To fixate or not to fixate in two-type annihilating branching random walks

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Abstract

We study a model of competition between two types evolving as branching random walks on \( \mathbb{Z}^d \). The two types are represented by red and blue balls respectively, with the rule that balls of different colour annihilate upon contact. We consider initial configurations in which the sites of \( \mathbb{Z}^d \) contain one ball each, which are independently coloured red with probability \( p \) and blue otherwise. We address the question of fixation, referring to the sites eventually settling for a given colour, or not. Under a mild moment condition on the branching rule, we prove that the process will fixate almost surely for \( p \neq 1/2 \), and that every site will change colour infinitely often almost surely for the balanced initial condition \( p = 1/2 \).

1 Introduction

Position, on each site of a connected graph \( G \), an urn. The urn may contain either red or blue balls, but not both at once. At the dawn of time (\( t = 0 \)), red and blue balls are distributed in the urns according to some rule. Each ball comes equipped with a unit-rate Poisson clock, and when a clock rings, the corresponding ball immediately sends an independent copy of itself to each of the urns at neighbouring sites (while the ball with the clock remains where it is). As red and blue balls may not exist together in the same urn, they annihilate on a one to one basis.

In the case that \( G \) is connected and finite, the authors together with Morris [2] have proved that the system of urns will eventually almost surely contain balls of only one colour. In the current paper we examine the process on the \( d \)-dimensional integer lattice \( \mathbb{Z}^d \), for \( d \geq 1 \), evolving from an initial configuration with a ball at each site, which independently from one another are coloured red with probability \( p \) and blue otherwise. Our main result shows that for \( p \neq 1/2 \) the colouring of the lattice induced by the urn process eventually fixates almost surely on a single colour, and that for \( p = 1/2 \) each

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site almost surely switches colour infinitely many times. We shall prove our results for a general family of branching mechanisms, further described below, of which the above mentioned nearest-neighbour rule is one example.

Models for systems of particles annihilating upon contact have a long history. The question of site recurrence in a one-dimensional system of random walkers annihilating upon contact was raised in the mid 1970s by Erdős and Ney [15]. Higher dimensional versions of the same problem was soon after considered by Griffeath [17] and Arratia [3], whereas survival for a version of the branching random walk where any two particles annihilate upon contact was studied by Bramson and Gray [11]. These problems concern a system of particles of a single type. Analogous models consisting of two types of particles have been suggested in the physics literature as descriptive for the inert chemical reaction $A + B \to \emptyset$, see [23, 26]. These models tend to require a different set of techniques for their analysis. In this setting, Bramson and Lebowitz [12, 13] derived the rate of decay of the density of particles in such a two-type model where particles perform simple random walks, and particles of different type annihilate upon contact. In more recent work, Cabezas, Rolla and Sidoravicius [14] addressed site recurrence, and proved that the origin is visited at arbitrarily large times.

The (discrete time) branching random walk first arose as a geometric interpretation of the evolution of generations in an age-dependent branching process, in work of Kingman [21], Biggins [5, 6] and Bramson [10]. Later work has explored important connections between branching random walks and their continuum counterpart, branching Brownian motion, to central objects in statistical physics such as spin glasses and the discrete planar Gaussian free field. For a more detailed discussion on these models and connections, we refer the reader to the monographs [9, 25, 27].

Very much in spirit of the work cited above, and further motivated by the corresponding question for Glauber dynamics of the Ising model (see, e.g., [16, 22]), we here address fixation for the annihilating urn system on $\mathbb{Z}^d$ starting from a stationary random initial configuration. In the monochromatic setting, in which all balls have the same colour, so there are no annihilations, the process we study corresponds to a continuous time branching random walk on the integer lattice. For this reason, we shall interchangeably refer to the model we consider as a competing urn scheme and as a two-type annihilating branching random walk. Our analysis of this process will be based on a combination of martingale techniques, and elements of Fourier analysis.

We proceed with a somewhat more formal description of the model we consider, and introduce some notation. We shall encode the presence of a red ball with the value $+1$ and the presence of a blue ball by the value $-1$. This encoding produces a bijection between particle configurations and integer-valued vectors indexed by $\mathbb{Z}^d$. Below, a configuration on $\mathbb{Z}^d$ will refer to a vector $\zeta = (\zeta_z)_{z \in \mathbb{Z}^d}$ of integers. A configuration $\zeta$ is said to be locally finite if $|\zeta_z| < \infty$ for all $z \in \mathbb{Z}^d$, and finite if the total number of particles $\|\zeta\| := \sum_{z \in \mathbb{Z}^d} |\zeta_z|$ is finite. (For typographical convenience, we occasionally write $\zeta(z)$ for $\zeta_z$.)

Let $\Phi$ be a probability measure on finite non-negative configurations on $\mathbb{Z}^d$, and let $\varphi$ denote a generic random configuration distributed according to $\Phi$. Given a locally finite initial configuration $\zeta$, we assign to each ball in $\zeta$ a clock independent from everything else. At the ring of a clock at position $z$, the corresponding ball makes an independent draw from the distribution $\Phi$, and positions new balls accordingly, translated by $z$. (The ball
with the clock is still assumed to remain where it is, although we shall comment on this restriction below.) As before, all children has the same colour as their parent, and if one or several balls are positioned in an urn with balls of opposite colour, then they annihilate one for one until all remaining balls are of the same colour. The nearest-neighbour rule (from [2]) described above thus corresponds to \( \Phi \) being the degenerate measure supported on the configuration consisting of one ball at each of the \( 2d \) neighbours to the origin. We shall at later occasions in the paper have more to say about the construction of the process as the need arises.

We say that \( \Phi \) is \textit{irreducible} if it is not supported on a proper subgroup of \( \mathbb{Z}^d \). We will assume throughout that \( \Phi \) is irreducible and that \( \| \varphi \| \) has finite mean, so that

\[
0 < \lambda := \mathbb{E}[\| \varphi \|] < \infty.
\]

(1.1)

**Remark 1.1.** We will for simplicity only consider initial configurations with at most one ball at each site. (Although more general cases might also be interesting, see Section 8.) In this case, and assuming (1.1), the process described above is well-defined and without explosions; more precisely, for any finite box \( B(0, r) := [-r, r]^d \) and any finite \( T \), there is a.s. (almost surely) only a finite number of balls appearing in \( B(0, r) \) at some time in \( [0, T] \), and as a consequence there is only a finite number of nucleations (branching events) and annihilations at any given site in a finite time interval. To see this, consider the (larger) process without annihilations, or equivalently the monochromatic version of our process, starting with at most one ball at each site, and note that then the expected number of balls at any given site at time \( t \) is at most \( e^{\lambda t} < \infty \), cf. (3.4) with \( p = 1 \). Formally, as we shall see in detail in the proof of Lemma 6.2, we may define the process as the a.s. limit of processes with finite initial configurations. Moreover, since the process has only a finite number of jumps at each site in each finite time interval, we may assume the standard convention that the process is right-continuous with left limits.

We aim in this paper to understand the evolution of the annihilating system on \( \mathbb{Z}^d \) starting from a stationary random initial configuration. To be precise, given \( p \in [0, 1] \) and \( d \geq 1 \), define a \( p \)-\textit{random Bernoulli colouring} of \( \mathbb{Z}^d \) as follows: for each \( z \in \mathbb{Z}^d \), the corresponding urn initially contains a single red ball with probability \( p \), and a single blue ball otherwise, all independently. Hence, the resulting configuration corresponds to an element in \( \{-1, 1\}^{\mathbb{Z}^d} \). We say that the two-type annihilating branching random walk \textit{fixates} if there exists a colour \( c \) such that every urn eventually contains only balls of colour \( c \). We define the density of red sites at time \( t \) as the limit, if it exists,

\[
\rho(t) := \lim_{n \to \infty} \frac{1}{(2n + 1)^d} \sum_{z \in [-n,n]^d} \mathbf{1}\{z \text{ red at time } t\}.
\]

(1.2)

1 Assuming that the offspring distribution is irreducible means no loss of generality, since we otherwise could consider the process on the subgroup \( G \) of \( \mathbb{Z}^d \) generated by the support of \( \Phi \); note that \( G \cong \mathbb{Z}^{d_1} \) for some \( d_1 \leq d \) and that the process on \( \mathbb{Z}^{d_1} \) then decomposes into independent copies of the process on \( G \), supported on different translates (cosets) of \( G \). We ignore the trivial case when the support of \( \Phi \) is \{0\}; then the urns are independent continuous-time branching processes, each with a fixed colour.

2 The times the different urns fixate are random and different; we do not claim that there is a single time when all urn have the same colour. (Indeed, since the system is infinite, we cannot expect this.)
As a measure on the displacement of balls in each nucleation, we define for \( r > 0 \),
\[
|||\varphi|||_r := \sum_{z \in \mathbb{Z}^d} |z|^r \varphi(z).
\] (1.3)

Our main theorem is the following. (By symmetry, it suffices to consider \( p \geq \frac{1}{2} \).)

**Theorem 1.2.** Let \( d \in \mathbb{N} \), and let \( \Phi \) be an irreducible probability measure on finite configurations on \( \mathbb{Z}^d \) such that \( E[|||\varphi|||^2_1] < \infty \), \( E[|||\varphi|||^2_2] < \infty \), and \( E[|||\varphi|||^{1+\epsilon}] < \infty \) for some \( \epsilon > 0 \). Then, for the competing urn scheme on \( \mathbb{Z}^d \) starting from a \( p \)-random Bernoulli colouring, almost surely:

(i) For \( p > \frac{1}{2} \) the system fixates; each urn is eventually red. Furthermore, the density of red urns, as defined in (1.2), exists for all \( t \geq 0 \) and tends to 1 as \( t \to \infty \).

(ii) For \( p = \frac{1}{2} \) every site changes colour infinitely often.

**Remark 1.3.** The proof shows that the exponent \( 4 + \epsilon \) may be reduced to \( 2 + \epsilon \) when \( d = 2 \), and to 2 for all \( d \geq 3 \). For \( L^2 \) convergence, the exponent 2 will suffice for all (irreducible) offspring distributions and in all dimensions \( d \geq 1 \). The same holds for Theorem 1.5 below, on which the proof of part (i) is based. For part (ii), which does not rely on Theorem 1.5, the exponent 2 will suffice for all (irreducible) offspring distributions in all dimensions \( d \geq 1 \). We do not know whether the conditions
\[
E[|||\varphi|||^2_1] < \infty \quad \text{and} \quad E[|||\varphi|||^2_2] < \infty
\] (1.4)
on the spatial displacement that figure in our theorems are necessary.

**Remark 1.4.** In the definition of the model, as offspring is produced, the parent is assumed to remain where it is. As a result, in the monochromatic version of the process, once a ball is born it remains in the same place at all future times. More generally we could assume that each ball lives for an exponentially distributed life time, at the end of which it reproduces according to \( \Phi \) and disappears. This is certainly more general, as \( \Phi \) could be specified to produce a copy of the parent in its place with probability one. In addition, this allows us, for instance, to consider models where the balls move according to continuous time random walks, which in each step branch with a non-zero probability. We shall in Section 7 describe how our results can be extended to cover also this setting.

Central in order to understand the annihilating process will be to closely examine the evolution of the monochromatic process (without annihilations), in which each site of \( \mathbb{Z}^d \) independently is initially occupied by a particle with probability \( p \in (0, 1] \) and otherwise empty. Let \( Y^p(t) = (Y^p_z(t))_{z \in \mathbb{Z}^d} \) be the configuration at time \( t \geq 0 \) of this process, where thus \( (Y^p_z(t))_{z \in \mathbb{Z}^d} \) are i.i.d. Bernoulli with parameter \( p \).

We shall prove the following result on the asymptotics of the monochromatic system.

**Theorem 1.5.** Let \( d \in \mathbb{N} \), and let \( \Phi \) be an irreducible probability measure on finite configurations on \( \mathbb{Z}^d \) such that \( E[|||\varphi|||^2_1] < \infty \), \( E[|||\varphi|||^2_2] < \infty \), and \( E[|||\varphi|||^{1+\epsilon}] < \infty \), for some \( \epsilon > 0 \). Then, for every \( p \in [0, 1] \) and every \( z \in \mathbb{Z}^d \), we have, with \( \lambda \) given by (1.1),
\[
\lim_{t \to \infty} e^{-\lambda t} Y^p_z(t) = p
\] (1.5)
almost surely and in \( L^2 \).
In the monochromatic process balls do not interact with each other, and in order to understand its asymptotics it will suffice to examine the evolution of each ball initially present in the system separately. For this purpose we let \( X_z(t) = (X_{z,x}(t))_{x \in \mathbb{Z}^d} \) be the configuration at time \( t \) of the process started with a single ball at \( z \), i.e., \( X_{z,x}(0) = \delta_{x,z} \). (These processes are obviously just translates of \( X_0(t) \), the evolution of a single ball started at the origin, but the collection of all of them will be useful in our arguments.) Note that the process \( (X_z(t))_{t \geq 0} \) is a multi-type continuous time Markov branching process with type space \( \mathbb{Z}^d \); see e.g. [4, Section V.7]. Moreover, the dynamics of the process is translation invariant, which in particular implies that \( \|X_z(t)\| \), the total number of balls in the system, evolves as a (single-type) continuous time Markov branching process in which each individual gets \( \|\varphi\| \) children with rate 1. The finite moment condition (1.1) is well-known to imply that the process is almost surely finite at all times, see [4, Section III.2]; in fact, it is easily seen that \( e^{-\lambda t} \|X_z(t)\| \) is a martingale, and thus, in particular,

\[
\mathbb{E} \|X_z(t)\| = e^{\lambda t} \mathbb{E} \|X_z(0)\| = e^{\lambda t},
\]

see e.g. [4, Section III.4 and Theorem III.7.1] or Lemma 2.1 below.

