SOME BANACH SPACE GEOMETRY

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1. Introduction

I have collected some standard facts about Banach spaces from various sources, see the references below for further results. Proofs are only given sometimes.

Notation. ε is an arbitrarily small positive number.

$L(X, Y)$ is the Banach space of bounded linear operators $T : X \to Y$.

$B(X)$ is the closed unit ball $\{x \in X : \|x\| \leqslant 1\}$.

$\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ is the one-point compactification of the discrete space $\mathbb{N}$.

All operators are bounded and linear.

A compact (topological) space always means a compact Hausdorff space. (We sometimes add "Hausdorff" explicitly for emphasis.)

2. Injective Banach spaces

Definition 2.1. Let $\lambda \geqslant 1$. A Banach space X is λ-*injective* if whenever Y and Z are Banach spaces with $Y \subseteq Z$ (isometrically), then every bounded linear operator $T : Y \to X$ can be extended to an operator $\tilde{T} : Z \to X$ with $\|\tilde{T}\| \leqslant \lambda \|T\|$.

A Banach space X is ∞-*injective* if it is λ-injective for some $\lambda < \infty$.

Example 2.2. The one-dimensional Banach space $\mathbb{R}$ or $\mathbb{C}$ is 1-injective. (This is the Hahn–Banach theorem.)

Example 2.3. ℓ^∞, and more generally $\ell^\infty(S)$ for any set S, is 1-injective. This follows by using the Hahn-Banach theorem on each coordinate.

Example 2.4. A finite-dimensional Banach space X is ∞-injective. This follows from Example 2.3, since X is isomorphic to $\ell^\infty_n := \ell^\infty([n])$, where $n = \dim X$.

Remark 2.5. When studying isometric properties of Banach spaces, the natural notion is 1-injective. On the other hand, when studying isomorphic properties, the natural notion is ∞-injective, which obviously is preserved by renorming the space. This is further shown by the simple Theorem 2.6 below, avoiding all estimates.

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More formally, using category theory terminology, "1-injective" is injective in category theoretic sense in the category BAN_1 of Banach spaces and linear contractions, while (by Theorem 2.6) "∞-injective" is injective in the category BAN_∞ of Banach spaces and bounded linear operators.

We use *injective* as an abbreviation of ∞-injective.

Warning: In several other papers, *injective* means 1-injective. (As said above, this is natural when studying isometric properties.)

Theorem 2.6. *A Banach space X is injective if whenever Y and Z are Banach spaces with $Y \subseteq Z$, then every bounded linear operator $T : Y \to X$ can be extended to an operator $\tilde{T} : Z \to X$.*

Proof. We may assume that Y is an isometric subspace of Z, i.e., $\| \|_Y = \| \|_Z$ on Y, since we otherwise may replace $\| \|_Y$ by the equivalent $\| \|_Z$.

Suppose that the extension property in the statement holds. We want to show that it is possible to choose $\tilde{T}$ such that there is a uniform norm estimate $\|\tilde{T}\| \leqslant \lambda \|T\|$ for all such Y, Z and T.

First, fix a pair Y, Z with $Y \subseteq Z$ (isometrically). The property says that the restriction map $L(Z, X) \to L(Y, X)$ is onto. Hence, the open mapping theorem shows that there exists a constant C such that every $T : Y \to X$ has an extension $\tilde{T} : Z \to X$ with $\|\tilde{T}\| \leqslant C\|T\|$. We let $C(Y, Z)$ be the infimum of the constants C for which this holds. (I do not know whether the minimum always is attained.)

Suppose that the set $\{C(Y, Z)\}$ is unbounded, where (Y, Z) ranges over all pairs of Banach spaces with $Y \subseteq Z$ (isometrically). Then there exists such pairs (Y_n, Z_n) with $C(Y_n, Z_n) \geqslant n$. Let $Y := \bigoplus_{n=1}^\infty Y_n$ and $Z := \bigoplus_{n=1}^\infty Z_n$ be the ℓ^1 sums of these spaces, and note that $Y \subseteq Z$ (isometrically).

