \to [0,1] \), where \( \Omega = (\Omega, F, \mu) \) is a probability space. We may always choose \( \Omega \) to be \([0,1]\) with Lebesgue measure, in the sense that any graphon is equivalent to a graphon defined on \([0,1]\), but it is often advantageous to use graphons defined on other probability spaces \( \Omega \) too.

The characterization of equivalence between graphons is known to be complicated. Any two graphons on the same space \( \Omega \) that are equal a.e. are equivalent, and every graphon is equivalent to any the pull-back of it by a measure preserving map (see below for definitions), but equivalence is not limited to this. See e.g. [8], [1], [5], [2] and [6].

Given a graph limit, it would be desirable to somehow define a canonical graphon representing it (at least up to equality a.e.); in other words, to define a canonical choice of a graphon in the corresponding equivalence class. In some special cases, this can be done in a natural way. For example, see [4], a graph limit that is the limit of a sequence of threshold graphs can always be represented by a graphon \( W(x,y) \) on \([0,1]\) that only takes values in \(\{0,1\}\), and furthermore is increasing in each coordinate separately (we say that a function \( f(x) \) is increasing if \( f(x) \leq f(y) \) when \( x \leq y \); moreover, two such graphons are equivalent if and only if they are a.e. equal. There is thus a canonical graphon representing each threshold graph limit.

Similarly, if a graphon \( W(x,y) \) defined on \([0,1]\) has a degree function

\[
D(x) = D_W(x) := \int_0^1 W(x,y) \, dy
\]  

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that is a strictly increasing function \([0, 1] \to [0, 1]\), then it is not difficult to show that any equivalent graphon that also has an increasing degree function is a.e. equal to \(W\); see Section 3 for details. Hence, a graphon with a strictly increasing degree function can be regarded as a canonical choice in its equivalence class.

Of course, not every graphon is equivalent to such a graphon; for example not a graphon with a constant degree function. Nevertheless, this leads to the following interesting question. We repeat that we use 'increasing' in the weak sense (also known as 'weakly increasing'): \(f\) is increasing if \(f(x) \leq f(y)\) when \(x \leq y\);

**Problem.** Given any graphon \(W\), does there exist an equivalent graphon on \([0, 1]\) with an increasing degree function (1.1)?

The purpose of this note is to show that this is not the case.

**Theorem 1.** There exists a graphon on \([0, 1]\) such that there is no equivalent graphon on \([0, 1]\) with a (weakly) increasing degree function.

We prove this theorem by giving a simple explicit example in (2.1). The example is similar to, and inspired by, standard examples such as [7, Example 7.11] showing that two equivalent graphons are not necessarily pull-backs of each other.

**Remark 2.** The analogue for finite graphs of the problem above for graphons is the trivial fact that the vertices of a graph can be ordered with (weakly) increasing vertex degrees. Note that there will always be ties, so even for a finite graph, this does not define a unique canonical labelling.

1.1. **Some notation.** \([0, 1]\] will, as above, be regarded as a probability space equipped with the Lebesgue measure and the Lebesque \(\sigma\)-field. (We might also use the Borel \(\sigma\)-field. For the present paper, this makes no difference; for other purposes, the choice of \(\sigma\)-field may have some importance.)

Let \((\Omega_1, \mathcal{F}_1, \mu_1)\) and \((\Omega_2, \mathcal{F}_2, \mu_2)\) be two probability spaces. A function \(\varphi : \Omega_1 \to \Omega_2\) is measure preserving if \(\mu_1(\varphi^{-1}(A)) = \mu_2(A)\) for any measurable \(A \subseteq \Omega_2\). If \(W\) is a graphon on \(\Omega_2\) and \(\varphi : \Omega_1 \to \Omega_2\) is measure preserving, then the pull-back \(W^{\varphi}\) is the graphon \(W^{\varphi}(x, y) := W(\varphi(x), \varphi(y))\) defined on \(\Omega_1\). As mentioned above, a pull-back \(W^{\varphi}\) is always equivalent to \(W\).