A large part of our work will consist in exploring the evolution of the process \( X_0(t) \) starting with a single ball in Section 2, and its implications for the monochromatic process \( Y_p(t) \), which is studied in Section 3, leading to a proof of Theorem 1.5. The analysis will be based on martingale techniques and elements of Fourier analysis. An important step in the argument is a precise variance estimate, which is stated as Proposition 3.1(ii).

We then return to the two-colour competition process, starting from a \( p \)-random Bernoulli colouring, which we describe by the vector \( Z(t) = (Z_x(t))_{x \in \mathbb{Z}^d} \). Although our main interest lies in the case of a random initial configuration, we will in the proofs consider various versions, and we thus allow an arbitrary initial configuration \( \zeta = (\zeta_x)_{x \in \mathbb{Z}^d} = \{\pm 1\}^{\mathbb{Z}^d} \), deterministic or random. (Thus \( \zeta_x = -1 \) means a blue ball at \( x \), 1 means a red ball and 0 means no ball.) We let \( Z(t, \zeta) \), for \( t \geq 0 \), denote the process started from \( \zeta \). In particular, the monochromatic process \( Y_p(t) \) equals (as a process) \( Z(t, \zeta) \) with \( \zeta = (\zeta_x)_{x \in \mathbb{Z}^d} \) independent Bernoulli with parameter \( p \), and \( X_z(t) \) corresponds to \( Z(t, \zeta) \) with \( \zeta = (\delta_{x,z})_{x \in \mathbb{Z}^d} \), whereas \( Z(t) \) itself corresponds to \( Z(t, \zeta) \) with \( \zeta \) being the \( p \)-random Bernoulli colouring whose entries are \( \pm 1 \)-valued and independent from one another.

We describe in Section 4 a coupling, previously employed in [2], that enables us to ignore annihilations and instead study a pair of (dependent) monochromatic processes, to which we can apply results from previous sections. Theorem 1.2(i) then is as an easy consequence of Theorem 1.5. The balanced case, Theorem 1.2(ii), will require a finer analysis of the order of fluctuations of the monochromatic process, which suitably comes out as a side while proving Theorem 1.5, together with a decoupling argument showing that the states of any finite set of sites are irrelevant for the long term evolution. Details are given in Sections 5 and 6, respectively.

At the end of this paper, in Section 7, we describe how to adapt our arguments to cover the more general version of our process where balls are assumed to die as they reproduce, cf. Remark 1.4. Finally, Section 8 contains some further directions and open problems.
2 The evolution of a single ball

In this section we analyze the evolution of a single ball, i.e., the process $X_z(t)$. By translation invariance, we may without loss of generality assume $z = 0$. Furthermore, in this section (only), we drop the index $z$ indicating the starting position and use the notation $\mathcal{X}(t) = (X_x(t))_{x \in \mathbb{Z}^d}$ for $\mathcal{X}_0(t)$. The analysis is based on a combination of Fourier analysis and a martingale approach.

2.1 Elements of Fourier analysis

We proceed with the study of the process $(\mathcal{X}(t))_{t \geq 0}$, evolving from a single ball initially at the origin. Recall that $\mathcal{X}(t) = (X_x(t))_{x \in \mathbb{Z}^d}$, for each $t \geq 0$, is an almost surely finite configuration on $\mathbb{Z}^d$. From a harmonic analysis point of view, the dual group of $\mathbb{Z}^d$ is the cycle group $T^d$, which we identify with $[-\pi, \pi]^d$. Hence, there is a natural correspondence between configurations on $\mathbb{Z}^d$ and certain complex-valued functions on $T^d$. More precisely, we define the Fourier transform $\hat{\mathcal{X}}_u(t)$ as

$$\hat{\mathcal{X}}_u(t) := \sum_{x \in \mathbb{Z}^d} e^{iu \cdot x} X_x(t), \quad u \in T^d. \quad (2.1)$$

We note that for $t = 0$ this definition yields $\hat{\mathcal{X}}_u(0) = 1$, and for $u = 0$ we obtain

$$\hat{\mathcal{X}}_0(t) = \sum_{x \in \mathbb{Z}^d} X_x(t) = \|\mathcal{X}(t)\|, \quad (2.2)$$

the total number of balls at time $t$. In general, the inequality $|\hat{\mathcal{X}}_u(t)| \leq \|\mathcal{X}(t)\|$ remains valid. $\mathcal{X}(t)$ may be recovered via the inversion formula:

$$X_x(t) = \int_{T^d} e^{-iu \cdot x} \hat{\mathcal{X}}_u(t) \, du, \quad (2.3)$$

where $du$ denotes the normalized Lebesgue measure $(2\pi)^{-d} du_1 \cdots du_d$ on $T^d$. (This is easily checked; plug in (2.1) and compute the integral.)

Denote by $\mu = (\mu(x))_{x \in \mathbb{Z}^d}$ the coordinate-wise expectation of $\Phi$, i.e. $\mu(x) := \mathbb{E}[\varphi(x)]$. Then, by (1.1),

$$\|\mu\| := \sum_{x \in \mathbb{Z}^d} \mu(x) = \sum_{x \in \mathbb{Z}^d} \mathbb{E} \varphi(x) = \mathbb{E} \sum_{x \in \mathbb{Z}^d} \varphi(x) = \mathbb{E} \|\varphi\| = \lambda < \infty. \quad (2.4)$$

Hence, also $\mu$ has a well-defined Fourier transform $\hat{\mu}(u) := \sum_{x \in \mathbb{Z}^d} e^{iu \cdot x} \mu(x)$; note that

$$\hat{\mu}(0) = \|\mu\| = \lambda. \quad (2.5)$$

(The Fourier transform $\zeta(u)$ of any finite configuration $\zeta$ on $\mathbb{Z}^d$ is defined analogously.)

As said in the introduction, it is well-known that $e^{-\lambda t} \hat{\mathcal{X}}_0(t) = e^{-\lambda t} \|\mathcal{X}(t)\|$ is a continuous-time martingale. We extend this to arbitrary $u \in T^d$ in the next lemma. Let

$$M_u(t) := e^{-\hat{\mu}(u)t} \hat{\mathcal{X}}_u(t). \quad (2.6)$$

In particular, by (2.5) and (2.2),

$$M_0(t) := e^{-\lambda t} \|\mathcal{X}(t)\|. \quad (2.7)$$
Lemma 2.1. The process $(M_u(t))_{t \geq 0}$ is a martingale for each $u \in T^d$. In particular,

$$E[\hat{X}_u(t)] = e^{\hat{\mu}(u)t}, \quad u \in T^d. \quad (2.8)$$

Taking $u = 0$ in Lemma 2.1 we recover, using (2.7), the fact noted above that $e^{-\lambda t}\|\mathcal{X}(t)\|$ is a martingale, and in particular, since also $\|\mathcal{X}(0)\| = 1$, that

$$E[\|\mathcal{X}(t)\|] = e^{\lambda t}, \quad (2.9)$$
i.e., that (1.6) holds.

Proof of Lemma 2.1. We prove first (2.8). Once (2.8) has been proven the martingale property will follow from the Markov and branching properties together with homogeneity in time and space. Hence, it will suffice to prove (2.8).

Note that, almost surely, no two clocks ever ring at the same time. If the clock rings for a ball at $z$, then $\hat{X}_u(t)$ jumps by (a copy of) $\varphi$ translated by the vector $z$. Hence, $\hat{X}_u(t)$ then jumps by

$$\Delta \hat{X}_u(t) = \sum_{y \in \mathbb{Z}^d} e^{iu \cdot (z+y)} \varphi(y) = e^{iu \cdot z} \hat{\varphi}(u), \quad (2.10)$$
and the expected jump of $\hat{X}_u(t)$, given that the clock rings for a ball at $z$, is

$$e^{iu \cdot z} E[\hat{\varphi}(u)] = e^{iu \cdot z} \hat{\mu}(u). \quad (2.11)$$
Since the number of balls at $z$ is $X_z(t)$, and each rings with intensity 1, this implies

$$\frac{d}{dt} E[\hat{X}_u(t)] = \sum_{z \in \mathbb{Z}^d} E[X_z(t)] e^{iu \cdot z} \hat{\mu}(u) = \hat{\mu}(u) E[\hat{X}_u(t)], \quad (2.12)$$
and (2.8) follows by the initial condition $\hat{X}_u(0) = 1$.

As another consequence of Lemma 2.1, we obtain a formula for the expected number of balls at a given position. Since $|\hat{X}_u(t)| \leq \|\mathcal{X}(t)\|$ and $E[\|\mathcal{X}(t)\|] < \infty$, we may combine the inversion formula (2.3), Fubini's theorem and (2.8) to obtain the expression

$$E[X_z(t)] = \int_{T^d} e^{-iu \cdot z} E[\hat{X}_u(t)] \, du = \int_{T^d} e^{-iu \cdot z} e^{\hat{\mu}(u)t} \, du. \quad (2.13)$$

2.2 Second moment analysis

To obtain higher moments of the process we will require a stronger assumption on the moments of $\Phi$. This is also where condition (1.4) on the displacement of $\Phi$ comes in.

We begin by noting that the condition $E[\|\varphi\|^2] < \infty$ implies that $E[\|\mathcal{X}(t)\|^2] < \infty$ for all $t \geq 0$, see [4, Corollary III.6.1] or [19, Theorem 6.3.6]. Since $|\hat{X}_u(t)| \leq \|\mathcal{X}(t)\|$ we have as a consequence that $E[M_u(t)^2] < \infty$ for all $u \in T^d$ and $t \geq 0$; in other words, $M_u(t)$ is a square-integrable martingale for every $u \in T^d$. The following proposition shows that under the condition $\text{Re} \hat{\mu}(u) > \frac{1}{2} \lambda$, this martingale is $L^2$-bounded.
Proposition 2.2. Assume that $\mathbb{E}[\|\varphi\|^2] < \infty$, and let $u \in \mathbb{T}^d$ be such that $\text{Re} \hat{\mu}(u) > \frac{1}{2} \lambda$. Then the process $(M_u(t))_{t \geq 0}$ is an $L^2$-bounded martingale; in particular, the limit $M_u^* := \lim_{t \to \infty} M_u(t)$ exists almost surely and in $L^2$. Furthermore, there exists a constant $C(u)$, which is uniformly bounded for $\text{Re} \hat{\mu}(u) - \frac{1}{2} \lambda \geq c$ for any $c > 0$, such that for all $t \geq 0$

$$
\mathbb{E} \left[ |M_u(t) - M_u^*|^2 \right] \leq C(u) \mathbb{E}[\|\varphi\|^2] e^{-\left(2\text{Re} \hat{\mu}(u) - \lambda\right)t},
$$

(2.14)

and if, in addition, $\mathbb{E}[\|\varphi\|^2] < \infty$ and $\mathbb{E}[\|\varphi\|^2_2] < \infty$, then for all $t \geq 0$

$$
\mathbb{E} \left[ |M_u(t) - M_0(t)|^2 \right] \leq C(u)|u|^2.
$$

(2.15)

The following lemma will be the first step towards the above proposition.

Lemma 2.3. For every $u, v \in \mathbb{T}^d$ and $t \geq 0$ we have

$$
\mathbb{E} \left[ M_u(t)M_v(t) \right] = 1 + \mathbb{E} \left[ \hat{\varphi}(u)\hat{\varphi}(v) \right] \int_0^t e^{\hat{\mu}(u+v)x - [\hat{\mu}(u) + \hat{\mu}(v)]x} \, dx.
$$

(2.16)

Proof. The two processes $(M_u(t))_{t \geq 0}$ and $(M_v(t))_{t \geq 0}$ are square-integrable martingales. Hence, their quadratic covariation $[M_u, M_v](t)$ is well-defined; see e.g. [24, Section II.6]. (Although not needed here, we note that it may be defined as the following limit, in probability [24, Theorem II.23],

$$
[M_u, M_v](t) := M_u(0)M_v(0) + \lim_{|P_n| \to 0} \sum_{k=1}^n (M_u(t_k) - M_u(t_{k-1}))(M_v(t_k) - M_v(t_{k-1})),
$$

(2.17)

where $(P_n)_{n \geq 1}$ is some sequence of partitions $0 = t_0 < t_1 < \cdots < t_n = t$ of $[0, t]$ with mesh max$_k |t_k - t_{k-1}|$ tending to zero.)

