# ON SEMI-RESTRICTED ROCK, PAPER, SCISSORS 

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#### Abstract

Spiro, Surya and Zeng (Electron. J. Combin. 2023) recently studied a semi-restricted variant of the well-known game Rock, Paper, Scissors; in this variant the game is played for $3 n$ rounds, but one of the two players is restricted and has to use each of the three moves exactly $n$ times. They find the optimal strategy, and they show that it results in an expected score for the unrestricted player $\Theta(\sqrt{n})$; they conjecture, based on numerical evidence, that the expectation is $\approx 1.46 \sqrt{n}$.

We analyse the result of the strategy further and show that the average is $\sim c \sqrt{n}$ with $c=3 \sqrt{3} / 2 \sqrt{\pi}=1.466$, verifying the conjecture. We also find the asymptotic distribution of the score, and compute its variance.


## 1. Introduction

A semi-restricted variant of the well-known game Rock, Paper, Scissors (RPS) was recently studied by Spiro, Surya and Zeng [6]. In the standard version of RPS, two players simultaneously select one of the three choices rock, paper, scissors, where paper beats rock, scissors beats paper, and rock beats scissors; if both select the same, the result is a draw. The game is symmetric, so there is obviously no advantage to any of the players. It is easy to see that the optimal strategy for both players is to choose randomly, with equal probability for each choice (see further Section 2.2).

In the semi-restricted variant in [6], two players R (restricted) and N (normal) agree to play $3 n$ rounds of RPS for some integer $n$, but R is restricted to choose rock, paper, and scissors exactly $n$ times each, while N plays without restriction. Clearly, the restriction is a disadvantage for R . (In particular, N will always win the last round, since R then has only one choice, and N knows which one.) How large is this disadvantage? More precisely, let $S_{n}$ be the final score of N , defined as the number of rounds won by $N$ minus the number lost. We assume (as [6]) that the objective of both players is the expectation $\mathbb{E} S_{n}$, which N wants as high as possible, while R wants the opposite. We assume also that both players play optimally, i.e., they use their optimal randomized strategy. (Which always exists by the theory of von Neumann [5], since RPS is a two-player zero-sum game; see further e.g. [4, Chapter 2].)

The main result of [6] is that the (unique) optimal strategy for R is to play greedily, i.e., as if each round were the last; see further Section 2.2. (This is far from obvious, and rather surprising.) It is also shown in [6] that with optimal strategies, the expected gain $\mathbb{E} S_{n}=\Theta(\sqrt{n})$, and it is asked [6, Question 21] whether $\mathbb{E} S_{n} \sim c \sqrt{n}$ for some constant $c>0$ as $n \rightarrow \infty$; [6] says further that numerical calculations for $n \leqslant 100$ suggest that this might hold with $c \approx 1.46$.

[^0]The main purpose of the present note is to verify this conjecture, and to identify the constant; we also find the asymptotic distribution of the final score $S_{n}$. The main results are collected in the two theorems below; the proofs (given in Sections 3 and Section 4) are based on the main result of [6], and an analysis of the strategy found there.

Theorem 1.1. For semi-restricted RPS played 3 n rounds, the expected score for $N$ with optimal plays for both players is, as $n \rightarrow \infty$,

$$
\begin{equation*}
\mathbb{E} S_{n} \sim \sqrt{\frac{27 n}{4 \pi}}=\frac{3 \sqrt{3}}{2 \sqrt{\pi}} \sqrt{n} \tag{1.1}
\end{equation*}
$$

The constant $3 \sqrt{3} /(2 \sqrt{\pi}) \doteq 1.4658$, which verifies also the numerical conjecture in [6].

For the asymptotic distribution, we give several alternative descriptions. See Sections 4 and 5 for uses of some of them.

Theorem 1.2. As $n \rightarrow \infty$, we have convergence in distribution, together with all moments,

$$
\begin{equation*}
n^{-1 / 2} S_{n} \xrightarrow{\mathrm{~d}} S^{*} \tag{1.2}
\end{equation*}
$$

where the limit $S^{*}$ can be described by any of the following equivalent formulas:
(i) We have

$$
\begin{equation*}
S^{*}=W+\max \left\{Z_{1}, Z_{2}, Z_{3}\right\} \tag{1.3}
\end{equation*}
$$

where $W, Z_{1}, Z_{2}, Z_{3}$ are jointly normal with $W$ independent of $\left(Z_{1}, Z_{2}, Z_{3}\right)$ and

$$
\begin{align*}
W & \in N(0,2)  \tag{1.4}\\
\left(Z_{1}, Z_{2}, Z_{3}\right) & \in N\left(0,\left(\begin{array}{rrr}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{array}\right)\right) \tag{1.5}
\end{align*}
$$

(ii) We have

$$
\begin{equation*}
S^{*}=W^{\prime}+\sqrt{3} \max \left\{Z_{1}^{\prime}, Z_{2}^{\prime}, Z_{3}^{\prime}\right\} \tag{1.6}
\end{equation*}
$$

where $W^{\prime}, Z_{1}^{\prime}, Z_{2}^{\prime}, Z_{3}^{\prime}$ are independent standard normal $N(0,1)$.
(iii) We have

$$
\begin{equation*}
S^{*}=\max \left\{Z_{1}^{\prime \prime}, Z_{2}^{\prime \prime}, Z_{3}^{\prime \prime}\right\} \tag{1.7}
\end{equation*}
$$

where $Z_{1}^{\prime \prime}, Z_{2}^{\prime \prime}, Z_{3}^{\prime \prime}$ are jointly normal with

$$
\left(Z_{1}^{\prime \prime}, Z_{2}^{\prime \prime}, Z_{3}^{\prime \prime}\right) \in N\left(0,\left(\begin{array}{lll}
4 & 1 & 1  \tag{1.8}\\
1 & 4 & 1 \\
1 & 1 & 4
\end{array}\right)\right)
$$

(iv) We have

$$
\begin{equation*}
S^{*}=W+R \cos \theta \tag{1.9}
\end{equation*}
$$

where $W, R, \theta$ are independent with $W \in N(0,2)$ as in (1.4), $R$ has a Rayleigh distribution with density $\frac{1}{2} r e^{-r^{2} / 4}, r>0$, and $\theta$ has a uniform distribution $U(0, \pi / 3)$.

