We collect some formulas related to the Gamma integral. (None of the formulas is new.) See also e.g. [1, Chapter 6] and [2, Section 5.9], where further results are given. (Several of the formulas below appear in [2], but we do not give individual references.)

All integrals are absolutely convergent unless we explicitly say otherwise.

We begin with the standard definition (Euler’s integral)

\[
\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x} \, dx, \quad \Re \alpha > 0.
\]

I. Extensions to \(\Re \alpha < 0\). For \(\Re \alpha < 0\), the integral in (1) does not converge, but if \(\Re \alpha \notin \mathbb{Z}\) we have the modifications

\[
\int_0^\infty (e^{-x} - 1) x^{\alpha-1} \, dx = \Gamma(\alpha), \quad -1 < \Re \alpha < 0,
\]

\[
\int_0^\infty (e^{-x} - 1 + x) x^{\alpha-1} \, dx = \Gamma(\alpha), \quad -2 < \Re \alpha < -1,
\]

and, in general, for any integer \(m \geq 0\),

\[
\int_0^\infty \left( e^{-x} - \sum_{k=0}^{m} \frac{(-x)^k}{k!} \right) x^{\alpha-1} \, dx = \Gamma(\alpha), \quad -m - 1 < \Re \alpha < -m.
\]

**Proof.** Denote the integral in (4) by \(I_{\alpha,m}\). Then an integration by parts gives

\[
\alpha I_{\alpha,m} = \left[ \left( e^{-x} - \sum_{k=0}^{m} \frac{(-x)^k}{k!} \right) x^\alpha \right]_0^\infty + I_{\alpha+1,m-1} = 0 + I_{\alpha+1,m-1}.
\]

For \(m = 0\) we have \(-1 < \Re \alpha < 0\) and then

\[
I_{\alpha+1,-1} = \int_0^\infty e^{-x} x^\alpha \, dx = \Gamma(\alpha + 1) = \alpha \Gamma(\alpha);
\]

thus (4) for \(m = 0\) follows from (5). (This is (2).) The general case now follows by (5) and induction. \(\square\)

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Next we note the following extension of (2).

$$\int_0^\infty (e^{-tx} - 1)x^{\alpha-1}\,dx = t^{-\alpha}\Gamma(\alpha), \quad -1 < \Re\alpha < 0, \ \Re t \geq 0. \tag{6}$$

**Proof.** For $t > 0$, this follows from (2) by a change of variables. The integral in (6) converges for $\Re t \geq 0$ and is a continuous function of $t$ in this half-plane, analytic in the open half-plane $\Re t > 0$. Hence the result follows by analytic continuation. \qed

II. $\sin$ and $\cos$.

$$\int_0^\infty x^{\alpha-1}\sin x\,dx = \sin \frac{\pi\alpha}{2} \Gamma(\alpha), \quad -1 < \Re\alpha < 0, \tag{7}$$

$$\int_0^\infty x^{\alpha-1}(1 - \cos x)\,dx = -\cos \frac{\pi\alpha}{2} \Gamma(\alpha), \quad -2 < \Re\alpha < 0. \tag{8}$$

**Proof.** For $-1 < \Re\alpha < 0$, these follow from (6) by taking $t = \pm i$ and using Euler’s formulas. (Alternatively, for real $\alpha$, by taking $t = -i$ and taking real and imaginary parts.) Then (8) extends to $\Re\alpha > -2$ by analytic continuation. \qed

In particular, taking $\alpha = -1$ in (8) yields the well-known

$$\int_0^\infty \frac{1 - \cos x}{x^2}\,dx = \frac{\pi}{2}. \tag{9}$$

In fact, (7) extends to $0 \leq \Re\alpha < 1$, although the integral no longer is absolutely convergent:

$$\int_0^\infty x^{\alpha-1}\sin x\,dx := \lim_{A \to \infty} \int_0^A x^{\alpha-1}\sin x\,dx = \sin \frac{\pi\alpha}{2} \Gamma(\alpha), \quad -1 < \Re\alpha < 1. \tag{10}$$