The process $\{M_u(t)M_v(t) - [M_u, M_v](t) : t \geq 0\}$ is again a martingale [24, Corollary 2 to Theorem II.27], which vanishes at $t = 0$ by definition, and thus

$$
\mathbb{E} \left[ M_u(t)M_v(t) \right] = \mathbb{E} \left[ [M_u, M_v](t) \right] \quad \text{for all } t \geq 0.
$$

(2.18)

Furthermore, $M_u(t)$ and $M_v(t)$ have finite variation on each compact time interval (since each realisation has piece-wise smooth trajectories). This implies [24, Theorems II.26 and II.28] that $[M_u, M_v](t)$ is a pure jump process with jumps given by

$$
\Delta [M_u, M_v](t) = \Delta M_u(t)\Delta M_v(t) = e^{-(\hat{\mu}(u) + \hat{\mu}(v))t} \Delta \hat{\varphi}_u(t)\Delta \hat{\varphi}_v(t).
$$

(2.19)

Similarly to the proof of Lemma 2.1, if the clock of a ball at $z$ rings, then by (2.10)

$$
\Delta \hat{\varphi}_u(t)\Delta \hat{\varphi}_v(t) = e^{i(u+v)\cdot z} \hat{\varphi}(u)\hat{\varphi}(v).
$$

(2.20)

Since the number of balls at $z$ is $X_z(t)$ and each rings with intensity 1, we obtain from the above and (2.8) that

$$
\frac{d}{dt} \mathbb{E} \left[ [M_u, M_v](t) \right] = \sum_{z \in \mathbb{T}^d} \mathbb{E}[X_z(t)]e^{-(\hat{\mu}(u) + \hat{\mu}(v))t}e^{i(u+v)\cdot z} \mathbb{E} \left[ \hat{\varphi}(u)\hat{\varphi}(v) \right]
$$

$$
= \mathbb{E}[\hat{\varphi}_u(t)\hat{\varphi}_v(t)]e^{-(\hat{\mu}(u) + \hat{\mu}(v))t} \mathbb{E} \left[ \hat{\varphi}(u)\hat{\varphi}(v) \right]
$$

$$
= e^{(\hat{\mu}(u+v) - \hat{\mu}(u) - \hat{\mu}(v))t} \mathbb{E} \left[ \hat{\varphi}(u)\hat{\varphi}(v) \right].
$$

(2.21)
Integrating (2.21) over the interval $[0,t]$, recalling that $[M_u,M_\nu](0) = M_u(0)M_\nu(0) = 1$, and then using (2.18) completes the proof.

**Proof of Proposition 2.2.** Note that the complex conjugates of $\hat{X}_u(t)$ and $\hat{\mu}(u)$ are given by $\hat{X}_{-u}(t)$ and $\hat{\mu}(-u)$, and consequently that

$$|M_u(t)|^2 = M_u(t)\bar{M_u(t)} = M_u(t)M_{-u}(t). \tag{2.22}$$

By Lemma 2.3 we find that

$$\mathbb{E}[|M_u(t)|^2] = 1 + \mathbb{E}[|\hat{\varphi}(u)|^2] \int_0^t e^{(\hat{\mu}(0) - \hat{\mu}(u) + \hat{\mu}(-u))x} \, dx \nonumber$$

$$= 1 + \mathbb{E}[|\hat{\varphi}(u)|^2] \int_0^t e^{(\lambda - 2\Re\hat{\mu}(u))x} \, dx. \tag{2.23}$$

Hence, for $u \in \mathbb{T}^d$ such that $2\Re\hat{\mu}(u) > \lambda$ the complex-valued martingale $(M_u(t))_{t \geq 0}$ is bounded in $L^2$. The existence of an almost sure and $L^2$ limit $M^*_u$ is now a consequence of the martingale convergence theorem. Moreover, as increments over disjoint time intervals for square-integrable martingales are uncorrelated, we have for any $s \geq t$ that

$$\mathbb{E}[|M_u(s)|^2] = \mathbb{E}[|M_u(s) - M_u(t)|^2] + \mathbb{E}[|M_u(t)|^2] \tag{2.24}$$

and hence by (2.23), since $\mathbb{E}[|\hat{\varphi}(u)|^2] \preceq \mathbb{E}[\|\varphi\|^2]$, that

$$\mathbb{E}[|M_u(s) - M_u(t)|^2] = \mathbb{E}[|M_u(s)|^2] - \mathbb{E}[|M_u(t)|^2] \preceq \mathbb{E}[\|\varphi\|^2] \int_0^s e^{(\lambda - 2\Re\hat{\mu}(u))x} \, dx. \tag{2.25}$$

Sending $s \to \infty$ thus yields (2.14).

Arguing for (2.15) we first observe that

$$|M_u(t) - M_0(t)|^2 = M_u(t)M_{-u}(t) + M_0(t)M_0(t) - M_u(t)M_0(t) - M_{-u}(t)M_0(t). \tag{2.26}$$

Hence, Lemma 2.3 gives that

$$\mathbb{E}[|M_u(t) - M_0(t)|^2] = \mathbb{E}[|\hat{\varphi}(u)|^2] \int_0^t e^{(\lambda - 2\Re\hat{\mu}(u))x} \, dx + \mathbb{E}[|\hat{\varphi}(0)|^2] \int_0^t e^{-\lambda x} \, dx \nonumber$$

$$- 2\Re\mathbb{E}[\hat{\varphi}(u)\hat{\varphi}(0)] \int_0^t e^{-\lambda x} \, dx \nonumber$$

$$= \mathbb{E}[|\hat{\varphi}(u)|^2] \int_0^t (e^{(\lambda - 2\Re\hat{\mu}(u))x} - e^{-\lambda x}) \, dx \nonumber$$

$$+ \mathbb{E}[|\hat{\varphi}(u) - \hat{\varphi}(0)|^2] \int_0^t e^{-\lambda x} \, dx. \tag{2.27}$$

Since $\mathbb{E}[|\hat{\varphi}(u)|^2] \preceq \mathbb{E}[\|\varphi\|^2]$, estimating the integrals leads to the upper bound

$$\mathbb{E}[|M_u(t) - M_0(t)|^2] \preceq \mathbb{E}[\|\varphi\|^2] \left( \frac{1}{2\Re\hat{\mu}(u) - \lambda} - \frac{1}{\lambda} \right) + \frac{1}{\lambda} \mathbb{E}[|\hat{\varphi}(u) - \hat{\varphi}(0)|^2] \nonumber$$

$$\preceq \mathbb{E}[\|\varphi\|^2] \left( \frac{2(\lambda - \Re\hat{\mu}(u))}{(2\Re\hat{\mu}(u) - \lambda)\lambda} \right) + \frac{1}{\lambda} \mathbb{E}[|\hat{\varphi}(u) - \hat{\varphi}(0)|^2]. \tag{2.28}$$
In order to obtain an upper bound of order $|u|^2$ we first note that

$$|e^{iu \cdot z} - 1| \leq |u \cdot z| \leq |u||z|,$$

(2.29)

using the mean-value theorem and Cauchy–Schwarz’ inequality. Hence, recalling (1.3),

$$|\hat{\varphi}(u) - \hat{\varphi}(0)| \leq \sum_{z \in \mathbb{Z}^d} |e^{iu \cdot z} - 1| \varphi(z) \leq |u| \sum_{z \in \mathbb{Z}^d} |z| \varphi(z) = |u||\varphi||_1.$$

(2.30)

Similarly we obtain, since $\mu(z) = \mathbb{E}[\varphi(z)],$

$$\lambda - \text{Re} \mu(u) = \hat{\mu}(0) - \text{Re} \hat{\mu}(u) = \sum_{z \in \mathbb{Z}^d} (1 - \cos(u \cdot z)) \mathbb{E}[\varphi(z)] \leq \frac{1}{2} |u|^2 \mathbb{E}[||\varphi||_2^1].$$

(2.31)

Hence, plugging (2.30) and (2.31) into (2.28) leaves us with

$$\mathbb{E}[|M_u(t) - M_0(t)|^2] \leq \mathbb{E}[||\varphi||^2] \frac{|u|^2 \mathbb{E}[||\varphi||^2_2]}{(2 \text{Re} \hat{\mu}(u) - \lambda)\lambda} + \frac{1}{\lambda} |u|^2 \mathbb{E}[||\varphi||^2_1] = O(|u|^2)$$

(2.32)

as required. □

### 2.3 Bounds on the spatial displacement of balls

We next consider the mean spatial distribution of $X(t)$. Recall that the expected total number of balls at time $t$ is $E[\|X(t)\|] = e^{\lambda t}$ by (2.9). Define

$$p_x(t) := \frac{\mathbb{E}[X_{-x}(t)]}{\mathbb{E}[\|X(t)\|]} = e^{-\lambda t} \mathbb{E}[X_{-x}(t)],$$

(2.33)

the proportion of the expected number of balls at time $t$ that are expected to be at $-x$, when starting (as always in this section) from a single ball at the origin. Note that $p_x(t)$ coincides with the expected contribution to the origin of a ball started at $x$. (The choice of $-x$ is just for notational convenience in later sections, e.g. in (3.9) and (6.2).) Note that, trivially by the definitions, for every $t \geq 0$, we have $p_x(t) \geq 0$ and

$$\sum_{x \in \mathbb{Z}^d} p_x(t) = 1.$$  

(2.34)

We shall next derive some key quantitative estimates that we shall use in later sections.

**Proposition 2.4.** Assume that $\Phi$ is irreducible and satisfies $E[||\varphi||^2] < \infty$, $E[||\varphi||^2_1] < \infty$ and $E[||\varphi||^2_2] < \infty$. Then, for $t \geq 1$,

(i) $\sup_{x \in \mathbb{Z}^d} p_x(t) = O(t^{-d/2}),$

(ii) $\sum_{x \in \mathbb{Z}^d} p_x(t)^2 = \Theta(t^{-d/2}),$

(iii) $e^{-2\lambda} \sum_{x \in \mathbb{Z}^d} \mathbb{E}[(X_x(t) - p_{-x}(t)||X(t)||)^2] = O(t^{-(d+2)/2}).$
Proof. We start with (i), and observe that by the definition (2.33) and (2.13) we have
\[ p_z(t) = e^{-\lambda t} E[X_z(t)] = \int_{\mathbb{T}^d} e^{i u z} e^{-(\lambda - \Re \tilde{\mu}(u)) t} \, du \leq \int_{\mathbb{T}^d} e^{-(\lambda - \Re \tilde{\mu}(u)) t} \, du. \]  

To further bound the integral we have the following well-known standard estimate, which highlights the importance of the irreducibility assumption. Note that a complementary upper bound (for all \( u \)) is given in (2.31), provided \( E[\|z\|_2] < \infty \).

Claim 1. Assume that \( \Phi \) is irreducible. Then, \( \Re \tilde{\mu}(u) < \lambda \) for all \( u \in \mathbb{T}^d \setminus \{0\} \), and there exists \( c > 0 \) such that
\[ \lambda - \Re \tilde{\mu}(u) \geq c |u|^2 \quad \text{for all } |u| \leq c. \]  

Proof of Claim. The statement is obtained by analyzing the identity
\[ \lambda - \Re \tilde{\mu}(u) = \tilde{\mu}(0) - \Re \tilde{\mu}(u) = \sum_{z \in \mathbb{Z}^d} \left[ 1 - \cos(u \cdot z) \right] \mu(z). \]  

We omit the details. \( \square \)

Since \( \Phi \) is irreducible, Claim 1 gives a constant \( c > 0 \) such that (2.36) holds. Let \( K_c := \{ u \in \mathbb{T}^d : |u| \geq c \} \). Since \( K_c \) is compact, Claim 1 and continuity of \( \tilde{\mu}(u) \) also gives a constant \( \gamma > 0 \) such that \( \lambda - \Re \tilde{\mu}(u) \geq \gamma \) on \( K_c \). Hence, together with (2.35),
\[ \sup_{z \in \mathbb{Z}^d} p_z(t) \leq \int_{|u| < c} e^{-c|u|^2 t} \, du + \int_{K_c} e^{-\gamma t} \, du = O(t^{-d/2}) + O(e^{-\gamma t}). \]  

This proves part (i).

The upper bound in (ii) is immediate from (i) and (2.34). Due to the identity in (2.35) we may for a lower bound use Parseval’s formula together with (2.31) to obtain that
\[ \sum_{z \in \mathbb{Z}^d} p_z(t)^2 = \int_{\mathbb{T}^d} \left| e^{-(\lambda - \Re \tilde{\mu}(u)) t} \right|^2 \, du \geq \int_{\mathbb{T}^d} e^{-C|u|^2 t} \, du = t^{-d/2} \int_{(-\pi t^{1/2}, \pi t^{1/2})^d} e^{-C|u|^2} \, du, \]  

where \( C = E[\|z\|_2^2]/2 \), which for \( t \geq 1 \) is bounded below by a constant times \( t^{-d/2} \).

For (iii) we recall (2.7) and the definition (2.33) of \( p_z(t) \), by which
\[ X_z(t) - p_{-z}(t) \|X(t)\| = X_z(t) - E[X_z(t)] M_0(t). \]  

Using the inversion formula (2.3) and (2.13) we find this equal to
\[ \int_{\mathbb{T}^d} e^{-iu \cdot z} [\tilde{X}_u(t) - e^{\tilde{\mu}(u)t} M_0(t)] \, du = \int_{\mathbb{T}^d} e^{-iu \cdot z} e^{\tilde{\mu}(u)t} [M_u(t) - M_0(t)] \, du. \]  

The right-hand side of (2.41) is the Fourier transform of a function on \( \mathbb{T}^d \). Hence, by Parseval’s formula, we obtain
\[ \sum_{z \in \mathbb{Z}^d} \|X_z(t) - p_{-z}(t) \|X(t)\|^2 = \int_{\mathbb{T}^d} \left| e^{\tilde{\mu}(u)t} [M_u(t) - M_0(t)] \right|^2 \, du. \]  

(2.42)
Taking expectation yields

$$\sum_{z \in \mathbb{Z}^d} \mathbb{E} \left[ |X(z) - p(z)| \|X(t)|^2 \right] = \int_{t^d} e^{2 \text{Re} \mu(t)} \mathbb{E} \left[ |M(t) - M(t)|^2 \right] du.$$  \hfill (2.43)

Let \( c > 0, K_c \) and \( \gamma > 0 \) be as above, so that \( \lambda - \text{Re} \mu(u) \geq c|u|^2 \) when \( |u| \leq c \) and \( \lambda - \text{Re} \mu(u) \geq \gamma \) on the complementary set \( K_c \). We may without loss of generality assume that \( c > 0 \) was chosen so that also \( \lambda - \text{Re} \mu(u) \leq \lambda/4 \) for \( |u| \leq c \) and that \( \gamma \leq \lambda/4 \).