Note that the normal variables $Z_{1}, Z_{2}, Z_{3}$ in (1.3) satisfy $Z_{1}+Z_{2}+Z_{3}=0$ a.s., as a consequence of (1.5); thus $\left(Z_{1}, Z_{2}, Z_{3}\right)$ lives in a 2 -dimensional space. See further the proof in Section 4.

The moment convergence in Theorem 1.2 enables us to find also asymptotics of higher moments of $S_{n}$, see Section 4.

In Section 5, we discuss the probability that the disadvantaged player R nevertheless wins the game; we compute it for the strategies used above, but leave the case of optimal play for this objective as an open problem.

## 2. Preliminaries

2.1. Notation. The three choices rock, paper, scissors will be numbered $1,2,3$; thus $i+1$ beats $i(\bmod 3)$.

The random variable $S(t)$ is the score of N after round $t=1, \ldots, 3 n$, i.e., the number of rounds won by N so far minus the number of rounds won by R. As in the introduction, $S_{n}:=S(3 n)$ is the score at the end of the game. (Except for $S_{n}$, we do not show $n$ explicitly in the notation, although $S(t)$ and many variables introduced below depend on $n$.)

If $X_{n}$ is a sequence of random variables, and $a_{n}$ a sequence of (positive) numbers, we write $X_{n}=O_{\mathrm{p}}\left(a_{n}\right)$ if the family $\left\{X_{n} / a_{n}\right\}$ is bounded in probability (also called tight), i.e., if for every $\varepsilon>0$ there exists $C$ such that $\mathbb{P}\left(\left|X_{n}\right|>C a_{n}\right)<\varepsilon$ for all $n$. Furthermore, we write $X_{n}=O_{L^{p}}\left(a_{n}\right)$ (where $p>0$ is a parameter) if the family $\left\{X_{n} / a_{n}\right\}$ is bounded in $L^{p}$, i.e., $\sup _{n} \mathbb{E}\left|X_{n} / a_{n}\right|^{p}<\infty$.

The basis vectors in $\mathbb{R}^{3}$ are denoted $\mathbf{e}_{1}:=(1,0,0), \mathbf{e}_{2}:=(0,1,0), \mathbf{e}_{3}:=(0,0,1)$.
$C$ denotes unspecified constants that may vary from one occurrence to the next; we use $C_{p}$ for constants that depends on the parameter $p$.

Unspecified limits are as $n \rightarrow \infty$.
2.2. The greedy strategy. Recall that in any two-person zero-sum game, each player has an optimal strategy which in general is randomized; the different alternatives are selected with some probabilities chosen such that they maximize the minimum over all strategies of the opponent of the expected gain; see [5] and e.g. [4].

As said above, it was shown by Spiro, Surya and Zeng [6] that in semi-restricted RPS, the best strategy of R is to play greedily, i.e., to analyse each round separately and use the optimal strategy for the expected score in that round. (This is far from obvious, since the best play in one specific round may be punished by lower expected score in later rounds; nevertheless, [6] shows that the expected later gains by any alternative strategy are offset by the immediate expected loss.) This optimal strategy for a single round is easy to find (as was done in [6]):
(i) If R still has all three choices available, then the optimal strategy is (obviously, by symmetry), to choose one of them randomly, with probability $1 / 3$ each. And the best strategy for N is the same. (This game was one of the examples in the original paper by von Neumann [5].) The outcome for N is $-1,0$, or +1 with probability $1 / 3$ each.
(ii) If R has only two choices available, say 1 (rock) and 2 (paper), then the game is described by the matrix in Figure 1. N should never play 1 (which in this case can lose but never win). A simple calculation shows [6] that the best strategy for R is to play 1 with probability $1 / 3$ and 2 with probability $2 / 3$; similarly N
plays 2 with probability $2 / 3$ and 3 with probability $1 / 3$. The expected gain for N is $1 / 3$.

|  | rock | paper | scissors |
| ---: | ---: | ---: | ---: |
| rock | 0 | 1 | -1 |
| paper | -1 | 0 | 1 |

Figure 1. Score matrix for N when R is restricted to \{rock, paper\}; rows show the move by $R$; columns the move by $N$.
(iii) If R has only one choice, then R has to play that, and N obviously plays the next choice $(\bmod 3)$ and is sure to win. Gain for N is 1 .

## 3. Analysis of the greedy strategy

Let $N_{t, i}$ be the number of times that R plays $i$ during rounds $1, \ldots, t$. The vector $\mathbf{N}_{t}=\left(N_{t, i}\right)_{i=1}^{3}$ then evolves as a random walk which changes character each time some $N_{t, i}$ hits $n$ and R thus cannot choose $i$ in the future. We let $T_{j}, j=1,2,3$, be the first time that R has used up $j$ of the three choices; in particular, $T_{3}:=3 n$, when the game ends.

Since R uses the greedy strategy described above, $\mathbb{N}_{t}$ evolves as follows, for $t=$ $0, \ldots, n$, starting at $\mathbf{N}_{0}=(0,0,0)$ :
I. A random walk $\mathbf{N}_{0}, \ldots, \mathbf{N}_{T_{1}}$ with increments that are independent and uniformly chosen from $\left\{\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}\right\}$, until

$$
\begin{equation*}
T_{1}:=\inf \left\{t: N_{t, i}=n \text { for some } i \in\{1,2,3\}\right\} \tag{3.1}
\end{equation*}
$$

II. A random walk $\mathbf{N}_{T_{1}}, \ldots, \mathbf{N}_{T_{2}}$ with increments chosen independently and randomly from the remaining two choices by the strategy above; for example, if $N_{t, 1}$ hits $n$ first, so $N_{T_{1}, 1}=n>N_{T_{1}, 2}, N_{T_{1}, 3}$, then the increments are chosen as $\mathbf{e}_{2}$ and $\mathbf{e}_{3}$ with probabilities $1 / 3$ and $2 / 3$. This goes on until

$$
\begin{equation*}
T_{2}:=\inf \left\{t: N_{t, i}=n \text { for at least two } i \in\{1,2,3\}\right\} \tag{3.2}
\end{equation*}
$$