**Proof.** Integration by parts yields, using (8) and letting $A \to \infty$,

$$\int_0^A x^{\alpha-1}\sin x\,dx = \left[ x^{\alpha-1}(1 - \cos x) \right]_0^A - (\alpha - 1) \int_0^A x^{\alpha-2}(1 - \cos x)\,dx$$

$$\quad \to 0 + (\alpha - 1) \cos \frac{\pi(\alpha - 1)}{2} \Gamma(\alpha - 1) = \sin \frac{\pi\alpha}{2} \Gamma(\alpha).$$

\qed

In particular, taking $\alpha = 0$ in (10) yields the conditionally convergent

$$\int_0^\infty \frac{\sin x}{x}\,dx := \lim_{A \to \infty} \int_0^A \frac{\sin x}{x}\,dx = \frac{\pi}{2}. \tag{11}$$

There is also a corresponding conditionally convergent cosine integral, related to (8):

$$\int_0^\infty x^{\alpha-1}\cos x\,dx := \lim_{A \to \infty} \int_0^A x^{\alpha-1}\cos x\,dx = \cos \frac{\pi\alpha}{2} \Gamma(\alpha), \quad 0 < \Re\alpha < 1. \tag{12}$$
Proof. Integration by parts yields, using (7) and letting $A \to \infty$, 
\[
\int_0^A x^{\alpha - 1} \cos x \, dx = [x^{\alpha - 1} \sin x]_0^A - (\alpha - 1) \int_0^A x^{\alpha - 2} \sin x \, dx \\
\to 0 - (\alpha - 1) \sin \frac{\pi (\alpha - 1)}{2} \Gamma(\alpha - 1) = \cos \frac{\pi \alpha}{2} \Gamma(\alpha). \]
\hfill \square

Another formula is:
(13) \[
\int_0^\infty \left( e^{-ax} - 1 + a \sin x \right) x^{\alpha - 1} \, dx \\
= \left( a^{-\alpha} + a \sin \frac{\pi \alpha}{2} \right) \Gamma(\alpha), \quad -2 < \Re \alpha < 0, \Re a \geq 0.
\]
Proof. If $-1 < \Re \alpha < 0$, this follows by (6) and (7). The general case follows by analytic continuation. \hfill \square

In particular, taking $\alpha = -1$ in (13) yields
(14) \[
\int_0^\infty \left( e^{-ax} - 1 + a \sin x \right) x^{-2} \, dx = a \log a, \quad \Re a \geq 0.
\]
Proof. If $f(a) := a^{-\alpha} + a \sin \frac{\pi \alpha}{2}$, then $f(-1) = 0$ and $f'(-1) = -a \log a$, and if $g(a) := 1/\Gamma(a) = a(a + 1)/\Gamma(a + 2)$, then $g(-1) = 0$ and $g'(-1) = -1$. The result follows by l'Hôpital’s rule. \hfill \square

III. Subtracting on $[0, 1]$ only.

(15) \[
\int_0^1 (e^{-x} - 1) x^{\alpha - 1} \, dx + \int_1^\infty e^{-x} x^{\alpha - 1} \, dx \\
= \int_0^\infty \left( e^{-x} - 1 \{x < 1\} \right) x^{\alpha - 1} \, dx = \Gamma(\alpha) - \alpha^{-1}, \quad -1 < \Re \alpha.
\]
Proof. For $\Re \alpha > 0$, this follows from (1). The general case $\Re \alpha > -1$ follows by analytic continuation. \hfill \square

In particular, taking $\alpha = 0$, 
(16) \[
\int_0^1 \frac{e^{-x} - 1}{x} \, dx + \int_1^\infty \frac{e^{-x}}{x} \, dx = \int_0^\infty \frac{e^{-x} - 1 \{x < 1\}}{x} \, dx = -\gamma.
\]
Proof. As $\alpha \to 0$, 
\[
\Gamma(\alpha) - \alpha^{-1} = \frac{\Gamma(\alpha + 1)}{\alpha} - 1 \to \Gamma'(1) = -\gamma. \hfill \square
\]

We have also similar results with $\sin x$ and $\cos x$ in the integral.

(17) \[
\int_0^1 x^{\alpha - 1} (\sin x - x \{x < 1\}) \, dx = \sin \frac{\pi \alpha}{2} \Gamma(\alpha) - \frac{1}{\alpha + 1}, \quad -3 < \Re \alpha < 1.
\]
Here the integral is absolutely convergent if $-3 < \Re \alpha < 0$, and otherwise conditionally convergent.