For \( |u| \leq c \) we use (2.36) and (2.15), and find that for some constant \( C_1 \),

$$e^{2 \text{Re} \mu(u)} \mathbb{E} \left[ |M(t) - M(t)|^2 \right] \leq C_1 e^{2(\lambda - c|u|^2)} |u|^2, \quad |u| \leq c.$$  \hfill (2.44)

Next we observe that for all \( u \), (2.23) implies that there exists a constant \( C_2 \) such that

$$\mathbb{E} \left[ |M(t) - M(t)|^2 \right] \leq 2 \mathbb{E} \left[ |M(t)|^2 \right] + 2 \mathbb{E} \left[ |M(t)|^2 \right] \leq C_2 \left[ 1 + \int_0^t e^{(\lambda - 2 \text{Re} \mu(u))x} du \right].$$  \hfill (2.45)

By distinguishing between the cases \( 2 \text{Re} \mu(u) \geq \frac{5}{4} \lambda \) and \( 2 \text{Re} \mu(u) \leq \frac{5}{4} \lambda \), we obtain from (2.45), rather crudely,

$$\mathbb{E} \left[ |M(t) - M(t)|^2 \right] \leq C_3 e^{\max \left\{ \frac{3}{2}(\lambda - 2 \text{Re} \mu(u)), \frac{3}{2} \lambda \right\} t}.$$  \hfill (2.46)

Hence, on \( K_c \),

$$e^{2 \text{Re} \mu(u)} \mathbb{E} \left[ |M(t) - M(t)|^2 \right] \leq C_3 e^{\max \left\{ \frac{3}{2}(\lambda - 2 \text{Re} \mu(u)), \frac{3}{2} \lambda \right\} t} \leq C_3 e^{(\lambda - \gamma)t}, \quad \mu \in K_c.$$  \hfill (2.47)

Combining (2.43) with the estimates (2.44) and (2.47) yields

$$e^{-2\lambda} \sum_{z \in \mathbb{Z}^d} \mathbb{E} \left[ |X(z) - p(z)| \|X(t)|^2 \right] \leq \int_{|u| < c} C_1 e^{-2c|u|^2} |u|^2 du + \int_{K_c} C_3 e^{-2\gamma t} du
= O(t^{-(d+2)/2}) + O(e^{-2\gamma t}).$$  \hfill (2.48)

This proves (iii) and thus completes the proof.

\( \square \)

3 The monochromatic process

We have defined the monochromatic process \( Y_p(t) = (Y_p^x(t))_{x \in \mathbb{Z}^d} \) as the process with independent Bernoulli distributed random initial values \( Y_p^x(0) \) with parameter \( p \). Our main goal in the present section is to prove Theorem 1.5. A key step will be to derive a variance bound that will be central also later. By translation invariance, \( Y_p^x(t) \) has the same distribution for all \( x \in \mathbb{Z}^d \), and we may consider only \( x = 0 \).

We begin by introducing a useful representation. Let \( \eta = (\eta_z)_{z \in \mathbb{Z}^d} \) be a vector of independent Bernoulli distributed entries (with parameter \( p \)). For each \( z \in \mathbb{Z}^d \), let, as above, \( X_z(t) = (X_{z,x}(t))_{x \in \mathbb{Z}^d} \) be the process started with a single ball at \( z \), and assume further that these processes are independent of each other and of \( \eta \). Then we can construct
the process \( Y_p \) as \( Y_p(t) = \sum_{x \in \mathbb{Z}^d} \eta_x X_x(t) \), i.e., the process which for \( x \in \mathbb{Z}^d \) and \( t \geq 0 \) is given by

\[
Y_p(x)(t) = \sum_{z \in \mathbb{Z}^d} \eta_z X_{x,z}(t),
\]

(3.1)

(This is because in the monochromatic process there are no annihilations and balls evolve independently.)

We next use this representation to prove the following key proposition.

**Proposition 3.1.** Assume that \( \Phi \) is irreducible and that \( p \in [0,1] \).

(i) If \( \lambda = E[\|\phi\|] < \infty \), then for every \( t \geq 0 \),

\[
E[e^{-\lambda Y_p^p(t)}] = p.
\]

(3.2)

(ii) If \( E[\|\phi\|^2] < \infty \), \( E[\|\phi\|^2_2] < \infty \) and \( E[\|\phi\|_2] < \infty \), and furthermore \( p > 0 \), then for some constant \( C = C(p, \Phi) \) and all \( t \geq 1 \),

\[
\text{Var}[e^{-\lambda Y_p^p(t)}] = C \sum_{z \in \mathbb{Z}^d} \eta_z p_z(\lambda) + O(t^{-(d+1)/2}) = \Theta(t^{-d/2}).
\]

(3.3)

**Proof of Proposition 3.1.** (i): By (3.1), independence, translation invariance, and (2.9), using an interchange of order of summation and expectation, that is justified since all variables are non-negative,

\[
E[Y_p^p(t)] = p \sum_{x \in \mathbb{Z}^d} E[X_{x,0}(t)] = p \sum_{x \in \mathbb{Z}^d} E[X_{0,-x}(t)] = p E[\|X_0(t)\|] = pe^{\lambda t},
\]

(3.4)

which yields (3.2).

(ii): Under the assumption that \( E[\|\phi\|^2] < \infty \), it follows by Proposition 2.2 (with \( u = \mathbf{0} \)) and (2.7) that for each \( z \in \mathbb{Z}^d \), the process \( \{e^{-\lambda t}\|X_z(t)\| : t \geq 0\} \) is an \( L^2 \)-bounded martingale. Hence, the limit

\[
W_z := \lim_{t \to \infty} e^{-\lambda t}\|X_z(t)\|
\]

(3.5)

exists almost surely and in \( L^2 \), and (2.9) implies

\[
E[W_z] = 1.
\]

(3.6)

Note that by (2.7),

\[
W_0 = \lim_{t \to \infty} M_0(t) = M_0^*.
\]

(3.7)

We decompose \( Y_p^p(t) \) in the following manner, using (3.1) and (2.34).

\[
e^{-\lambda t} Y_p^p(t) - p = \sum_{z \in \mathbb{Z}^d} \eta_z e^{-\lambda t}(X_{z,0}(t) - p_z(X_z(t)))
\]

\[
+ \sum_{z \in \mathbb{Z}^d} p_z(t) \eta_z(e^{-\lambda t}\|X_z(t)\| - W_z)
\]

\[
+ \sum_{z \in \mathbb{Z}^d} p_z(t)(\eta_z W_z - p).
\]

(3.8)
We will prove below that the sums converge in $L^2$, and that their values as elements of $L^2$ are independent of the order of summation, so the decomposition is well-defined. Moreover, although we don’t really need this, the proof also shows that for any given fixed order of summation (given by a fixed enumeration of $\mathbb{Z}^d$), the sums converge a.s.

Denote the three sums on the right-hand side of (3.8) by $\Sigma_1(t)$, $\Sigma_2(t)$ and $\Sigma_3(t)$. Note first that by translation invariance, (2.33) and (2.9),

$$E[X_{z,0}(t)] = E[X_{0,-z}(t)] = e^{\lambda t}p_z(t) = p_z(t)E[\|X_z(t)\|].$$ \hfill (3.9)

Hence, the terms in the first sum have zero mean. The same holds for the terms in the second and third sums too by (2.9) and (3.6). It follows that each of the three sums consists of independent terms with zero mean; hence a sufficient (and necessary) condition for the existence of the sum, in $L^2$ and almost surely (for any fixed order of summation), is that the sum of the variance of the summands is finite. We state this well-known standard result formally for easy reference.

Claim 2 (Kolmogorov). Let $\xi_1, \xi_2, \ldots$ be independent zero mean random variables and let $S_n$ denote the sum of the first $n$ of them. If $\sum_{k=1}^{\infty} \text{Var}(\xi_k) < \infty$, then $S_\infty := \lim_{n \to \infty} S_n$ exists almost surely and in $L^2$, and

$$\text{Var}(S_\infty) = \sum_{k=1}^{\infty} \text{Var}(\xi_k).$$ \hfill (3.10)

Proof of Claim. For the existence of the limit, see e.g. [18, Lemma 6.5.2 and Theorem 6.5.2]. The formula then follows since $E[S_\infty^2] = \lim_{n \to \infty} E[S_n^2]$. \hfill \Box

We treat the three terms in (3.8) separately and in order, obtaining estimates of the variance and at the same time showing the existence of the sums $\Sigma_j(t)$ in $L^2$ and a.s.

First, Claim 2 and translation invariance show (assuming it is finite) that

$$\text{Var} [\Sigma_1(t)] = p e^{-2\lambda t} \sum_{z \in \mathbb{Z}^d} E \left[ (X_{z,0}(t) - p_z(t)\|X_z(t)\|)^2 \right].$$

$$= p e^{-2\lambda t} \sum_{z \in \mathbb{Z}^d} E \left[ (X_{0,-z}(t) - p_z(t)\|X_0(t)\|)^2 \right].$$ \hfill (3.11)

By Proposition 2.4(iii) the right-hand side is indeed finite, so $\Sigma_1(t)$ is well-defined and (3.11) justified. By the same proposition, we find that, for some $C_1 < \infty$,

$$\text{Var} [\Sigma_1(t)] = E [\Sigma_1(t)^2] \leq C_1 t^{-(d+2)/2}.$$ \hfill (3.12)

Similarly, Claim 2 yields

$$\text{Var} [\Sigma_2(t)] = p \sum_{z \in \mathbb{Z}^d} p_z(t)^2 E \left[ (e^{-\lambda t}\|X_z(t)\| - W_z)^2 \right].$$ \hfill (3.13)

where we note that by translation invariance, (2.7), (3.7), and Proposition 2.2,

$$E \left[ (e^{-\lambda t}\|X_z(t)\| - W_z)^2 \right] = E \left[ (e^{-\lambda t}\|X_0(t)\| - W_0)^2 \right] = E \left[ (M_0(t) - M_0)^2 \right] \leq C_2 e^{-\lambda t}.$$ \hfill (3.14)
Recalling (2.34), we see from (3.14) that the sum in (3.13) converges, so $\Sigma_3(t)$ is well-defined, and furthermore
\[
\text{Var}[\Sigma_2(t)] = E[\Sigma_2(t)^2] \leq C_2 e^{-\lambda t}. \quad (3.15)
\]

Finally, by Proposition 2.2 the variables $W_n \overset{d}{=} W_0 = M_0^*$ exist in $L^2$, so Claim 2 gives
\[
\text{Var}[\Sigma_3(t)] = \sum_{n \in \mathbb{Z}^d} p_n(t)^2 E[(\eta_n W_n - \bar{p})^2] = (p E[W_0^2] - \bar{p}^2) \sum_{n \in \mathbb{Z}^d} p_n(t)^2. \quad (3.16)
\]

Note that $E[W_0^2] \geq E[W_0]^2 = 1$. Furthermore, equality would imply $W_0 = 1$ a.s., and thus the martingale $e^{-\lambda t}||\lambda(t)||$ would be constant, i.e., $||\lambda(t)|| = e^{\lambda t}$ a.s. for every $t \geq 0$, which is absurd. Hence, $E[W_0^2] > 1$, and thus $p E[W_0^2] - \bar{p}^2 > 0$ for all $p \in (0,1]$. Proposition 2.4(ii) and (3.16) thus show that
\[
\text{Var}[\Sigma_3(t)] = E[\Sigma_3(t)^2] = \Theta(t^{-d/2}), \quad t \geq 1. \quad (3.17)
\]

Examining the variance estimates (3.12), (3.15) and (3.17), we conclude that $\Sigma_3(t)$ has a variance of larger order than the other two sums. We conclude that the variance of $\Sigma_3(t)$ is the dominating term in the variance of $Y_0^p(t)$; more precisely, (3.8) and Minkowski’s inequality imply, using (3.12) and (3.15),
\[
\left|\left(\text{Var}[e^{-\lambda t}Y_0^p(t)]\right)^{1/2} - \left(E[\Sigma_3(t)^2]\right)^{1/2}\right| \leq \left|\left(E[\Sigma_1(t)^2]\right)^{1/2} + \left(E[\Sigma_2(t)^2]\right)^{1/2}\right| = O(t^{-(d+2)/4}). \quad (3.18)
\]

The first equality in (3.3) now follows by (3.16) and (3.17) (with $C = p E[W_0^2] - \bar{p}^2 > 0$). The second equality follows for large $t$ by Proposition 2.4(ii); it trivially extends to all $t \geq 1$ since $\text{Var}[e^{-\lambda t}Y_0^p(t)]$, as a simple consequence of (3.1), is bounded below by some positive constant for every bounded interval $[1, T]$.

\begin{remark}
In fact, it is easy to see, e.g. as a consequence of [4, III.4.(5)], that $E[W_0^2] = 1 + E[||\varphi||^2]/\lambda$, and thus $C = p E[||\varphi||^2]/\lambda + p - \bar{p}^2$. Note also that the bounds (3.12) and (3.15) are uniform in $p$.
\end{remark}

\begin{proof}[Proof of Theorem 1.5] The case $p = 0$ is trivial, so we assume $p > 0$. By translation invariance, we may assume $x = 0$. Proposition 3.1 immediately yields $L^2$ convergence in (1.5), so it remains to establish almost sure convergence. We shall show, for every fixed $\delta > 0$, that
\[
e^{-\lambda \delta n}Y_0^p(\delta n) \overset{a.s.}{\longrightarrow} p \quad \text{as } n \to \infty. \quad (3.19)
\]

Since $Y_0^p(t)$ is non-decreasing in $t$, this implies that, a.s.,
\[
e^{-\lambda \delta} p \leq \liminf_{t \to \infty} e^{-\lambda t}Y_0^p(t) \leq \limsup_{t \to \infty} e^{-\lambda t}Y_0^p(t) \leq e^{\lambda \delta} p. \quad (3.20)
\]

Hence, a.s. (3.20) holds for all rational $\delta > 0$, which implies $\lim_{t \to \infty} e^{-\lambda t}Y_0^p(t) = p$.