III. A deterministic walk $\mathbf{N}_{T_{2}}, \ldots, \mathbf{N}_{T_{3}}$ where all increments are $\mathbf{e}_{i}$ for the only $i$ that still has $N_{t, i}<n$.
We note that the expected gain for N is (assuming optimal play) 0 for each step in phase I, $1 / 3$ for each step in phase II, and 1 for each step in phase III, so the expected score for N is $\mathbb{E}\left[\left(T_{2}-T_{1}\right) / 3+T_{3}-T_{2}\right]$; we will analyse this more carefully below and also both bound and asymptotically describe the random fluctuations. We do this by analysing the constrained random walk $\mathbf{N}_{t}$ and the stopping times $T_{1}$ and $T_{2}$ in some detail. A central role is palyed by the (somewhat arbitrary) non-random time

$$
\begin{equation*}
T_{0}:=3 n-3\left\lceil n^{2 / 3}\right\rceil . \tag{3.3}
\end{equation*}
$$

3.1. Phase I: until $T_{1}$. Let $\left(\boldsymbol{\xi}_{t}\right)_{t=1}^{\infty}$ be an i.i.d. sequence of random vectors with the distribution $\mathbb{P}\left(\boldsymbol{\xi}_{t}=\mathbf{e}_{i}\right)=1 / 3$ for $i=1,2,3$. We may assume that $\mathbf{N}_{t}-\mathbf{N}_{t-1}=\boldsymbol{\xi}_{t}$ for $1 \leqslant t \leqslant T_{1}$. Let

$$
\begin{equation*}
\mathbf{N}_{t}^{\prime}=\left(N_{t, i}^{\prime}\right)_{i=1}^{3}:=\sum_{u=1}^{t} \boldsymbol{\xi}_{u}, \quad t \geqslant 0 \tag{3.4}
\end{equation*}
$$

thus $\mathbf{N}_{t}^{\prime}=\mathbf{N}_{t}$ for $t \leqslant T_{1}$. (We may interpret $\boldsymbol{\xi}_{t}$ and $\mathbf{N}_{t}^{\prime}$ as how R would have played if the restriction had not existed.) In particular, for $t \leqslant T_{1}$ we have $N_{t, i}^{\prime}=N_{t, i} \leqslant n$
for all $i$, and for $t \geqslant T_{1}$ we have $\max _{i} N_{t, i}^{\prime} \geqslant \max _{i} N_{T_{1}, i}=n$; thus $T_{1}$ is also the time that $\max _{i} N_{t, i}^{\prime}$ hits $n$.

At time $T_{0}$, the central limit theorem shows that

$$
\begin{equation*}
N_{T_{0}, i}^{\prime}=\frac{1}{3} T_{0}+O_{\mathrm{p}}\left(n^{1 / 2}\right)=n-n^{2 / 3}+O_{\mathrm{p}}\left(n^{1 / 2}\right) . \tag{3.5}
\end{equation*}
$$

This is w.h.p. (with high probability, i.e., with probability $1-o(1)$ as $n \rightarrow \infty)<n$ for each $i$, and thus w.h.p. $T_{1}>T_{0}$. More precisely, the Chernoff inequality (e.g. in the version in [2, Remark 2.5]) yields

$$
\begin{equation*}
\mathbb{P}\left(T_{1} \leqslant T_{0}\right) \leqslant \sum_{i=1}^{3} \mathbb{P}\left(N_{T_{0}, i}^{\prime} \geqslant n\right)=3 \mathbb{P}\left(N_{T_{0}, 1}^{\prime}-\frac{1}{3} T_{0} \geqslant\left\lceil n^{2 / 3}\right\rceil\right) \leqslant e^{-2 n^{4 / 3} / T_{0}} \leqslant e^{-n^{1 / 3}} \tag{3.6}
\end{equation*}
$$

Hence, this probability decreases faster than any polynomial, which means that we can ignore the event $T_{1} \leqslant T_{0}$ also when calculating moments below (since the random variables we consider all are deterministically $O(n)$ ).

Similarly, concentrating on the time after $T_{0}$, define

$$
\begin{equation*}
M^{\prime}:=\max _{i=1,2,3} \max _{T_{0} \leqslant t \leqslant 3 n}\left|N_{t, i}^{\prime}-N_{T_{0}, i}^{\prime}-\frac{1}{3}\left(t-T_{0}\right)\right| . \tag{3.7}
\end{equation*}
$$

By classical results on moment convergence in the central limit theorem together with Doob's inequality (since $N_{t, i}^{\prime}-N_{T_{0}, i}^{\prime}-\frac{1}{3}\left(t-T_{0}\right)$ is a martingale), see for example [1, Theorem 7.5.1, Corollary 3.8.2, and Theorem 10.9.4], we have, for any $p>1$

$$
\begin{align*}
\mathbb{E}\left(M^{\prime}\right)^{p} & \leqslant \sum_{i=1}^{3} \mathbb{E} \max _{T_{0} \leqslant t \leqslant 3 n}\left|N_{t, i}^{\prime}-N_{T_{0}, i}^{\prime}-\frac{1}{3}\left(t-T_{0}\right)\right|^{p} \\
& \leqslant C_{p} \sum_{i=1}^{3} \mathbb{E}\left|N_{3 n, i}^{\prime}-N_{T_{0}, i}^{\prime}-\frac{1}{3}\left(3 n-T_{0}\right)\right|^{p} \\
& \leqslant C_{p}\left(3 n-T_{0}\right)^{p / 2} \leqslant C_{p} n^{p / 3} . \tag{3.8}
\end{align*}
$$

Consequently,

$$
\begin{equation*}
M^{\prime}=O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.9}
\end{equation*}
$$

for every $p<\infty$.
We introduce some further notation. Let

$$
\begin{equation*}
X_{i}:=N_{T_{0}, i}^{\prime}-\mathbb{E} N_{T_{0}, i}^{\prime}=N_{T_{0}, i}^{\prime}-\frac{1}{3} T_{0} . \tag{3.10}
\end{equation*}
$$