Suppose that $n \geqslant 1$ and that $T : Y_n \to X$. We extend T to an operator $U : Y \to X$ by defining $U(y_1, \dots,) := T(y_n)$; then $\|U\| = \|T\|$. We may extend U to an operator $\tilde{U} : Z \to X$ with norm $\|\tilde{U}\| \leqslant (C(Y, Z) + \varepsilon)\|U\| = (C(X, Y) + \varepsilon)\|T\|$ and the restriction to $Z_n \subseteq Z$ yields an extension $\tilde{T} : Z_n \to X$ of T with $\|\tilde{T}\| \leqslant \|\tilde{U}\| \leqslant (C(X, Y) + \varepsilon)\|T\|$. Hence $C(Y_n, Z_n) \leqslant C(Y, Z)$ for every n, which contradicts the assumption that $C(Y_n, Z_n) \geqslant n$.

Consequently, $\{C(Y, Z)\}$ is bounded, and then X is λ-injective for any $\lambda > \sup\{C(Y, Z)\}$. ☐

The definition is equivalent to the existence of suitable projections, and to an extension property when X is a subspace of another space. This holds both for the isometric and isomorphic versions, i.e., with and without norm estimates, and we state this as a theorem containing both versions. See further e.g. [6] and [9].

Theorem 2.7. *The following properties of a Banach space X are equivalent.*

(i) *X is [λ-]injective.*
(ii) *If $X \subseteq Z$ [isometrically], where Z is another Banach spaces, then there is a projection $Z \to X$ [of norm $\leqslant \lambda$].*

(iii) *If $X \subseteq Z$ [isometrically], where Z is another Banach space, then every bounded linear operator $T : X \to Y$, where Y is another Banach space, can be extended to an operator $\tilde{T} : Z \to Y$ [with norm $\|\tilde{T}\| \leqslant \lambda \|T\|$].*

Proof. (i) $\Longrightarrow$ (ii): Extend the identity $X \to X$ to $P : Z \to X$.

(ii) $\Longrightarrow$ (i): X can be embedded isometrically into some $\ell^\infty(S)$. (E.g. by taking $S = B(X^*)$.) Let $P : \ell^\infty(S) \to X$ be the projection given by (ii). If $Y \subseteq Z$ are Banach spaces and $T : Y \to X \subseteq \ell^\infty(S)$, then T can be regarded as a map $Y \to \ell^\infty(S)$, which by Example 2.3 can be extended (isometrically) to an operator $\tilde{T} : Z \to \ell^\infty(S)$. Then $P\tilde{T} : Z \to X$ is the desired extension.

(ii) $\Longrightarrow$ (iii): If P is such a projection, T may be extended by TP.

(iii) $\Longrightarrow$ (ii): Extend the identity $X \to X$ to $P : Z \to X$. □

Corollary 2.8. *The following properties of a Banach space X are equivalent.*

(i) *X is injective.*
(ii) *Every subspace isomorphic to X of an arbitrary Banach space Z is complemented.*
(iii) *Every subspace isometric to X of an arbitrary Banach space Z is complemented.*

Proof. This is a reformulation of Theorem 2.7(i) $\Longleftrightarrow$ (ii). □

Corollary 2.9. *If X is injective, then a closed subspace of X is injective if and only if it is complemented.*

Proof. An injective subspace is always complemented by Corollary 2.8.

Conversely, let $W \subseteq X$ be a complemented subspace, where X is injective. Then there exists a projection $P : X \to W$, and we argue as in the proof of Theorem 2.7: If $Y \subseteq Z$ are Banach spaces and $T : Y \to W \subseteq X$, then T can be regarded as a map $Y \to X$, which by assumption can be extended to an operator $\tilde{T} : Z \to X$. Then $P\tilde{T} : Z \to W$ is the desired extension. □

There is a characterization of the 1-injective Banach spaces. (There is no known characterization of general injective Banach spaces, or of λ-injective Banach spaces for a given $\lambda > 1$.)

A topological Hausdorff space is *extremally disconnected* if the closure of an open set is open. See e.g. [5, Section 6.2, pp. 368–369].

Theorem 2.10 (Kelley [8], Hasumi [7])**.** *The following are equivalent, for a Banach space X.*

(i) *X is 1-injective.*
(ii) *X is isometric to $C(K)$ for some extremally disconnected compact Hausdorff space K.* □

In the real case, this can be elaborated as follows.