Thus, fix $\delta > 0$. In order to show (3.19), we again use the decomposition (3.8) and show that $\Sigma_1(\delta n) \overset{a.s.}{\longrightarrow} 0$, $\Sigma_2(\delta n) \overset{a.s.}{\longrightarrow} 0$, and $\Sigma_3(\delta n) \overset{a.s.}{\longrightarrow} 0$ as $n \to \infty$.
First, (3.12) shows that
\[ E \left[ \sum_{n=1}^{\infty} \Sigma_1(\delta n)^2 \right] = \sum_{n=1}^{\infty} E \left[ \Sigma_1(\delta n)^2 \right] \leq C_1 \sum_{n=1}^{\infty} (\delta n)^{-d/2} < \infty. \]  
(3.21)
In particular, a.s., \( \sum_{n=1}^{\infty} \Sigma_1(\delta n)^2 < \infty \) and thus \( \lim_{n \to \infty} \Sigma_1(\delta n) = 0 \).

Similarly, (3.15) implies that a.s. \( \lim_{n \to \infty} \Sigma_2(\delta n) = 0 \).

To complete the proof of the theorem, it remains to show that also a.s. \( \sum_{n=1}^{\infty} \Sigma_3(\delta n)^2 \) as \( n \to \infty \). For \( d \geq 3 \) this follows from (3.17) just like for \( \Sigma_1(t) \) and \( \Sigma_2(t) \). For \( d = 1, 2 \) we need to argue differently, which requires a stronger moment condition.

Consider any \( d \geq 1 \) and fix \( r \geq 2 \) with \( rd > 4 \) such that \( E[\|\varphi\|^r] < \infty \); by our assumption we may always take \( r = 4 + \varepsilon \), but if \( d = 2 \) it suffices to take \( r = 2 + \varepsilon \) and if \( d \geq 3 \) we may take \( r = 2 \), which justifies the claims in Remark 1.3.

Note that the assumption \( E[\|\varphi\|^r] < \infty \) implies \( E[W_z^r] < \infty \); see [8, Theorems 1 and 3] (or [7, Corollary to Theorem 5], applied to the Galton–Watson process \( \|X_\varepsilon(n)\| \), and [4, Corollary III.6.1]). Rosenthal’s inequality, see [18, Theorem 3.9.1], then gives
\[ E \left[ |\Sigma_3(t)|^r \right] \leq C \sum_{z \in \mathbb{Z}^d} p_z(t)^r \left( \sum_{z \in \mathbb{Z}^d} p_z(t)^2 \right)^{r/2} + C \left( \sum_{z \in \mathbb{Z}^d} p_z(t)^2 \right)^{r/2}, \]  
(3.22)
where the constant \( C \) may vary one line from another. By Proposition 2.4 and (2.34) we obtain the further upper bound
\[ C \sup_{z \in \mathbb{Z}^d} p_z(t)^{r-1} \sum_{z \in \mathbb{Z}^d} p_z(t) + (Ct^{-d/2})^{r/2} \leq Ct^{-(r-1)d/2} + Ct^{-rd/4}. \]  
(3.23)
Since \( r - 1 \geq r/2 \), we conclude from (3.22) and (3.23) that
\[ E \left[ |\Sigma_3(t)|^r \right] \leq Ct^{-rd/4}. \]  
(3.24)
Since \( rd > 4 \), we obtain
\[ E \left[ \sum_{n=1}^{\infty} |\Sigma_3(\delta n)|^r \right] = \sum_{n=1}^{\infty} E \left[ |\Sigma_3(\delta n)|^r \right] < \infty, \]  
(3.25)
which implies that \( \sum_{n=1}^{\infty} |\Sigma_3(\delta n)|^r \) is a.s. finite, and thus that \( \Sigma_3(\delta n) \to 0 \) as \( n \to \infty \), a.s.

We have shown that each of the three sums \( \Sigma_j(t) \) on the right-hand side of (3.8) a.s. tends to 0 for \( t = n\delta \to \infty \), which as said above yields (3.19) and completes the proof of Theorem 1.5.

4 A conservative version of the annihilating system

We now, finally, turn to the two-colour competition process. We introduce a conservative version of the process, previously explored in [2]. In this process, red and blue balls branch
and get offspring as in the competition process described in the introduction, but when a red and a blue ball meet, instead of annihilating, the two balls merge to form a purple ball. Each purple ball in the system continues to branch independently and according to the same rule as red and blue. Purple balls, however, do not interact with other balls. Consequently, we recover the competition process by ignoring all purple balls.

Let \( R_x(t) \), \( B_x(t) \), and \( P_x(t) \) be the numbers of red, blue, and purple balls, respectively, at site \( x \in \mathbb{Z}^d \) at time \( t \) in the above conservative process, and let \( R(t) := (R_x(t))_{x \in \mathbb{Z}^d} \), \( B(t) := (B_x(t))_{x \in \mathbb{Z}^d} \), and \( P(t) := (P_x(t))_{x \in \mathbb{Z}^d} \) be the corresponding vectors. Then the competition process is given by \( R(t) = R(t) - B(t) \). Furthermore, we use the common notation \( x_+ := \max(x, 0) \) and \( x_- := \max(-x, 0) \) for real \( x \), and extend this component-wise to vectors \( \zeta = (\zeta_x)_{x \in \mathbb{Z}^d} \). Then, in particular, \( R(t) = R(t)_+ \) and \( B(t) = R(t)_- \).

The crucial facts about this conservative process are stated in the following lemma. The lemma will in the coming sections allow us to apply the previous results for the monochromatic process in order to prove Theorem 1.2.

**Lemma 4.1.** If the process starts with (deterministic) initial configuration \( \mathcal{Z}(0) = \zeta \), then \( R(t) + P(t) \) is an instance of the monochromatic process started with \( \zeta_+ \), and \( B(t) + P(t) \) is an instance of the monochromatic process started with \( \zeta_- \). Furthermore, for all \( t \geq 0 \),

\[
\mathcal{Z}(t) = R(t) - B(t) = [R(t) + P(t)] - [B(t) + P(t)].
\]

**Proof.** Immediate from the definitions. \( \square \)

Note that the two monochromatic processes \( R(t) + P(t) \) and \( B(t) + P(t) \) are not independent.

### 4.1 A technical digression on the construction of the process

Sometimes we shall require some care in the construction of the competition process. In the proofs below we therefore (without loss of generality) assume the following.

We label each ball (regardless of its colour) by a finite string \((z, i_1, i_2, \ldots, i_m)\) with \( z \in \mathbb{Z}^d \), \( m \geq 0 \) and \( i_j \in \mathbb{N} \), such that the ball initially at \( z \) (if any) is labelled by \((z)\), and if a ball has label \((z, i_1, i_2, \ldots, i_m)\), then its children are labelled by \((z, i_1, i_2, \ldots, i_m, i)\) for \( i = 1, 2, \ldots \) (in some fixed order). This gives each ball a unique label. Furthermore, we assume that we have a Poisson clock for each possible label; these clocks are independent of each other and of the initial configuration. Moreover, each clock is equipped with one realization of the random offspring configuration \( \varphi \) for each ring of the clock. We now define the process with each ball using the corresponding clock and the copies of \( \varphi \) provided by that clock. (Ticks and tocks of unused clocks are ignored.) Furthermore, when a ball annihilates another, and there are several balls at that site that may be chosen for annihilation, we chose the one that comes first according to some fixed rule, for example the oldest ball at that site, or the ball with smallest label in lexicographic order (using an arbitrary but fixed order on \( \mathbb{Z}^d \)). Note that all randomness in the process \( \mathcal{Z}(t) \) now lies in the clocks and the initial configuration; \( \mathcal{Z}(t) \) is a deterministic function of these. Moreover, all clocks may be assumed to start ticking at the dawn of time, and are thus completely independent of the initial configuration.
The conservative process is defined in the same way, using the same clocks; when two balls merge to a purple ball, one of them is newborn and the other one is chosen among the existing balls at the site by the same rule as for annihilations in the competition process; furthermore, the purple ball inherits, for definiteness, the label (and thus the clock) of the ball that existed at the site before merging.

5 Fixation for \( p \neq 1/2 \)

In this section we prove fixation of the competing urn scheme starting from an unbalanced initial configuration, as stated in part (i) of Theorem 1.2. Throughout this section \( Z(t) \) will describe the evolution of the system starting from the \( p \)-random Bernoulli colouring. We begin with two preliminary results.

**Lemma 5.1.** Assume \( E \| \varphi \| < \infty \). For every \( T < \infty \) and \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( p \in [0,1] \) and interval \( I \subseteq [0,T] \) of length at most \( \delta \), we have

\[
P(Z_0(t) \text{ is not constant for } t \in I) < \varepsilon. \tag{5.1}
\]

**Proof.** Using Lemma 4.1 (and replacing \( \varepsilon \) by \( \varepsilon/2 \)), we see that it suffices to prove the corresponding result for the monochromatic process \( Y_p(t) \). Let \( [a,b] \subseteq [0,T] \). Since the monochromatic process is (weakly) increasing at each site, we obtain using Markov's inequality together with (3.2) and its proof,

\[
P(Y_p^0(t) \text{ is not constant for } t \in [a,b]) = P(Y^0_p(b) > Y^0_p(a)) \leq E[Y^0_p(b) - Y^0_p(a)] = p(e^{\lambda b} - e^{\lambda a}) \leq (b - a)e^{\lambda T}.
\]

(5.2)

The result follows by taking \( \delta \) small enough.

**Proposition 5.2.** Assume \( E \| \varphi \| < \infty \). Then, for the competing urn scheme on \( \mathbb{Z}^d \) starting from a \( p \)-random Bernoulli colouring, almost surely, the density \( \rho(t) \) of red urns, as defined in (1.2), exists for all \( t \geq 0 \) and

\[
\rho(t) = P(Z_0(t) > 0). \tag{5.3}
\]

**Proof.** First consider a fixed \( t \geq 0 \). We use a standard type of argument. That the limit in (1.2) exists, almost surely, for a fixed \( t \geq 0 \) is a consequence of translation invariance and the (multivariate) ergodic theorem (see e.g. [20, Theorem 10.12]). Using the construction of \( Z(t) \) in Section 4.1, \( Z(t) \) is a measurable deterministic function of the clocks and the initial colouring. Furthermore, changing a finite number of the clocks and initial colours can only affect \( Z_x(t) \) for finitely many \( x \), a.s., which will not change the limit (1.2). Thus \( \rho(t) \) is measurable with respect to the corresponding tail \( \sigma \)-field, and the Kolmogorov 0–1 law implies that \( \rho(t) \) is a.s. equal to a deterministic constant. Finally, by taking expectations in (1.2) and using the bounded convergence theorem, a.s.,

\[
\rho(t) = E \rho(t) = \lim_{n \to \infty} \frac{1}{(2n + 1)^d} \sum_{x \in [-n,n]^d} P(Z_0(t) > 0) = P(Z_0(t) > 0).
\]

(5.4)
This establishes (5.3) for a fixed $t \geq 0$.

We next show how to extend this equality to all $t \geq 0$ simultaneously. Define the upper and lower densities $\bar{\rho}(t)$ and $\underline{\rho}(t)$ as in (1.2) but using $\limsup$ and $\liminf$, respectively. These are thus always defined, and a.s. equal to each other and given by (5.3).

Given an interval $I$, define similarly $\bar{\rho}_+(I)$ as the upper density of sites that are red for some $t \in I$, and $\underline{\rho}_-(I)$ as the lower density of points that are red for all $t \in I$. The argument just given for $\rho(t)$ shows also that these densities a.s. exist and are equal to the corresponding probabilities at 0. Let $T < \infty$ and $\varepsilon > 0$, and let $\delta$ be as in Lemma 5.1. Then (5.1) implies that for any fixed interval $I \subseteq [0, T]$ of length at most $\delta$ we have a.s.

$$ \bar{\rho}_+(I) - \underline{\rho}_-(I) < \varepsilon \quad (5.5) $$

Furthermore, for all $t \in I$,

$$ \underline{\rho}_-(I) \leq \rho(t) \leq \bar{\rho}(t) \leq \bar{\rho}_+(I), \quad (5.6) $$

and thus by (5.5), a.s.,

$$ \sup_{t \in I} (\bar{\rho}(t) - \rho(t)) \leq \bar{\rho}_+(I) - \underline{\rho}_-(I) < \varepsilon. \quad (5.7) $$

By covering $[0, T]$ by a finite number of intervals of length at most $\delta$ we conclude that a.s.

$$ \sup_{t \in [0, T]} (\bar{\rho}(t) - \rho(t)) < \varepsilon, \quad (5.8) $$

and sending $\varepsilon \to 0$ and $T \to \infty$ shows that a.s. $\bar{\rho}(t) = \rho(t)$ for all $t$ simultaneously.