Note for later use that

$$
\begin{equation*}
X_{1}+X_{2}+X_{3}=\sum_{i=1}^{3} N_{T_{0}, i}^{\prime}-T_{0}=0 \tag{3.11}
\end{equation*}
$$

Furthermore, let

$$
\begin{equation*}
X_{\max }:=\max _{i=1,2,3} X_{i} \tag{3.12}
\end{equation*}
$$

Condition on the event $T_{1}>T_{0}$, which has probability $1-o(1)$. Then, $N_{T_{0}}^{\prime}=N_{T_{0}}$. Moreover, we may take $t=T_{1}$ in (3.7) and obtain, using (3.10),

$$
\begin{equation*}
N_{T_{1}, i}=N_{T_{1}, i}^{\prime}=N_{T_{0}, i}^{\prime}+\frac{1}{3}\left(T_{1}-T_{0}\right)+O\left(M^{\prime}\right)=X_{i}+\frac{1}{3} T_{1}+O\left(M^{\prime}\right) . \tag{3.13}
\end{equation*}
$$

Hence, recalling the definitions of $T_{1}$ and $X_{\max }$,

$$
\begin{equation*}
n=\max _{i} N_{T_{1}, i}=\max _{i} X_{i}+\frac{1}{3} T_{1}+O\left(M^{\prime}\right)=X_{\max }+\frac{1}{3} T_{1}+O\left(M^{\prime}\right) \tag{3.14}
\end{equation*}
$$

Consequently,

$$
\begin{equation*}
T_{1}=3 n-3 X_{\max }+O\left(M^{\prime}\right) \tag{3.15}
\end{equation*}
$$

and thus, using (3.9),

$$
\begin{equation*}
T_{1}=3 n-3 X_{\max }+O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.16}
\end{equation*}
$$

This was derived conditioned on $T_{1}>T_{0}$, but by (3.6) and the comment after it, (3.16) holds also unconditionally.

Furthermore, for every $i \in\{1,2,3\}$, by (3.13) and (3.9),

$$
\begin{equation*}
N_{T_{1}, i}-\frac{1}{3} T_{1}=X_{i}+O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.17}
\end{equation*}
$$

and thus by (3.16)

$$
\begin{equation*}
n-N_{T_{1}, i}=n-\frac{1}{3} T_{1}-X_{i}+O_{L^{p}}\left(n^{1 / 3}\right)=X_{\max }-X_{i}+O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.18}
\end{equation*}
$$

Thus, at time $T_{1}$, when R runs out of one of the three choices, she has approximatively $X_{\max }-X_{i}$ left of each other choice $i$.

To find the score in Phase I, consider first the score at $T_{0}$, and condition again on $T_{1}>T_{0}$. Then in each round up to $T_{0}$, R plays normally and thus R and N win with probability $1 / 3$ each, and draw otherwise; thus $\Delta S(t):=S(t)-S(t-1) \in\{ \pm 1,0\}$ with probability $1 / 3$ each. Consequently, the central limit theorem shows that, since $\mathbb{E} \Delta S(t)=0$ and $\operatorname{Var} \Delta S(t)=2 / 3$, and $T_{0} \sim 3 n$,

$$
\begin{equation*}
\frac{S\left(T_{0}\right)}{n^{1 / 2}} \xrightarrow{\mathrm{~d}} N(0,2), \quad \text { as } n \rightarrow \infty, \tag{3.19}
\end{equation*}
$$

together with all moments. Moreover, since also N is assumed to use the optimal strategy, which for these $t$ means uniformly randomly, the score in each round is independent of the choices made by R , and thus of the vectors $\mathbf{N}_{t}$. Consequently, $S\left(T_{0}\right)$ is independent of $\left(X_{1}, X_{2}, X_{3}\right)$. We conditioned here on $T_{1}>T_{0}$, but in the unlikely event $T_{1} \leqslant T_{0}$, we may modify $S\left(T_{0}\right)$ (similarly as we defined $\mathbf{N}^{\prime}$ above) and define a sum $S^{\prime}\left(T_{0}\right)$ that is independent of $\left(X_{1}, X_{2}, X_{3}\right)$ and satisfies $S^{\prime}\left(T_{0}\right)=S\left(T_{0}\right)$ whenever $T_{1}>T_{0}$, and thus, by (3.6), (rather coarsely)

$$
\begin{equation*}
S\left(T_{0}\right)=S^{\prime}\left(T_{0}\right)+O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.20}
\end{equation*}
$$

For $T_{0}<t \leqslant T_{1}$, we still have the same distribution of $\Delta S(t)$, and by the same argument as in (3.8), if we condition on $T_{1}>T_{0}$, then

$$
\begin{equation*}
S\left(T_{1}\right)-S\left(T_{0}\right)=O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.21}
\end{equation*}
$$

By (3.6), this holds also unconditionally.
3.2. Phase II: $T_{1}$ to $T_{2}$. Since the entire game is symmetric under cyclic permutations of the three choices rock, paper, scissors, we may for the next phase assume that R first uses up all $n$ rock, i.e., that $N_{T_{1}, 1}=0$. Note, however, that the game is not symmetric under odd permutations, so having made this assumption, choices 2 (paper) and 3 (scissors) play different roles, since 3 beats 2 .

By the discussion of the greedy strategy in Section 2.2 , for $t \in\left[T_{1}, T_{2}\right.$ ), R should play randomly and choose 2 or 3 with probabilities $1 / 3$ and $2 / 3$. We argue as in the preceding subsection (and therefore omit some details); we now let $\left(\boldsymbol{\eta}_{t}\right)_{1}^{\infty}$
be an i.i.d. sequence of random vectors with $\mathbb{P}\left(\boldsymbol{\eta}_{t}=\mathbf{e}_{i}\right)=p_{i}$ for $i=1,2,3$, with $\left(p_{1}, p_{2}, p_{3}\right)=\left(0, \frac{1}{3}, \frac{2}{3}\right)$, and we assume as we may that $\mathbf{N}_{t}-\mathbf{N}_{t-1}=\boldsymbol{\eta}_{t}$ for $T_{1}<t \leqslant T_{2}$. Let

$$
\begin{equation*}
\mathbf{N}_{t}^{\prime \prime}=\left(N_{t, i}^{\prime \prime}\right)_{i=1}^{3}:=\mathbf{N}_{T_{1}}+\sum_{u=T_{1}+1}^{t} \boldsymbol{\eta}_{u}, \quad t \geqslant T_{1} \tag{3.22}
\end{equation*}
$$