**Proof.** This follows from (7) when $-1 < \Re \alpha < 0$, and extends to $-3 < \Re \alpha < 0$ by analytic continuation (with absolutely convergent integrals). The case $-1 < \Re \alpha < 1$ follows similarly from (10). □

In particular, taking $\alpha = 0, -1$ and $-2$, cf. (11),

\begin{align*}
\int_0^\infty \frac{x^{\alpha-1} \sin x - 1 \{x < 1\}}{x^3} \, dx &= 1 - \frac{\pi}{4}, \\
\int_0^\infty \frac{x^{\alpha-1} \sin x - 1 \{x < 1\}}{x^2} \, dx &= 1 - \gamma, \\
\int_0^\infty \frac{x^{\alpha-1} \sin x - 1 \{x < 1\}}{x} \, dx &= \frac{\pi}{2} - 1,
\end{align*}

(18)\hspace{1cm} (19)\hspace{1cm} (20)

**Proof.** As $\varepsilon \to 0$,

\begin{align*}
\sin \frac{\pi \varepsilon}{2} \Gamma(\varepsilon) - \frac{1}{\varepsilon + 1} &\to \frac{\pi}{2} - 1, \\
\sin \frac{\pi(\varepsilon - 1)}{2} \Gamma(\varepsilon - 1) - \frac{1}{\varepsilon} &= \sin \frac{\pi(\varepsilon - 1)}{2} \left( (\varepsilon - 1)^{-1} \Gamma(\varepsilon + 1) - 1 \right) \\
&= \frac{\pi}{2} \left( \sin \frac{\pi(1 - \varepsilon)}{2} (1 - \varepsilon)^{-1} \Gamma(\varepsilon + 1) - 1 \right) \\
&\to \frac{d}{d\varepsilon} \left( \sin \frac{\pi(1 - \varepsilon)}{2} (1 - \varepsilon)^{-1} \Gamma(\varepsilon + 1) \right) \bigg|_{\varepsilon = 0} \\
&= -\frac{\pi}{2} \cos \frac{\pi}{2} + 1 + \Gamma'(1) = 1 - \gamma,
\end{align*}

and

\begin{align*}
\sin \frac{\pi(\varepsilon - 2)}{2} \Gamma(\varepsilon - 2) - \frac{1}{\varepsilon - 1} &= \sin \frac{\pi \varepsilon}{2} \Gamma(\varepsilon + 1) + \frac{1}{1 - \varepsilon} \to \frac{\pi}{4} + 1.
\end{align*}

\hfill □

Similarly for $\cos x$, with the integral absolutely convergent for $-2 < \Re \alpha < 0$ and conditionally convergent for $0 \leq \Re \alpha < 1$:

\begin{align*}
\int_0^\infty x^{\alpha-1} (\cos x - 1 \{x < 1\}) \, dx &= \cos \frac{\pi \alpha}{2} \Gamma(\alpha) - \frac{1}{\alpha}, \quad -2 < \Re \alpha < 1.
\end{align*}

(21)

**Proof.** This follows from (8) when $-2 < \Re \alpha < 0$. The case $0 < \Re \alpha < 1$ follows directly from (12). The general case follows by integration by parts.
and (17), which yield
\[
\int_0^\infty x^{\alpha-1} (\cos x - 1 \{x < 1\}) \, dx
\]
\[
= - \int_0^\infty (\alpha - 1) x^{\alpha-2} (\sin x - x 1 \{x < 1\} - 1 1 \{x \geq 1\}) \, dx
\]
\[
= - (\alpha - 1) \left( \sin \frac{\pi(\alpha - 1)}{2} \Gamma(\alpha - 1) - \frac{1}{\alpha} - 1 \right) + \int_1^\infty (\alpha - 1) x^{\alpha-2} \, dx
\]
\[
= \frac{\sin \frac{\pi(1 - \alpha)}{2}}{\alpha} \Gamma(\alpha) + \frac{\alpha - 1}{\alpha} - 1
\]
\[
= \cos \frac{\pi \alpha}{2} \Gamma(\alpha) - \frac{1}{\alpha}.
\]
\[\square\]