Furthermore, write for convenience $f(t) := \mathbb{P}(Z_0(t) > 0)$, so $\rho(t) = f(t)$ a.s. for each fixed $t$. Fix again an interval $I$ as above. Then (5.1) implies that for any $s, t \in I$, $|f(s) - f(t)| < \varepsilon$. Fix $s \in I$. Since (5.6) and (5.5) imply that a.s.

$$ |\bar{\rho}_+(I) - f(s)| = |\bar{\rho}_+(I) - \bar{\rho}(s)| \leq \bar{\rho}_+(I) - \underline{\rho}_-(I) < \varepsilon, \quad (5.9) $$

it follows that a.s.,

$$ \sup_{t \in I} |\bar{\rho}_+(I) - f(t)| \leq |\bar{\rho}_+(I) - f(s)| + \sup_{t \in I} |f(s) - f(t)| < 2\varepsilon \quad (5.10) $$

and thus, using (5.6) and (5.5) again, a.s.

$$ \sup_{t \in I} |\bar{\rho}(t) - f(t)| < 2\varepsilon + |\bar{\rho}(s) - \bar{\rho}_+(I)| < 2\varepsilon + (\bar{\rho}_+(I) - \underline{\rho}_-(I)) < 3\varepsilon. \quad (5.11) $$

By covering $[0, T]$ by a finite number of intervals of length at most $\delta$ we conclude that a.s. the same holds for $I$ replaced by $[0, T]$, and then sending $\varepsilon \to 0$ and $T \to \infty$ shows that a.s. $\bar{\rho}(t) = f(t)$ for all $t \geq 0$. Hence, a.s., $\rho(t) = \bar{\rho}(t) = f(t)$ for all $t$ simultaneously. \qed

**Proof of Theorem 1.2(i).** We first use Lemma 4.1 to show that the origin fixates (to red), i.e., that $Z_0(t) > 0$ for all large $t$, almost surely. Fix $\varepsilon > 0$ such that $2p - 1 > 3\varepsilon$. By Theorem 1.5 there exists a.s. a (random) finite $T_0$ such that, for all $t \geq T_0$,

$$ e^{-M}(R_0(t) + P_0(t)) > p - \varepsilon \quad \text{and} \quad e^{-M}(B_0(t) + P_0(t)) < 1 - p + \varepsilon, \quad (5.12) $$

and hence (4.1) yields $Z_0(t) > \varepsilon e^{M} > 0$, as required. By translation invariance, a.s. every site $x$ fixates.

In particular, this implies that $\mathbb{P}(Z_0(t) > 0) \to 1$ as $t \to \infty$. The statement about the density now follows from Proposition 5.2, which completes the proof of Theorem 1.2(i). \qed
6 Non-fixation at $p = 1/2$

The goal of this section is to prove part (ii) of Theorem 1.2, stating that the competing urn system started from a balanced initial Bernoulli colouring does not fixate. Let us first present a brief sketch of the proof. We start with the intuition that the state of the origin at time $t = 1$ is unlikely to dictate the state of the origin at time $t \gg 1$. Taking this intuition to its logical conclusion we should be able to choose a fast growing sequence of times $t_1, t_2, \ldots$ such that the state of the origin at time $t_n$ is approximately independent from its states at times $t_1, \ldots, t_{n-1}$. One would therefore expect the origin to be red for infinitely many of the times $t_n$ and blue for infinitely many of the times $t_n$, which would complete the proof.

In order to make this rigorous we show that the state of the origin at time $t = 1$ mostly depends on the descendants of balls which start near the origin. On the contrary, at time $t \gg 1$ the state at the origin depends on descendants of balls from a much larger region, while balls originating near the origin contribute little. This will allow us to define a growing sequence of scales $r_1, r_2, \ldots$ such that the state of the origin at time $t_n$ may be well approximated by considering only the descendants of balls initially in the annulus $[-r_n+1, r_n+1]^d \setminus [-r_n, r_n]^d$. Since these annuli are disjoint, this will allow us to ‘decouple’ the state of the origin at times $t_1, t_2, \ldots$.

We implement this approach in Section 6.3. An important ingredient will be to understand the likely order of magnitude of the number of balls at the origin. While the order of magnitude does not grow as fast as $e^{\lambda t}$ (as in the case $p \neq 1/2$), it is still likely to be at the order of its standard deviation, which is $t^{-d/4} e^{\lambda t}$. This will be obtained via a second moment approach, resting on the quantitative bounds for the monochromatic process obtained in Proposition 3.1; see Section 6.2 for details. We first present some notation and some useful lemmas.

6.1 Notation and some lemmas

In the proof we will consider several versions of the process with different initial values. We use throughout this section the detailed construction of the process described in Section 4.1. Since the clocks exist independently of the balls and colours in the initial configuration, this yields for any collection of initial configurations $\{\zeta\}$ a coupling of the corresponding copies $\{Z(t, \zeta)\}$ of the process.

For configurations, we use the product order on $\mathbb{Z}^d$, and write $\zeta \leq \zeta'$ for configurations $\zeta = (\zeta_x)_x$ and $\zeta' = (\zeta'_x)_x$ if and only if $\zeta_x \leq \zeta'_x$ for every $x \in \mathbb{Z}^d$. We let $|(x_1, \ldots, x_d)| := \max_i |x_i|$, the $l^\infty$-norm on $\mathbb{R}^d$, and set

$$B(0, r) := [-r, r]^d = \{x : |x| \leq r\}, \quad \text{for } r \geq 0. \quad (6.1)$$

For a configuration $\zeta$, let $\zeta^{<r}$ and $\zeta^{>r}$ denote the restrictions of $\zeta$ to $B(0, r)$ and its complement, respectively. (That is, the configurations given by $\zeta^{<r}_x = \zeta_x \cdot 1\{|x| \leq r\}$ and $\zeta^{>r}_x = \zeta_x \cdot 1\{|x| > r\}$.) Finally, let $\zeta^{(r, r')} := (\zeta^{>r})^{<r'}$.

We note that the expected configuration at a given time is a linear function of the initial configuration; we state this formally for the number of balls at $0$. 

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Lemma 6.1. For any deterministic initial configuration \( \zeta \in \{-1, 0, 1\}^Z \),

\[
E[Z_0(t, \zeta)] = e^{\lambda t} \sum_{\mathbf{z} \in Z^d} p_{\mathbf{z}}(t) \zeta_\mathbf{z}.
\] (6.2)

**Proof.** First, consider the monochromatic case; say \( \zeta \geq 0 \), so all balls are red. Then all balls evolve independently, so \( Z_0(t, \zeta) = \sum_{\mathbf{z} \in Z^d} X_{\mathbf{z}, 0}(t) \zeta_\mathbf{z} \), cf. (3.1), and (6.2) follows by linearity (since all terms are non-negative) and (3.9).

In general, we introduce purple balls as in Section 4 and use Lemma 4.1. Then, by (4.1) and using (6.2) for each of the monochromatic processes \( R(t) + P(t) \) and \( B(t) + P(t) \),

\[
E[Z_0(t, \zeta)] = E[R_0(t) + P_0(t)] - E[B_0(t) + P_0(t)] = e^{\lambda t} \sum_{\mathbf{z} : \zeta_\mathbf{z} = 1} p_{\mathbf{z}}(t) - e^{\lambda t} \sum_{\mathbf{z} : \zeta_\mathbf{z} = -1} p_{\mathbf{z}}(t),
\] (6.3)

which is well-defined since both sums are finite, and (6.2) follows.

We state some technical lemmas that assume the above coupling \( \{Z(t, \zeta)\} \) over different starting configurations (together with the construction in Section 4). The lemmas will be used later in this section with random initial configurations, but for the proofs the reader may consider only deterministic initial configurations, since the random case follows by conditioning. The first lemma also justifies the claim in Remark 1.1, that the process can be defined by first considering finite initial configurations and then taking a limit.

**Lemma 6.2.** For any initial configuration \( \zeta \in \{-1, 0, 1\}^Z \), using the coupling above, for every \( r, T < \infty \) there exists a.s. a (random) \( L < \infty \) such that \( Z_x(t, \zeta') = Z_x(t, \zeta \leq r') \) for all \( x \in B(0, r), t \in [0, T] \) and \( r' \geq L \).

We postpone the proof, and first present a lemma stating that the process is monotone in the initial configuration.

**Lemma 6.3.** Let \( \zeta \) and \( \zeta' \) be two initial configurations with \( \zeta \leq \zeta' \). Then, using the coupling above, a.s. \( Z(t, \zeta) \leq Z(t, \zeta') \) for all \( t \geq 0 \).

**Proof of Lemma 6.3, finite case.** The inequality \( \zeta \leq \zeta' \) means that every red ball in \( \zeta \) exists (with the same label) also in \( \zeta' \), and every blue ball in \( \zeta' \) exists also in \( \zeta \). (There may also be further red balls in \( \zeta' \) and blue balls in \( \zeta \).) We claim that this holds at all later times \( t \geq 0 \) too, and thus \( Z(t, \zeta') \leq Z(t, \zeta') \). In fact, since balls with the same label obey the same clock, this property is preserved at each nucleation (including accompanying annihilations, since they follow a fixed rule), as is easily verified. (If a nucleation results in a red ball in the process starting from \( \zeta \), then the nucleation will result in a red ball with the same label also in the process started from \( \zeta' \).) If \( \zeta \) and \( \zeta' \) are finite, the result now follows by induction over the number of nucleations (which then is finite in every finite interval, cf. Remark 1.1).

We postpone the proof of the general case.

**Proof of Lemma 6.2.** For (notational) simplicity, we show the result only for \( x = 0 \); the general case is the same with trivial modifications.
Let \( r_1 < r_2 \) and let
\[
\zeta'_y := \begin{cases} 
\zeta_y, & |y| \leq r_1, \\
1, & r_1 < |y| \leq r_2, \\
0, & r_2 < |y|.
\end{cases}
\] (6.4)

Then \( \zeta^{r'} \leq \zeta'_r \) for every \( r' \in [r_1, r_2] \), so by the already proved finite case of Lemma 6.3,
\[
Z(t, \zeta^{r'}) \leq Z(t, \zeta'), \quad r' \in [r_1, r_2], \ t \in [0, T].
\] (6.5)

In particular,
\[
Z_x(t, \zeta^{r_1}) \leq Z_x(t, \zeta'), \quad \text{for all } x \in \mathbb{Z}^d.
\] (6.6)

Let \( \mathcal{E}_+(r_1, r_2) \) be the event that \( Z_0(t, \zeta^{r_1}) < Z_0(t, \zeta^{r'}) \) for some \( t \in [0, T] \) and \( r' \in [r_1, r_2] \). On the event \( \mathcal{E}_+(r_1, r_2) \), using (6.5), \( Z_0(t, \zeta^{r_1}) + 1 \leq Z_0(t, \zeta^{r'}) \leq Z_0(t, \zeta') \) and thus, using (6.6),
\[
Z(t, \zeta^{r_1}) + \delta_{x,0} \leq Z(t, \zeta').
\] (6.7)

An induction argument, as in the proof of Lemma 6.3, shows that (6.7) implies
\[
Z(T, \zeta^{r_1}) + \delta_{x,0} \leq Z(T, \zeta').
\] (6.8)

In particular, \( Z_0(T, \zeta^{r_1}) + 1 \leq Z_0(T, \zeta') \). Consequently, Markov’s inequality yields, together with (6.6) and Lemma 6.1,
\[
\mathbb{P}(\mathcal{E}_+(r_1, r_2)) \leq \mathbb{P}(Z_0(T, \zeta') - Z_0(T, \zeta^{r_1}) \geq 1) \leq \mathbb{E}[Z_0(T, \zeta') - Z_0(T, \zeta^{r_1})] \\
= e^{\lambda T} \sum_{r_1 < |y| \leq r_2} p_y(T).
\] (6.9)

By symmetry, the same inequality holds for the event \( \mathcal{E}_-(r_1, r_2) \) that \( Z_0(t, \zeta^{r_1}) > Z_0(t, \zeta^{r'}) \) for some \( t \in [0, T] \) and \( r' \in [r_1, r_2] \). Consequently, letting \( r_2 \to \infty \),
\[
\mathbb{P}(\mathcal{E}_+(r_1, \infty) \cup \mathcal{E}_-(r_1, \infty)) \leq 2e^{\lambda T} \sum_{|x| > r_1} p_x(T),
\] (6.10)

which tends to 0 as \( r_1 \to \infty \), and on the complementary event, \( Z_0(t, \zeta^{r'}) = Z_0(t, \zeta^{r_1}) \) for all \( t \in [0, T] \) and \( r' \geq r_1 \). This proves that a.s. \( Z_0(t, \zeta^{r'}) \) is the same function of \( t \in [0, T] \) for all large \( r' \), and the extension to general \( x \) implies that \( Z(t, \zeta^{r'}) \) converges in \( D[0, \infty) \mathbb{Z}^d \) as \( r' \to \infty \) (in a strong sense). We can thus define \( Z(t, \zeta) \) as this limit, which verifies the claim in Remark 1.1. Furthermore, with this definition the proof of the lemma is now complete.

\[ \square \]

**Proof of Lemma 6.3, conclusion.** The general case follows immediately from the finite case proved above and Lemma 6.2.
We next show that the contribution to the origin at time $t$ coming from balls in $B(0, r)$ is at most order $t^{-d/2}e^{\lambda t}$.