Then $\mathbf{N}_{t}^{\prime \prime}=\mathbf{N}_{t}$ for $T_{1} \leqslant t \leqslant T_{2}$. Let

$$
\begin{equation*}
M^{\prime \prime}:=\max _{i=1,2,3} \max _{T_{1} \leqslant t \leqslant 3 n}\left|N_{t, i}^{\prime \prime}-N_{T_{1}, i}^{\prime \prime}-p_{i}\left(t-T_{1}\right)\right| \tag{3.23}
\end{equation*}
$$

If we again condition on $T_{1}>T_{0}$, we obtain, by conditioning on $T_{1}$ and arguing as in (3.8) and using $3 n-T_{1}<3 n-T_{0}=O\left(n^{2 / 3}\right)$,

$$
\begin{equation*}
M^{\prime \prime}=O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.24}
\end{equation*}
$$

By (3.6) again, this holds also unconditionally. We obtain from (3.23) and (3.24), taking $t=T_{2}$, for every $i$,

$$
\begin{equation*}
N_{T_{2}, i}=N_{T_{2}, i}^{\prime \prime}=N_{T_{1}, i}+p_{i}\left(T_{2}-T_{1}\right)+O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.25}
\end{equation*}
$$

and thus, by (3.18),

$$
\begin{align*}
n-N_{T_{2}, i} & =n-N_{T_{1}, i}-p_{i}\left(T_{2}-T_{1}\right)+O_{L^{p}}\left(n^{1 / 3}\right) \\
& =X_{\max }-X_{i}-p_{i}\left(T_{2}-T_{1}\right)+O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.26}
\end{align*}
$$

We have assumed $N_{T_{1}, 1}=n$, and then $T_{2}$ is the first $t$ such that $N_{t, 2}=n$ or $N_{t, 3}=n$. In particular, (3.26) implies

$$
\begin{equation*}
0=\min _{i=2,3}\left(n-N_{T_{2}, i}\right)=\min _{i=2,3}\left(X_{\max }-X_{i}-p_{i}\left(T_{2}-T_{1}\right)\right)+O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.27}
\end{equation*}
$$

Consequently,

$$
\begin{equation*}
\min _{i=2,3}\left(X_{\max }-X_{i}-p_{i}\left(T_{2}-T_{1}\right)\right)=O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.28}
\end{equation*}
$$

It follows that also

$$
\begin{equation*}
\min _{i=2,3} p_{i}^{-1}\left(X_{\max }-X_{i}-p_{i}\left(T_{2}-T_{1}\right)\right)=O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.29}
\end{equation*}
$$

which can be written

$$
\begin{equation*}
\min _{i=2,3} p_{i}^{-1}\left(X_{\max }-X_{i}\right)-\left(T_{2}-T_{1}\right)=O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.30}
\end{equation*}
$$

Thus

$$
\begin{equation*}
T_{2}-T_{1}=\min _{i=2,3} \frac{X_{\max }-X_{i}}{p_{i}}+O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.31}
\end{equation*}
$$

We repeat that this holds assuming that choice 1 is the first to be used up by R .
In this phase, the gain $\Delta S(t)$ of N has expectation $1 / 3$ in each round (and its absolute value is bounded by 1 , so all moments are bounded); moreover, the gains in different rounds are i.i.d. Hence, similarly to (3.8) again, the central limit theorem with moment convergence together with Doob's inequality yields

$$
\begin{equation*}
S\left(T_{2}\right)-S\left(T_{1}\right)=\frac{1}{3}\left(T_{2}-T_{1}\right)+O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.32}
\end{equation*}
$$

3.3. Phase III: $T_{2}$ to $T_{3}$. This phase is deterministic, and not very fun to play (at least not for R ): R has only one choice, and N wins every round. The total gain for N in this phase are thus, using (3.16),

$$
\begin{equation*}
S\left(T_{3}\right)-S\left(T_{2}\right)=T_{3}-T_{2}=T_{3}-T_{1}-\left(T_{2}-T_{1}\right)=3 X_{\max }-\left(T_{2}-T_{1}\right) \tag{3.33}
\end{equation*}
$$

3.4. Collecting the gains. By (3.21), (3.32), and (3.33), the final score of N is

$$
\begin{equation*}
S_{n}=S\left(T_{3}\right)=S\left(T_{0}\right)+3 X_{\max }-\frac{2}{3}\left(T_{2}-T_{1}\right)+O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.34}
\end{equation*}
$$

where furthermore $T_{2}-T_{1}$ is given by (3.31) when choice 1 (rock) is the first to be used up by R. We develop (3.34) as follows.

Lemma 3.1. We have

$$
\begin{equation*}
S_{n}-S\left(T_{0}\right)=\max \left\{X_{1}+2 X_{2}, X_{2}+2 X_{3}, X_{3}+2 X_{1}\right\}+O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.35}
\end{equation*}
$$

Proof. We may again, by symmetry, suppose that R first uses up 1. Typically, this is the case when $X_{\max }=X_{1}$, but it is possible that $X_{1}$ is not the maximum. (Then $N_{t, 1}$ is not the largest at $t=T_{0}$, but $N_{t, 1}$ overtakes the other two components and hits $n$ first.) In any case, $N_{T_{2}, 1}=N_{T_{1}, 1}=n$, and thus (3.26) yields, recalling $p_{1}=0$,

$$
\begin{equation*}
0=n-N_{T_{2}, 1}=X_{\max }-X_{1}+O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.36}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
X_{\max }=X_{1}+O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.37}
\end{equation*}
$$