2. **The example**

Our example is the graphon

\[
W(x, y) := \begin{cases} 
4xy, & x, y \in (0, \frac{1}{2}), \\
1/2, & x + y > 3/2, \\
0, & \text{otherwise}.
\end{cases}
\]  

(2.1)

Note that the degree function is given by

\[
\mathcal{D}(x) := \int_0^1 W(x, y) \, dy = \begin{cases} 
\frac{1}{2}x, & x \in (0, \frac{1}{2}), \\
\frac{1}{2}(x - \frac{1}{2}), & x \in (\frac{1}{2}, 1).
\end{cases}
\]  

(2.2)

Suppose that \(W\) is equivalent to a graphon \(W_1\) on \([0, 1]\) that has an increasing degree function \(\mathcal{D}_1(x) := \int_0^1 W_1(x, y) \, dy\); we will show that this leads to a contradiction.
The equivalence \( W \equiv W_1 \) implies by [1, Corollary 2.7], see also [7, Corollary 10.35] and [6, Theorem 8.6], that there exist a probability space \((\Omega, \mu)\) and two measure preserving maps \( \varphi, \psi : \Omega \to [0, 1] \) such that \( W^\varphi = W_1^\psi \) a.e., i.e.,

\[
W(\varphi(x), \varphi(y)) = W_1(\psi(x), \psi(y)), \quad \text{a.e. \ on } \Omega^2. \tag{2.3}
\]

(The probability space \((\Omega, \mu)\) can be taken as \([0, 1]\) with Lebesgue measure, but we have no need for this. Instead, we prefer to use the notation \(\Omega\) and \(\mu\) to distinguish between this space and \([0, 1]\), which hopefully will make the proof easier to follow.)

Since \( \varphi \) and \( \psi \) are measure preserving, we have for every Borel measurable \( f \geq 0 \) on \([0, 1]\),

\[
\int_0^1 f(x) \, dx = \int_\Omega f(\varphi(x)) \, d\mu(x) = \int_\Omega f(\psi(x)) \, d\mu(x). \tag{2.4}
\]

We use this repeatedly below.

In particular, (2.3) and (2.4) imply that for a.e. \( x \in \Omega \)

\[
\mathcal{D}(\varphi(x)) = \int_0^1 W(\varphi(x), y) \, dy = \int_\Omega W(\varphi(x), \varphi(y)) \, d\mu(y)
= \int_\Omega W_1(\psi(x), \psi(y)) \, d\mu(y) = \int_0^1 W_1(\psi(x), y) \, dy = \mathcal{D}_1(\psi(x)). \tag{2.5}
\]

Hence, for every real \( r \in (0, \frac{1}{4}] \), using (2.2),

\[
\lambda \{ x \in [0, 1] : \mathcal{D}_1(x) \leq r \} = \mu \{ x \in \Omega : \mathcal{D}_1(\psi(x)) \leq r \}
= \mu \{ x \in \Omega : \mathcal{D}(\varphi(x)) \leq r \} = \lambda \{ x \in [0, 1] : \mathcal{D}(x) \leq r \} = 4r. \tag{2.6}
\]

Since we have assumed that \( \mathcal{D}_1 \) is increasing, this implies

\[
\mathcal{D}_1(x) = x/4, \quad x \in (0, 1). \tag{2.7}
\]

Define

\[
h(x) := \lambda \{ y : W(x, y) \notin \{0, \frac{1}{2}\} \} \begin{cases} \frac{1}{2}, & x \in (0, \frac{1}{2}), \\ 0, & x \in (\frac{1}{2}, 1), \end{cases} \tag{2.8}
\]

and, similarly,

\[
h_1(x) := \lambda \{ y : W_1(x, y) \notin \{0, \frac{1}{2}\} \}. \tag{2.9}
\]

Then (2.3) implies, similarly to (2.5), for a.e. \( x \in \Omega \),

\[
h(\varphi(x)) = \lambda \{ y : W(\varphi(x), y) \notin \{0, \frac{1}{2}\} \}
= \mu \{ y : W(\varphi(x), \varphi(y)) \notin \{0, \frac{1}{2}\} \}
= \mu \{ y : W_1(\psi(x), \psi(y)) \notin \{0, \frac{1}{2}\} \}
= \lambda \{ y : W_1(\psi(x), y) \notin \{0, \frac{1}{2}\} \} = h_1(\psi(x)). \tag{2.10}
\]

This will yield our contradiction. We first calculate \( h_1 \).