**Lemma 6.4.** For every $r \geq 1$ and $\delta > 0$ there exists $C > 0$ such that for all $t \geq 1$ and every $\zeta \in \{-1, 0, 1\}^{\mathbb{Z}^d}$ we have, using the coupling above,

$$
\mathbb{P}\left(\left|Z_0(t, \zeta) - Z_0(t, \zeta^{>r})\right| > Ct^{-d/2}e^{\lambda t}\right) < \delta.
$$

**Proof.** Recall that in $\zeta^{>r}$, all $\zeta_x$ with $|x| \leq r$ have been reset to 0. Fix $r$ and define $\zeta^+$ by instead letting $\zeta^+_x := 1$ when $|x| \leq r$, and as before $\zeta^+_x = \zeta_x$ otherwise. Then $\zeta^+ \geq \zeta$ and $\zeta^+ \geq \zeta^{>r}$, and thus by Lemma 6.3, $Z_0(t, \zeta^+) \geq Z_0(t, \zeta)$ and $Z_0(t, \zeta^+) \geq Z_0(t, \zeta^{>r})$. Consequently, the triangle inequality and Lemma 6.1 (applied four times) give

$$
\mathbb{E}\left|Z_0(t, \zeta) - Z_0(t, \zeta^{>r})\right| \leq \mathbb{E}\left[Z_0(t, \zeta^+) - Z_0(t, \zeta)\right] + \mathbb{E}\left[Z_0(t, \zeta^+) - Z_0(t, \zeta^{>r})\right] \\
\leq 3e^{\lambda t} \sum_{x \in B(0, r)} p_x(t) \leq 3(2r + 1)^d e^{\lambda t} \sup_x p_x(t).
$$

The result follows by Proposition 2.4(i) and Markov’s inequality.

\[ \square \]

### 6.2 A second moment analysis

In this subsection we return to consider the process starting from the random-symmetric Bernoulli colouring, that is, in which each site is given a ball whose colour is determined by a fair coin flip. We aim to prove the following bound on deviations of $Z_0(t)$.

**Proposition 6.5.** There exists a constant $c > 0$ such that for all $t > 1/c$ we have

$$
\mathbb{P}\left(e^{-\lambda t}Z_0(t) > ct^{-d/4}\right) > c.
$$

To show this, we shall use the following generic lemma, which is a conditional version of the Paley–Zygmund inequality.

**Lemma 6.6.** Let $X$ be a random variable and $\mathcal{F}$ a sub-$\sigma$-field (on some probability space). Suppose that $\mathbb{E}[X^2] \leq K$ and $\mathbb{P}\left(\mathbb{E}[X \mid \mathcal{F}] \geq 1/K\right) \geq 1/K$ for some constant $K$. Then

$$
\mathbb{P}\left(X > 1/(2K)\right) \geq 1/(4K^5).
$$

**Proof.** Let $F$ be the event that $\mathbb{E}[X \mid \mathcal{F}] \geq 1/K$ and let $E$ be the event that both $\mathbb{E}[X \mid \mathcal{F}] \geq 1/K$ and $X > 1/(2K)$. Cauchy–Schwartz gives

$$
\mathbb{E}[X 1_E] \leq \|X\|_2 1_E \|_2 \leq K^{1/2} \mathbb{P}(E)^{1/2}.
$$

On the other hand we have that

$$
\mathbb{E}[X 1_F] \geq \frac{\mathbb{P}(F)}{K}
$$

and, since $X \leq 1/2K$ on $F \setminus E$,

$$
\mathbb{E}[X 1_{F \setminus E}] \leq \frac{\mathbb{P}(F \setminus E)}{2K} \leq \frac{\mathbb{P}(F)}{2K}.
$$

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Thus, since $P(F) \geq 1/K$ by assumption,
\[
\mathbb{E}[X 1_E] \geq \frac{P(F)}{2K} \geq \frac{1}{2K^2},
\] (6.18)
which combined with (6.15) gives
\[
K^{1/2} P(E)^{1/2} \geq \frac{1}{2K^2}.
\] (6.19)
It follows that $P(E) \geq 1/4K^5$, and the result follows.

We also need an estimate of $\text{Var}[Z_0(t)]$.

**Lemma 6.7.** For all $t \geq 1$,
\[
\text{Var}[e^{-\lambda t} Z_0(t)] = O(t^{-d/2}).
\] (6.20)

**Proof.** We use the conservative process with purple balls and (4.1), recalling that both
$R_0(t) + P_0(t)$ and $B_0(t) + P_0(t)$ have the same distribution as $Y_0^p(t)$ with $p = 1/2$. Hence,
by Proposition 3.1(ii),
\[
\text{Var}[Z_0(t)] \leq 4 \text{Var}[Y_0^{1/2}(t)] = O(t^{-d/2} e^{2\lambda t}).
\] (6.21)

We are now in position to proceed with the proof of the proposition.

**Proof of Proposition 6.5.** Let $\zeta$ be a random symmetric Bernoulli colouring and let $\mathcal{F}$ be the $\sigma$-field generated by $\zeta$. Since the clocks are independent of $\zeta$, Lemma 6.1 yields
\[
\mathbb{E}[e^{-\lambda t} Z_0(t) \mid \mathcal{F}] = \sum_{z \in \mathbb{Z}^d} p_{z}(t) \zeta_z =: S(t).
\] (6.22)

$S(t)$ is a sum of independent random variables with mean 0, so using Proposition 2.4,
\[
\text{Var}[S(t)] = \sum_{z \in \mathbb{Z}^d} p_{z}(t)^2 = \Theta(t^{-d/2}).
\] (6.23)
We next claim that, as $t \to \infty$, we have
\[
S(t)/\sqrt{\text{Var}[S(t)]} \to N(0,1) \text{ in distribution.}
\] (6.24)

To see this, we use the central limit theorem with the Lyapunov condition that
\[
\beta(r,t) := \left( \text{Var}[S(t)] \right)^{-r/2} \sum_{z \in \mathbb{Z}^d} \mathbb{E}[|p_{z}(t) \zeta_z|^r] = o(1) \text{ as } t \to \infty,
\] (6.25)
for some $r > 2$; see e.g. [18, Theorem 7.2.2 and 7.2.4]. (The central limit theorem is usually
stated for finite sums, but extends immediately to $L^2$-convergent sums by truncation and

the Cramér–Slutsky theorem.) We verify the Lyapounov condition (6.25) with \( r = 3 \). Then, by Proposition 2.4(i)

\[
\sum_{z \in \mathbb{Z}^d} \mathbb{E}[|p_z(t)\zeta_z|^3] = \sum_{z \in \mathbb{Z}^d} p_z(t)^3 \leq \sup_z p_z(t)^2 = O(t^{-d}).
\] (6.26)

Hence, (6.23) and (6.26) yield \( \beta(3, t) = O(t^{-d/4}) \), which verifies (6.25), so (6.24) holds.

To complete the proof, let \( X := e^{-\lambda t} Z_0(t)/\sqrt{\text{Var}[S(t)]} \). Recall that \( Z_0(t) \) has mean zero, so we may from Lemma 6.7 and (6.23) deduce that \( \mathbb{E}[e^{-2\lambda t} Z_0(t)^2] \leq K \text{Var}[S(t)] \) for some constant \( K \); in other words, \( \mathbb{E}[X^2] \leq K \). Increasing \( K \) if necessary, it follows from (6.22) and (6.24) that

\[
\mathbb{P} (\mathbb{E}[X \mid \mathcal{F}] \geq 1/K) = \mathbb{P} \left( S(t) \geq \sqrt{\text{Var}[S(t)]/K} \right) \geq 1/K
\] (6.27)

for all large \( t \). Lemma 6.6 therefore shows that

\[
\mathbb{P} \left( e^{-\lambda t} Z_0(t) > \sqrt{\text{Var}[S(t)]/(2K)} \right) \geq 1/(4K^3).
\] (6.28)

Since \( \sqrt{\text{Var}[S(t)]} = \Theta(t^{-d/4}) \) by (6.23), the proof is complete. \( \square \)

### 6.3 A decoupling argument and conclusion of the proof

We now complete the proof of part (ii) of Theorem 1.2. We began Section 6 with an overview of the proof, including the idea that we would consider a sequence of times \( t_1, t_2, \ldots \) and scales \( r_1, r_2, \ldots \) such that the state of the origin at time \( t_n \) mostly depends on the descendants of balls which start in the annulus \( B(0, r_{n+1}) \setminus B(0, r_n) \). We now implement this idea rigorously. In addition to the sequences of times and scales we define an auxiliary sequence \((C_i)_{i \geq 1}\) which controls the contribution at the origin of balls descending from within a growing sequence of regions.

**Proof of Theorem 1.2(ii).** Throughout the proof \( \zeta \) denotes a random symmetric Bernoulli colouring, and \( c > 0 \) is the constant from Proposition 6.5. Let \( t_0 = 1/c \) and let \( r_0 = 0 \). We now define the sequences \((r_i)_{i \geq 1}\), \((C_i)_{i \geq 1}\) and \((t_i)_{i \geq 1}\) sequentially. For \( i \geq 1 \), choose

(i) \( r_i > r_{i-1} \), using Lemma 6.2, such that

\[
\mathbb{P} \left( Z_0(t_{i-1}, \zeta) \neq Z_0(t_{i-1}, \zeta^{\leq r_1}) \right) \leq 2^{-i};
\] (6.29)

(ii) \( C_i > 0 \), using Lemma 6.4, such that for every \( t \geq 1 \) and \( r \geq r_i \),

\[
\mathbb{P} \left( \left| Z_0(t, \zeta^{\leq r}) - Z_0(t, \zeta^{(r_i, r)}) \right| > C_i t^{-d/2} e^{\lambda t} \right) \leq 2^{-i}.
\] (6.30)

(iii) \( t_i > t_{i-1} + 1 \) such that \( t_i^{d/4} > 3e^{-1} C_i \), so that, by Proposition 6.5, we have

\[
\mathbb{P} \left( Z_0(t_i, \zeta) > 3C_i t_i^{-d/2} e^{\lambda t_i} \right) \geq c.
\] (6.31)
In particular, (6.30) yields
\[ P\left( |Z_0(t_i, \zeta^{(r_i, r_{i+1})}) - Z_0(t_i, \zeta_{(r_i, r_{i+1})})| > C_i t_i^{-d/2} e^{-\lambda_i t_i} \right) \leq 2^{-i}. \] (6.32)

Hence, using also (6.29),
\[ P\left( |Z_0(t_i, \zeta) - Z_0(t_i, \zeta_{(r_i, r_{i+1})})| > C_i t_i^{-d/2} e^{-\lambda_i t_i} \right) \leq 2^{-i} + 2^{-i-1} \leq 2^{1-i}. \] (6.33)

Thus, (6.31) implies
\[ P\left( |Z_0(t_i, \zeta_{(r_i, r_{i+1})}) > 2C_i t_i^{-d/2} e^{\lambda_i t_i} \right) \geq c - 2^{1-i}. \] (6.34)

Next, we define two ‘failure’ events, of which (at least) one must occur for the origin to be blue at all large times.

- Let \( F_1 \) be the event that for infinitely many \( i \geq 1 \) we have
  \[ |Z_0(t_i, \zeta) - Z_0(t_i, \zeta_{(r_i, r_{i+1})})| > C_i t_i^{-d/2} e^{-\lambda_i t_i}. \] (6.35)

- Let \( F_2 \) be the event that for at most finitely many \( i \geq 1 \) we have
  \[ Z_0(t_i, \zeta_{(r_i, r_{i+1})}) > 2C_i t_i^{-d/2} e^{\lambda_i t_i}. \] (6.36)

By (6.33), it follows that (6.35) occurs with probability at most \( 2^{1-i} \). Hence, by the Borel–Cantelli lemma, we have that \( P(F_1) = 0 \). Moreover, by (6.34), (6.36) occurs with probability at least \( c/2 \) for large \( i \). Note that the processes \( Z(t, \zeta_{(r_i, r_{i+1})}) \), for \( i \geq 1 \), are mutually independent by our construction. Hence the events in (6.36) are independent, and thus the other Borel–Cantelli lemma implies that \( P(F_2) = 0 \).

Let \( I_1 \) and \( I_2 \) denote the sets of \( i \)'s for which (6.35) and (6.36) occur, respectively. We have shown that \( I := I_2 \setminus I_1 \) is infinite almost surely. To complete the proof, we note that for each \( i \in I \) we have
\[ Z_0(t_i, \zeta_{(r_i, r_{i+1})}) \geq Z_0(t_i, \zeta_{(r_i, r_{i+1})}) - C_i t_i^{-d/2} e^{\lambda_i t_i} > C_i t_i^{-d/2} e^{\lambda_i t_i} > 0. \]

Hence, there are a.s. arbitrarily large times \( t \) such that \( Z_0(t, \zeta) > 0 \) and thus \( 0 \) is red. By symmetry there are a.s. also arbitrarily large \( t \) with \( 0 \) being blue. This completes the proof of part (ii) of Theorem 1.2.

\[ \square \]

7 Dealing with death

As we have defined our process, at each ring of a clock, the corresponding ball produces offspring according to \( \Phi \), and remains itself where it was. Consequently, in the monochromatic version of our process, once a ball is born, it remains at its position at all future times. We shall in this section describe briefly how the results obtained for this process can be extended to allow balls to die (disappear) as they reproduce. (Recall Remark 1.4.) This is obviously more general, since we can let the parent be replaced by a copy of itself.
In particular, this extension allows us to consider models where the balls move around at random, such as the standard (continuous time, discrete space) branching random walk where particles perform independent simple symmetric random walks, and in each step, with some probability, split in two or more independent copies.