We obtain from $(3.34),(3.31)$ and $(3.37)$, recalling $p_{2}=\frac{1}{3}$ and $p_{3}=\frac{2}{3}$,

$$
\begin{align*}
S_{n}-S\left(T_{0}\right) & =3 X_{\max }-\frac{2}{3}\left(T_{2}-T_{1}\right)+O_{L^{p}}\left(n^{1 / 3}\right) \\
& =3 X_{\max }-\min \left\{2\left(X_{\max }-X_{2}\right),\left(X_{\max }-X_{3}\right)\right\}+O_{L^{p}}\left(n^{1 / 3}\right) \\
& =\max \left\{X_{\max }+2 X_{2}, 2 X_{\max }+X_{3}\right\}+O_{L^{p}}\left(n^{1 / 3}\right) \\
& =\max \left\{X_{1}+2 X_{2}, 2 X_{1}+X_{3}\right\}+O_{L^{p}}\left(n^{1 / 3}\right) . \tag{3.38}
\end{align*}
$$

Furthermore, (3.11) implies that $X_{\max } \geqslant 0$ and that, using also (3.37),

$$
\begin{align*}
2 X_{1}+X_{3} & =3 X_{1}+X_{2}+2 X_{3}=3 X_{\max }+X_{2}+2 X_{3}+O_{L^{p}}\left(n^{1 / 3}\right) \\
& \geqslant X_{2}+2 X_{3}+O_{L^{p}}\left(n^{1 / 3}\right) \tag{3.39}
\end{align*}
$$

Hence (3.38) yields (3.35) in the case when R first uses up choice 1. By symmetry (3.35) holds in general.

We may now summarize the analysis in the following limit result.
Lemma 3.2. As $n \rightarrow \infty$,

$$
\begin{equation*}
n^{-1 / 2} S_{n} \xrightarrow{\mathrm{~d}} S^{*}:=W+\max \left\{V_{1}+2 V_{2}, V_{2}+2 V_{3}, V_{3}+2 V_{1}\right\}, \tag{3.40}
\end{equation*}
$$

together with all moments, where $W, V_{1}, V_{2}, V_{3}$ are jointly normal with $W$ independent of $\left(V_{1}, V_{2}, V_{3}\right)$ and

$$
\begin{gather*}
W \in N(0,2)  \tag{3.41}\\
\left(V_{1}, V_{2}, V_{3}\right) \in N\left(0,\left(\begin{array}{rrr}
\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\
-\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}
\end{array}\right)\right) . \tag{3.42}
\end{gather*}
$$

Proof. The random vectors $\boldsymbol{\xi}_{t}$ in (3.4) are i.i.d. with $\mathbb{E} \boldsymbol{\xi}_{t}=0$ and covariance matrix (regarding $\boldsymbol{\xi}_{t}$ as a column vector)

$$
\mathbb{E} \boldsymbol{\xi}_{t}^{\operatorname{tr}} \boldsymbol{\xi}_{t}=\Sigma:=\left(\begin{array}{rrr}
\frac{2}{9} & -\frac{1}{9} & -\frac{1}{9}  \tag{3.43}\\
-\frac{1}{9} & \frac{2}{9} & -\frac{1}{9} \\
-\frac{1}{9} & -\frac{1}{9} & \frac{2}{9}
\end{array}\right)
$$

Since $T_{0} \sim 3 n$ by (3.3), the central limit theorem yields, recalling (3.10),

$$
\begin{equation*}
n^{-1 / 2}\left(X_{1}, X_{2}, X_{3}\right) \xrightarrow{\mathrm{d}}\left(V_{1}, V_{2}, V_{3}\right) \in N(0,3 \Sigma), \tag{3.44}
\end{equation*}
$$

which agrees with (3.42). Similarly, as noted in (3.19), $n^{-1 / 2} S\left(T_{0}\right) \xrightarrow{\mathrm{d}} W$. Furthermore, by (3.20) we may here replace $S\left(T_{0}\right)$ be the approximation $S^{\prime}\left(T_{0}\right)$ which, as noted above, is independent of $\left(X_{1}, X_{2}, X_{3}\right)$. Hence,

$$
\begin{equation*}
n^{-1 / 2}\left(S\left(T_{0}\right), X_{1}, X_{2}, X_{3}\right) \xrightarrow{\mathrm{d}}\left(W, V_{1}, V_{2}, V_{3}\right), \tag{3.45}
\end{equation*}
$$

and thus (3.35) and the continuous mapping theorem yield (3.40). Moreover, all moments converge in the central limit theorems (3.44) and (3.19) [1, Theorem 7.5.1], and it follows (e.g. using uniform integrability) that all moments converge also in (3.45) and (3.40).

In the following section, we give several alternative expressions for the limit $S^{*}$.

## 4. The distribution of the limit

Proof of Theorem 1.2. We have proved the convergence (1.2) in Lemma 3.2, and it remains only to show that the limit in (3.40) can be expressed as in (i)-(iv). (Note that only the distribution matters.)
(i): Define $Z_{1}:=V_{1}+2 V_{2}, Z_{2}:=V_{2}+2 V_{3}$, and $Z_{3}=V_{3}+2 V_{1}$. Then Lemma 3.2 shows that (1.2) holds, and a simple calculation shows that $\left(Z_{1}, Z_{2}, Z_{3}\right)$ has the distribution (1.5).
(iii): Define $Z_{i}^{\prime \prime}:=W+Z_{i}, i=1,2,3$. Then (1.3) yields (1.7), and (1.4)-(1.5) yield (1.8).
(ii): We may write $W=W^{\prime}+\widetilde{W}$, where $W^{\prime}, \widetilde{W} \in N(0,1)$, and $W^{\prime}$ and $\widetilde{W}$ are independent of each other and of $\left(Z_{1}, Z_{2}, Z_{3}\right)$. Define $Z_{i}^{\prime}:=\left(\widetilde{W}+Z_{i}\right) / \sqrt{3}, i=$ $1,2,3$. Then (1.3) yields (1.6), and it follows from (1.5) that the covariance matrix of $\left(Z_{1}^{\prime}, Z_{2}^{\prime}, Z_{3}^{\prime}\right)$ is the identity matrix; thus the jointly normal variables $W^{\prime}, Z_{1}^{\prime}, Z_{2}^{\prime}, Z_{3}^{\prime}$ are independent $N(0,1)$.
(iv): It follows from (1.5) that $Z_{1}+Z_{2}+Z_{3}$ has variance 0 , and thus $Z_{1}+Z_{2}+Z_{3}=$ 0 a.s., so $\left(Z_{1}, Z_{2}, Z_{3}\right)$ has really a 2 -dimensional normal distribution. In fact, if $\boldsymbol{\zeta}=\left(\zeta_{1}, \zeta_{2}\right)$ is a centered normal distribution in $\mathbb{R}^{2}$ with $\operatorname{Var} \zeta_{1}=\operatorname{Var} \zeta_{2}=2$ and $\operatorname{Cov}\left(\zeta_{1}, \zeta_{2}\right)=0$, then we can construct $\left(Z_{1}, Z_{2}, Z_{3}\right)$ with the desired distribution (1.5) by