In particular, taking \(\alpha = 0\) and \(-1\), with the first integral conditionally convergent and the second absolutely convergent,
\[
(22) \quad \int_0^\infty \frac{\cos x - 1 \{x < 1\}}{x} \, dx = -\gamma,
\]
\[
(23) \quad \int_0^\infty \frac{\cos x - 1 \{x < 1\}}{x^2} \, dx = 1 - \frac{\pi}{2}.
\]

**Proof.** As \(\varepsilon \to 0\),
\[
\cos \frac{\pi \varepsilon}{2} \Gamma(\varepsilon) - \frac{1}{\varepsilon} = \frac{\cos \frac{\pi \varepsilon}{2} \Gamma(\varepsilon + 1) - 1}{\varepsilon}
\]
\[
\to \frac{d}{d\varepsilon} \left( \cos \frac{\pi \varepsilon}{2} \Gamma(\varepsilon + 1) \right) \bigg|_{\varepsilon=0}
\]
\[
= \Gamma'(1) = -\gamma
\]
and
\[
\cos \frac{\pi (\varepsilon - 1)}{2} \Gamma(\varepsilon - 1) - \frac{1}{\varepsilon - 1} = \frac{\sin \frac{\pi \varepsilon}{2} \Gamma(\varepsilon + 1)}{(\varepsilon - 1)\varepsilon} + \frac{1}{1 - \varepsilon} \to -\frac{\pi}{2} + 1.
\]
\[\square\]

**IV. Differences for different exponents.**
\[
(24) \quad \int_0^\infty (e^{-ax} - e^{-bx}) x^{\alpha-1} \, dx = (a^{-\alpha} - b^{-\alpha}) \Gamma(\alpha), \quad \text{Re} \alpha > -1, \text{Re} \alpha > 0, \text{Re} b > 0.
\]

**Proof.** If \(\text{Re} \alpha > 0\) and \(a > 0, b > 0\), this follows immediately from (1) by separating the integral into two and changing variables. The case \(\text{Re} \alpha > 0\) now follows by analytic continuation in \(a\) and \(b\), and this extends to \(\text{Re} \alpha > -1\) by analytic continuation in \(\alpha\). (Cf. also (6).) \[\square\]

In particular, taking \(\alpha = 0\) we find:
\[
(25) \quad \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \, dx = \log b - \log a = \log \frac{b}{a}, \quad \text{Re} a, \text{Re} b > 0.
\]
V. Another formula for $\gamma$.

(26) \[ \int_0^\infty \left( \frac{1}{1-e^{-x}} - \frac{1}{x} \right) e^{-x} \, dx = \int_0^\infty \left( \frac{e^{-x}}{1-e^{-x}} - \frac{e^{-x}}{x} \right) \, dx = \gamma. \]

Proof. We have, using (25),

\[ \int_0^\infty \left( \frac{1}{1-e^{-x}} - \frac{1}{x} \right) e^{-x} \, dx = \int_0^\infty \frac{e^{-x} - 1 + x}{x(1-e^{-x})} e^{-x} \, dx \]

\[ = \int_0^\infty \frac{e^{-x} - 1 + x}{x} \sum_{n=1}^\infty e^{-nx} \, dx \]

\[ = \sum_{n=1}^\infty \int_0^\infty \left( \frac{e^{-(n+1)x} - e^{-nx}}{x} + e^{-nx} \right) \, dx \]

\[ = \sum_{n=1}^\infty \left( \log n - \log(n+1) + \frac{1}{n} \right) = \lim_{N \to \infty} \left( \sum_{n=1}^N \frac{1}{n} - \log(N+1) \right) = \gamma. \]

VI. Other powers in the exponent. The change of variables $x = y^{1/\beta}$ yields immediately

(27) \[ \int_0^\infty x^{\alpha-1} e^{-x^{\beta}} \, dx = \frac{1}{\beta} \Gamma\left( \frac{\alpha}{\beta} \right), \quad \text{Re} \, \alpha > 0. \]

and, in particular,

(28) \[ \int_0^\infty e^{-x^{\beta}} \, dx = \Gamma(1 + 1/\beta), \quad \beta > 0. \]

REFERENCES