So, consider the model in which particles have an exponentially distributed life time, at the end of which they are removed and replaced by configuration \( \varphi \), shifted to the position of the particle, drawn from \( \Phi \). We assume, as before, that \( \Phi \) is an irreducible probability measure on finite non-negative (but not necessarily non-empty) configurations on \( \mathbb{Z}^d \), satisfying \( 1 < \mathbb{E}[||\varphi||] < \infty \). Under the condition that \( \mathbb{E}[||\varphi||] > 1 \), then the total number \( ||X(t)|| \), when starting from a single ball at the origin, is a supercritical branching process.\(^3\) We outline below how our arguments may be adapted to cover this more general family of processes. (As before, we assume that \( \mathbb{E}[||\varphi||^{4+\varepsilon}] < \infty \) and that (1.4) holds where appropriate.)

There are four places at which our arguments need modification. First, in Section 2, we need to compensate for the death of particles. Write \( \varphi' \) for the change caused as a clock rings. Then \( \varphi' = \varphi - \delta x,0 \) and \( \mu'(x) = \mu(x) - \delta x,0 \). Similarly, redefine \( \lambda := \mathbb{E}[||\varphi||] - 1 > 0 \). By replacing \( \varphi \) and \( \mu \) by \( \varphi' \) and \( \mu' \) throughout Section 2, then all results continue to hold for the more general family of processes. (Note, in particular, how the expression \( \lambda - \text{Re} \tilde{\mu}(u) \) is unaffected by these changes.)

Second, in Section 3, when \( Y^p(t) \) is no longer non-decreasing, we need an argument to deduce (3.20) from (3.19). A simple large deviation estimate will suffice, since if \( Y^p(t) \) changes significantly during a short time span, then a greater than expected number of clock rings must have occurred. To make this formal we introduce the events

\[
A_n := \{ e^{-\lambda \delta n} Y^p_0(\delta n) \in (p e^{-\lambda \delta}, p e^{\lambda \delta}) \},
\]
\[
B_n := \{ e^{-\lambda \delta} Y^p_0(t) \geq p(1 - 2\delta) e^{-2\lambda \delta} \text{ for all } t \in [\delta n, \delta(n+1)] \},
\]
\[
C_n := \{ e^{-\lambda \delta} Y^p_0(t) \leq p(1 + 2\delta) e^{2\lambda \delta} \text{ for all } t \in [\delta n, \delta(n+1)] \}.
\]

It will thus suffice to show that for every \( \delta \in (0, \frac{1}{2}) \) a.s. the events \( B_n \) and \( C_n \) will occur for all but finitely many \( n \). Let \( M := Y^p_0(\delta n) \) be the number of balls present at the origin at time \( \delta n \), and let \( y_n := p e^{\lambda \delta(n-1)} \). On the event \( A_n \cap B^c_n \), \( M > y_n \), and of these \( M \) balls at least \( M - (1 - 2\delta)y_n \) must die before time \( \delta(n+1) \). Since each ball dies with probability \( 1 - e^{-\delta} < \delta \), Chebyshev’s inequality implies, conditioned on \( M > y_n \),

\[
\mathbb{P}(A_n \cap B^c_n \mid M) \leq \frac{M}{(M - (1 - 2\delta)y_n - \delta M)^2} = \frac{M}{((1 - \delta)M - (1 - 2\delta)y_n)^2}.
\]

The right-hand side is decreasing in \( M \geq y_n \), and thus, for all such \( M \),

\[
\mathbb{P}(A_n \cap B^c_n \mid M) \leq \frac{y_n}{((1 - \delta)y_n - (1 - 2\delta)y_n)^2} = C' e^{-\lambda \delta n}.
\]

\(^3\)Note that we above have assumed, implicitly, that \( \Phi \) is supported on nonempty configurations, as the contrary would simply correspond to a rescaling of time. This is no longer assumed here, resulting in the possible extinction in the process evolving from a single ball. Extinction will, of course, not be possible when starting from an infinite starting configuration.
Furthermore, this holds trivially for \( M < y_n \) too, since the conditional probability then is 0. Consequently, \( \mathbb{P}(A_n \cap B'_n) \leq C'e^{-\lambda \delta n} \), and the Borel–Cantelli lemma shows that a.s. the event \( A_n \cap B'_n \) occurs for only finitely many \( n \).

Similarly, with \( y'_n := pe^{\lambda \delta (n+2/2)} \), on the event \( A_{n+1} \cap C'_n \) the number of balls at the origin exceeds \((1 + 2\delta)y'_n \) at some point during the time interval. Let \( \tau \) be the first time that this happens, and \( N > (1 + 2\delta)y'_n \) the number of balls at that time. At least \( N - y'_n \) of these balls must die before time \( \delta(n + 1) \). Conditioned on \( \tau \) and \( N \), each ball dies with probability less than \( \delta \), and Chebyshev’s inequality yields, for \( N \geq y''_n := (1 + 2\delta)y'_n \),

\[
\mathbb{P}(A_{n+1} \cap C'_n \mid \tau, N) \leq \frac{N}{(1 - \delta)(N - y'_n)^2} \leq \frac{y''_n}{((1 - \delta)y'_n - y_n)^2} = C''e^{-\lambda \delta n}.
\]

Hence \( \mathbb{P}(A_{n+1} \cap C'_n) \leq C''e^{-\lambda \delta n} \), so the event \( A_{n+1} \cap C'_n \) occurs for only finitely many \( n \).

Since \( A_n \) a.s. occurs for all large \( n \) by (3.19), it follows that for every \( \delta > 0 \) a.s.

\[
p(1 - 2\delta)e^{-2\lambda \delta} \leq \liminf_{t \to \infty} e^{-\lambda T}Y(t)_{\lambda} \leq \limsup_{t \to \infty} e^{-\lambda T}Y(t)_{\lambda} \leq p(1 + 2\delta)e^{2\lambda \delta}.
\]

This completes the proof of Theorem 1.5 in this more general setting.

Next, we see how to prove Lemma 5.1 in the more general setting. As before, it will suffice to consider the monochromatic process \( Y_{\lambda}(t) \). Let \( A \) denote the event that a ball present at the origin at time \( a \) dies before time \( b \), and let \( B \) denote the event that a ball not present at the origin at time \( a \) arrives at the origin before time \( b \). Moreover, let \( N \) denote the number of balls that arrive at the origin during the interval \([a, b], D \) the number of balls already present at time \( a \) that die before time \( b \), and \( D' \) the number of balls that arrive after time \( a \) and die before time \( b \). Then

\[
\mathbb{E}[D] \leq (b - a)\mathbb{E}[Y_{\lambda}(a)] \quad \text{and} \quad \mathbb{E}[D'] \leq (b - a)\mathbb{E}[N].
\]

In addition, \( N = Y_{\lambda}(b) - Y_{\lambda}(a) + D + D' \), so under the assumption that \( b - a \leq 1/2 \),

\[
\mathbb{E}[N] \leq 2\mathbb{E}[Y_{\lambda}(b) - Y_{\lambda}(a) + D] \leq 2(e^{\lambda b} - e^{\lambda a}) + 2(b - a)e^{\lambda a} \leq 4(b - a)(\lambda + 1)e^{\lambda T}.
\]

The proof of the lemma is now concluded as before, using Markov’s inequality.

Finally, we see how to adapt the proof of Lemma 6.2. We need a bound on the event \( \mathcal{E}_+(r_1, r_2) \). The induction in the proof of the finite case of Lemma 6.3 still holds, and shows that (6.5) holds in the strong sense that every red ball in \( Z(t, \zeta_{<r'}) \) exists also in \( Z(t, \zeta) \), and conversely for blue balls. For times \( t \) such that \( Z_0(t, \zeta_{<r'}) < Z_0(t, \zeta') \) and \( Z_0(t, \zeta') > 0 \), define the excess ball (at time \( t \)) as the red ball with smallest label (in a fixed order) that is at \( 0 \) in \( Z(t, \zeta) \) but does not exist in \( Z(t, \zeta_{<r'}) \); if \( Z_0(t, \zeta_{<r'}) < Z_0(t, \zeta') \leq 0 \) define the excess ball as the blue ball with smallest label that is at \( 0 \) in \( Z(t, \zeta_{<r'}) \) but does not exist in \( Z(t, \zeta') \). For completeness, if \( Z_0(t, \zeta_{<r'}) = Z_0(t, \zeta') \), define the excess ball as an extra (non-existing) ball with its own clock. Let \( \tau \) denote the first time at which \( Z_0(t, \zeta_{<r_1}) < Z_0(t, \zeta_{<r'}) \) (with \( \tau = \infty \) if this never happens), and note that \( \mathcal{E}_+(r_1, r_2) = \{ \tau \leq T \} \). Let \( F \) be the event that the excess ball does not die in the interval \([\tau, T] \). Then, on the event \( \mathcal{E}_+(r_1, r_2) \cap F \), (6.7) holds for \( t = \tau \) and the induction argument in the proof
of Lemma 6.3 implies that also (6.8) holds. Consequently, Markov’s inequality yields that the bound (6.9) holds for $\mathbb{P}(E_+ \cap (r_1, r_2) \cap F)$. To complete the proof it suffices to note that, since $\tau$ is a stopping time and $E_+(r_1, r_2)$ is determined by $\tau$,

$$\mathbb{P}(F \mid \tau) = e^{-(T-\tau)} \geq e^{-T}$$

(7.10)

and

$$\mathbb{P}(E_+(r_1, r_2) \cap F) = \mathbb{E}[\mathbf{1}_{E_+(r_1, r_2)} \mathbb{P}(F \mid \tau)] \geq e^{-T} \mathbb{P}(E_+(r_1, r_2))$$

(7.11)

The rest of the proof is the same as before.

8 Open problems and further directions

We round off with some open problems and suggested directions for further study, inspired by the results above. We give also some comments on possible extensions, some of which seem easy, but we leave them for the reader to check.

The problems may be considered for general branching rules, much like in the present paper, but in some cases (such as for the first question) it may make more sense for a specific branching rule (such as the nearest-neighbour rule, in which $\varphi$ is the deterministic configuration that puts a ball at each of the $2d$ neighbours of the origin). In some cases we even expect that the answer to the question may depend on the branching rule, much opposed to the results reported in this paper.

1. For $d = 1$, what is the length of a typical monochromatic interval?

2. For $p > 1/2$, at what rate does the density of blue sites tend to zero?

3. For $p = 1/2$, at what rate does a site change colour?

4. For $p = 1/2$, how may balls are contained at the origin at a given time? Proposition 6.5 provides a partial answer, and Lemma 6.7 a matching upper bound. Is it true that $|Z_0(t)| = \Theta(t^{-d/4}e^{\lambda t})$ with high probability, or does the density of times for which it holds tend to 1 as $t \to \infty$?

5. Are the moment conditions in Theorems 1.2 and 1.5 necessary? In particular, is the condition $\mathbb{E}[\|\varphi\|^2] < \infty$, instead of $\mathbb{E}[\|\varphi\|^{4+\varepsilon}] < \infty$, sufficient for the conclusion of Theorem 1.5 to hold?

6. We have in this paper considered competition between two types. It would be interesting to extend our results to three or more competing types. We believe that it may be challenging to find a substitute for the conservative process described in Section 4. Problems of a similar character were suggested also in [2].

7. We assumed throughout the paper that the initial configuration has at most one ball at each site. We can more generally consider initial configurations $(\zeta_x)_{x \in \mathbb{Z}^d}$ where the $\zeta_x$ are i.i.d. with an arbitrary distribution. We expect that the results above generalize rather easily under some moment condition on $\zeta_x$, but we have not checked the details. We expect that it is less straightforward to adapt our techniques
to allow the different types to jump at different rates, or reproduce according to different rules.

8. In the model studied by Bramson and Lebowitz [12, 13], no particles are born, and particles move according to independent continuous-time symmetric random walks. This can be regarded as an extreme case of our model (not covered above), where balls die as in Section 7 and the offspring $\varphi$ consists of a single ball. For this model Cabezas, Rolla and Sidoravicius [14] have shown that, under weak assumptions, a.s. there exist arbitrarily large times when the origin is occupied. A more detailed conjecture, which seems to be open, would be that, starting from a $p$-random Bernoulli initial colouring, the origin is a.s. visited by both colours infinitely many times when $p = 1/2$, but not when $p > 1/2$.

9. Consider the urn process on $\mathbb{Z}^d$ run from an initial configuration with a single red and a single blue ball. Under what conditions will both red and blue balls remain in the system at all times (so-called coexistence) with positive probability? For $d = 1$, in the event of coexistence, under what conditions does an ‘interface’, that is a macroscopic division, between red and blue exist, and how does it evolve over time? What is the analogous higher-dimensional phenomenon? Some progress have been made to these questions for a related model by Ahlberg, Angel and Kolesnik [1].

References


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