$$
\begin{equation*}
Z_{i}:=\mathbf{f}_{i} \cdot \boldsymbol{\zeta} \tag{4.1}
\end{equation*}
$$

where $\mathbf{f}_{1}:=(1,0), \mathbf{f}_{2}:=\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \mathbf{f}_{3}:=\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$. We define $R:=|\boldsymbol{\zeta}|$ and $\theta:=\arg \left(\zeta_{1}+\mathrm{i} \zeta_{2}\right) \in[-\pi, \pi)$; thus

$$
\begin{equation*}
\boldsymbol{\zeta}=(R \cos \theta, R \sin \theta) \tag{4.2}
\end{equation*}
$$

and it follows from (4.1) by simple calculations (which are made even simpler by identifying $\mathbb{R}^{2}$ and $\mathbb{C}$ and regarding $\zeta$ as a complex random variable) that

$$
\begin{equation*}
Z_{1}=R \cos \theta, \quad Z_{2}=R \cos (\theta-2 \pi / 3), \quad Z_{3}=R \cos (\theta+2 \pi / 3) . \tag{4.3}
\end{equation*}
$$

The normal distribution of $\boldsymbol{\zeta}$ is rotationally symmetric, and thus, as is well-known, $R$ and $\theta$ are independent, with $\theta$ uniformly distributed on $[-\pi, \pi)$; furthermore, $R$ has the Rayleigh distribution stated in the theorem. To find the distribution of $Z_{\max }:=\max \left(Z_{1}, Z_{2}, Z_{3}\right)$, we may by symmetry condition on $Z_{\max }=Z_{1}$, which by (4.3) is equivalent to $\theta \in[-\pi / 3, \pi / 3]$, and since $\cos \theta$ is an even function, we may further restrict to $\theta \in[0, \pi / 3]$. Then $Z_{\max }=Z_{1}=R \cos \theta$, and thus (1.9) follows from (1.3)

Proof of Theorem 1.1. By the moment convergence in Theorem 1.2, it suffices to find $\mathbb{E} S^{*}$. For this we use (1.9). We have $\mathbb{E} W=0$, and by simple calculations

$$
\begin{align*}
\mathbb{E} R & =\int_{0}^{\infty} \frac{1}{2} r^{2} e^{-r^{2} / 4} \mathrm{~d} r=\sqrt{\pi},  \tag{4.4}\\
\mathbb{E} \cos \theta & =\frac{3}{\pi} \int_{0}^{\pi / 3} \cos \vartheta \mathrm{~d} \vartheta=\frac{3 \sqrt{3}}{2 \pi} . \tag{4.5}
\end{align*}
$$

Hence, by the independence,

$$
\begin{equation*}
\mathbb{E} S^{*}=\mathbb{E} R \cdot \mathbb{E} \cos \theta=\frac{3 \sqrt{3}}{2 \sqrt{\pi}} \doteq 1.4658075 . \tag{4.6}
\end{equation*}
$$

Remark 4.1. Alternatively, we can use (1.6) and conclude

$$
\begin{equation*}
\mathbb{E} S^{*}=\sqrt{3} \mathbb{E} \max \left\{Z_{1}^{\prime}, Z_{2}^{\prime}, Z_{3}^{\prime}\right\}, \tag{4.7}
\end{equation*}
$$

where the right-hand side contains the expectation of the maximum of three i.i.d. standard normal variables which is known to be $3 /(2 \sqrt{\pi})$ [3].

Higher moments of $S^{*}$ can be computed in the same way. For example, we have

$$
\begin{equation*}
\mathbb{E}\left(S^{*}\right)^{2}=\mathbb{E} W^{2}+\mathbb{E} R^{2} \mathbb{E} \cos ^{2} \theta=2+4\left(\frac{1}{2}+\frac{3 \sqrt{3}}{8 \pi}\right)=4+\frac{3 \sqrt{3}}{2 \pi} \doteq 4.82699 \tag{4.8}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\operatorname{Var} S^{*}=4+\frac{6 \sqrt{3}-27}{4 \pi} \doteq 2.67840 \tag{4.9}
\end{equation*}
$$

Hence, we have

$$
\begin{equation*}
\operatorname{Var} S_{n} \sim\left(4+\frac{6 \sqrt{3}-27}{4 \pi}\right) n \tag{4.10}
\end{equation*}
$$

## 5. The probability of winning

Finally, we note that we may also calculate the asymptotic probability that R wins the game, in spite of her restriction, i.e., that the final score $S_{n}<0$. (Recall that $S_{n}$ is the score for N .)
Theorem 5.1. If both players use the strategies above, then the probability that $R$ wins has as $n \rightarrow \infty$ the limit

$$
\begin{equation*}
\mathbb{P}\left(S_{n}<0\right) \rightarrow \frac{3 \arccos (1 / 4)-\pi}{4 \pi}=\frac{\arccos (11 / 16)}{4 \pi} \doteq 0.064677 . \tag{5.1}
\end{equation*}
$$

Proof. By Theorem 1.2, we have $\mathbb{P}\left(S_{n}<0\right) \rightarrow \mathbb{P}\left(S^{*}<0\right)$ (since $S^{*}$ has a continuous distribution, e.g. by (1.3)). We compute this probability using Theorem 1.2(iii). By (1.7), we have

$$
\begin{equation*}
S^{*}<0 \Longleftrightarrow Z_{i}^{\prime \prime}<0 \forall i \tag{5.2}
\end{equation*}
$$