If \( 0 < a < b < 1 \), then, using (2.7), (2.4), (2.10), (2.5), and (2.4) again,

\[
\int_a^b h_1(x) \, dx = \int_0^1 h_1(x) \mathbf{1}_{\left\{ \frac{a}{4} < \mathcal{D}_1(x) < \frac{b}{4} \right\}} \, dx
\]
\[
\begin{align*}
= & \int_{\Omega} h_1(\psi(x))1\left\{ \frac{a}{4} < \mathcal{D}_1(\psi(x)) < \frac{b}{4} \right\} \, d\mu(x) \\
= & \int_{\Omega} h(\varphi(x))1\left\{ \frac{a}{4} < \mathcal{D}(\varphi(x)) < \frac{b}{4} \right\} \, d\mu(x) \\
= & \int_0^1 h(x)1\left\{ \frac{a}{4} < \mathcal{D}(x) < \frac{b}{4} \right\} \, dx.
\end{align*}
\]

However, by (2.8) and (2.2),
\[
\int_0^1 h(x)1\left\{ \frac{a}{4} < \mathcal{D}(x) < \frac{b}{4} \right\} \, dx = \frac{1}{2} \int_0^{1/2} 1\left\{ \frac{a}{4} < \mathcal{D}(x) < \frac{b}{4} \right\} \, dx = \frac{1}{2} \lambda(a, b) = \frac{b-a}{4}.
\]

Consequently, (2.11) and (2.12) show that for every \( a \in (0, 1) \) and \( \varepsilon \in (0, 1 - a) \),
\[
\frac{1}{\varepsilon} \int_a^{a+\varepsilon} h_1(x) \, dx = 1 - \frac{\varepsilon}{4} = \frac{1}{4}.
\]

However, by the Lebesgue differentiation theorem, as \( \varepsilon \to 0 \), this converges a.e. to \( h_1(x) \). Hence,
\[
h_1(x) = \frac{1}{4} \quad \text{a.e. } x \in [0, 1].
\]

We may now complete the proof. It follows from (2.14) that \( h_1(\psi(x)) = \frac{1}{4} \) a.e. on \( \Omega \), while (2.8) implies that \( h(x) \neq \frac{1}{4} \) a.e. on \( [0, 1] \), and thus \( h(\varphi(x)) \neq \frac{1}{4} \) a.e. on \( \Omega \). Thus (2.10) yields a contradiction.

Consequently, there is no graphon \( W_1 \) equivalent to \( W \) with increasing degree function.

\[\square\]

3. Strictly increasing degree functions

In this section, we give a proof of the following result, mentioned in the introduction. This result is not new; it is mentioned in Delmas, Dherzin and Sciauveau [3] (without proof), and it may also have been observed earlier. We do not know any published proof, so we give one for completeness.

**Theorem 3.** If \( W(x,y) \) is a graphon defined on \([0,1]\) such that its degree function \( \mathcal{D}(x) \) is a strictly increasing function \([0,1] \to [0,1] \), then any equivalent graphon that also has a strictly increasing degree function is a.e. equal to \( W \).

**Proof.** Suppose that \( W_1 \) is an equivalent graphon on \([0,1]\) that has a strictly increasing degree function \( \mathcal{D}_1 \). As in Section 2, there exists a probability space \((\Omega, \mu)\) and measure preserving maps \( \varphi, \psi : \Omega \to [0,1] \) such that (2.3)–(2.5) hold. By (2.5), for a.e. \( x, y \in \Omega \),
\[
\varphi(x) < \varphi(y) \implies \mathcal{D}(\varphi(x)) < \mathcal{D}(\varphi(y)) \implies \mathcal{D}_1(\psi(x)) < \mathcal{D}_1(\psi(y)) \implies \psi(x) < \psi(y).
\]

We may interchange \( W \) and \( W_1 \) and thus, for a.e. \( x, y \),
\[
\varphi(x) < \varphi(y) \iff \psi(x) < \psi(y).
\]
Consequently, for a.e. \( x \in \Omega \),
\[
\varphi(x) = \lambda\{t \in [0,1] : t < \varphi(x)\} = \mu\{y \in \Omega : \varphi(y) < \varphi(x)\}
\]
\[
= \mu\{y \in \Omega : \psi(y) < \psi(x)\} = \lambda\{t \in [0,1] : t < \psi(x)\} = \psi(x).
\]
This together with (2.3) shows that 
\[
W(\varphi(x), \varphi(y)) = W_1(\varphi(x), \varphi(y)) \quad \text{a.e. on } \Omega^2,
\]
and a final use of the fact that \( \varphi \) is measure preserving shows that 
\[
W(s, t) = W_1(s, t) \quad \text{for a.e. } s, t \in [0,1].
\]
\( \square \)

**Remark 4.** Theorem 3 can easily be slightly extended to show that also there is no equivalent graphon with a weakly but not strictly increasing degree function. We omit the proof.

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**References**


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