We may, similarly to (4.1), construct $Z_{i}^{\prime \prime}$ as

$$
\begin{equation*}
Z_{i}^{\prime \prime}:=\hat{\mathbf{f}}_{i} \cdot \hat{\boldsymbol{\zeta}} \tag{5.3}
\end{equation*}
$$

where $\widehat{\boldsymbol{\zeta}}$ is a standard normal distribution in $\mathbb{R}^{3}$, and $\widehat{\mathbf{f}}_{1}, \widehat{\mathbf{f}}_{2}, \widehat{\mathbf{f}}_{3}$ are three vectors in $\mathbb{R}^{3}$ such that

$$
\widehat{\mathbf{f}}_{i} \cdot \widehat{\mathbf{f}}_{j}= \begin{cases}4, & i=j  \tag{5.4}\\ -1, & i \neq j\end{cases}
$$

By (5.3), the condition (5.2) means that $\widehat{\boldsymbol{\zeta}}$ lies in the intersection of three open halfspaces $H_{1}, H_{2}, H_{3}$, which are bounded by hyperplanes orthogonal to $\widehat{\mathbf{f}}_{1}, \widehat{\mathbf{f}}_{2}$ and $\widehat{\mathbf{f}}_{3}$. The angle between any two of these vectors is, by $(5.4), \alpha:=\arccos (-1 / 4)$. Hence, the interior angle between any of the two hyperplanes is $\beta:=\pi-\alpha=\arccos (1 / 4)$, and thus the intersection of the unit sphere and $H_{1} \cap H_{2} \cap H_{3}$ is a spherical triangle $\Delta$ with all three angles $\beta$. Consequently, the area $|\Delta|$ of $\Delta$ is $3 \beta-\pi$. The distribution of $\hat{\boldsymbol{\zeta}}$ is rotationally symmetric, and thus we may project $\hat{\boldsymbol{\zeta}}$ onto the unit sphere, and find, recalling that the area of the sphere is $4 \pi$,

$$
\begin{equation*}
\mathbb{P}\left(Z_{\max }<0\right)=\mathbb{P}\left(\widehat{\zeta} \in H_{1} \cap H_{2} \cap H_{3}\right)=\frac{|\Delta|}{4 \pi}=\frac{3 \beta-\pi}{4 \pi}=\frac{3 \arccos (1 / 4)-\pi}{4 \pi} \tag{5.5}
\end{equation*}
$$

Finally, note that

$$
\begin{equation*}
\cos (3 \beta-\pi)=-4 \cos ^{3} \beta+3 \cos \beta=-4\left(\frac{1}{4}\right)^{3}+3 \cdot \frac{1}{4}=\frac{11}{16} \tag{5.6}
\end{equation*}
$$

Theorem 5.1 assumes that the players use their optimal strategies for maximizing the expected gain; if they instead want to maximize the probability of winning (but do not care about how much they win or lose), the optimal strategies are presumably different (see Example 5.3), and most likely much more complex; hence we do not know whether (5.1) holds or not in that case.

Problem 5.2. Suppose that both players want to maxime $\mathbb{P}$ (win) $-\mathbb{P}($ lose $)$. What is (asymptotically) the probability that R wins?

It is possible that the asymptotic answer is the same as in Theorem 5.1, although the probabilities for finite $n$ presumably are different. (See Example 5.3 for $n=1$.) It might seem likely that a strategy that gives one of the players a significantly lower expected score will also give a lower probability that this score is positive. However, Example 5.4 shows that strategies with the same expectation still might give different distributions of the score and therefore different probabilities of winning, so it seems that there is no simple solution to Problem 5.2.

Example 5.3. Here is simple example showing that, at least for $n=1$, the greedy strategy is not the optimal strategy if the objective is to win, as in Problem 5.2. Let $n=1$, and suppose that both players choose scissors in the first round. This is a draw, and then R cannot win the game, since she knows that N will win the last round. In the second round R plays the game in Figure 1; if she wants to minimize
the probability of losing the entire game, the best strategy is to play rock or paper with equal probabilities, and not with the probabilities in Section 2.2 that minimize the expected loss.

Example 5.4. Suppose that R uses the greedy strategy above, but that N plays only rock until $T_{1}$, and then follows the strategy above. (This is obviously a risky strategy if $R$ would guess it, but we assume that $R$ is a mathematician and knows that the greedy strategy is proven to be optimal, and therefore sticks to it.) Since R plays randomly even when N does not, this leads to exactly the same behaviour of $\mathbf{N}(t)$ as in the analysis above, and also $S(t)$ behaves as above for $t \leqslant T_{1}$, but now $\mathbf{N}(t)$ and $S(t)$ are dependent. In fact, for $t \leqslant T_{1}$, we have $S(t)=N_{t, 3}-N_{t, 2}$. In particular, $S\left(T_{0}\right)=X_{3}-X_{2}$, and the argument in the proof of Lemma 3.2 yields

$$
\begin{equation*}
n^{-1 / 2} S_{n} \xrightarrow{\mathrm{~d}} S^{\prime}:=V_{3}-V_{2}+\max \left\{V_{1}+2 V_{2}, V_{2}+2 V_{3}, V_{3}+2 V_{1}\right\} . \tag{5.7}
\end{equation*}
$$

We have $V_{3}-V_{2}=-\left(V_{1}+2 V_{2}\right)$, and thus $S^{\prime} \geqslant 0$ with a point mass (by symmetry in the maximum in (5.7)) $\mathbb{P}\left(S^{\prime}=0\right)=1 / 3$. In this case, we cannot immediately find the limit of $\mathbb{P}\left(S_{n}<0\right)$, but if the strategy is perturbed a little, and N plays normally for the first $\varepsilon_{n} n$ rounds with $\varepsilon_{n} \rightarrow 0$ very slowly, it can be seen that $\left.\mathbb{P}\left(S_{n}<0\right) \rightarrow \frac{\mathbb{P}}{\underset{T}{P}} S^{\prime}=0\right)=1 / 6$.

In this case, the new strategy for N is worse for him; it gives the same expected score but a lower probability that the score is positive (given that R plays greedily). However, it suggests that there also might be other strategies that instead increase the probability that N wins.

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