Theorem 2.11 (Nachbin [12], Goodner [6], Kelley [8])**.** *For a real Banach space X, the following are equivalent:*

(i) *X is 1-injective.*
(ii) *X is isometric to $C(K)$ for some extremally disconnected compact space K.*
(iii) *The closed balls in X have the binary intersection property. (I.e., any family of closed balls that intersect pairwise has non-empty intersection.)*
(iv) *X is isometric to some $C(K)$ (K compact) such that the unit ball is a complete Boolean lattice. (I.e., any family $\{f_\alpha\}$ has a l.u.b.)*
(v) *X is isometric to some $C(K)$ (K compact) such that any family $\{f_\alpha\} \subset X$ that has an upper bound has a l.u.b.* □

Example 2.12. A finite-dimensional Banach space X is 1-injective if and only if it is isometric to ℓ^∞_n for some n. (Cf. Example 2.4.)

Note that if $C(K)$ and $C(L)$ are isometric, then K and L are homeomorphic (Theorem 4.3 below). Hence:

Corollary 2.13. *If K is compact, then $C(K)$ is 1-injective if and only if K is extremally disconnected.* □

Corollary 2.14. *If X is a commutative C^*-algebra with maximal ideal space Δ, then X is 1-injective if and only if Δ is extremally disconnected.*

Proof. X is isometric to $C(\Delta)$ by the Gelfand transform. □

Corollary 2.15. *The following properties of a Banach space X are equivalent.*

(i) *X is injective.*
(ii) *X is isomorphic (or isometric) to a complemented subspace of $\ell^\infty(S)$ for some set S.*
(iii) *X is isomorphic (or isometric) to a complemented subspace of $C(K)$ for some extremally disconnected compact space K.*

Proof. (i) $\implies$ (ii): As in the proof of Theorem 2.7, X is isometric to a subspace of $\ell^\infty(S)$ for some set S, and this subspace is complemented by Corollary 2.8.

(ii) $\implies$ (iii): $\ell^\infty(S)$ is isometric to some such $C(K)$ by Example 2.3 and Theorem 2.10.

(iii) $\implies$ (i): By Theorem 2.10 and Corollary 2.9. □

Although there is no characterization of infinite-dimensional injective Banach spaces, they are known to have, or not have, several properties.

Theorem 2.16. *Let X be an infinite-dimensional injective Banach space.*

(i) *X contains a complemented subspace isomorphic to ℓ^∞.*
(ii) *X is not separable, not reflexive, not WCG (weakly compactly generated).*

Proof. (i): By Corollary 2.15(iii) and [4, Corollary VI.2.11], X contains a subspace isomorphic to ℓ^∞, and this subspace is complemented by Corollary 2.8 since ℓ^∞ is injective (Example 2.3).

(ii): Follows from (i), since ℓ^∞ does not have these properties. □

3. Injective dual spaces

For a dual space $X = W^*$, a linear operator $Y \to W^*$ is the same as a bilinear form $Y \times W \to \mathbb{R}$ or $\mathbb{C}$. This leads to the following, se [16, Corollary 2.12].

Theorem 3.1. *The following properties of a Banach space W are equivalent:*

(i) *W^* is [λ-]injective.*
(ii) *Whenever Y and Z are Banach spaces with $Y \subseteq Z$ (isometrically), every bounded bilinear form α on $Y \times W$ can be extended to a bounded bilinear form $\tilde{\alpha}$ on $Z \times W$ [with norm $\|\tilde{\alpha}\| \leqslant \lambda\|\alpha\|$].*
(iii) *Whenever Y and Z are Banach spaces with $Y \subseteq Z$ (isometrically), the projective tensor norms $\| \|_{Y\widehat{\otimes}W}$ and $\| \|_{Z\widehat{\otimes}W}$ are equivalent on $Y \otimes W$ [with $\|u\|_{Y\widehat{\otimes}W} \leqslant \lambda\|u\|_{Z\widehat{\otimes}W}$].*
(iv) *Whenever Y and Z are Banach spaces with $Y \subseteq Z$ (isometrically), the projective tensor product $Y\widehat{\otimes}W$ is a closed subspace of $Z\widehat{\otimes}W$ [with $\|u\|_{Y\widehat{\otimes}W} \leqslant \lambda\|u\|_{Z\widehat{\otimes}W}$ for $u \in Y\widehat{\otimes}W$].*

Proof. (i) $\iff$ (ii): By the comment before the theorem.

(ii) $\iff$ (iii): A bounded bilinear form on $Y \times W$ is the same as a bounded linear form on $Y \otimes W$ with the projective tensor norm $\| \|_{Y\widehat{\otimes}W}$. The equivalence now follows easily using the Hahn-Banach theorem.

(iii) $\iff$ (iv): This is immediate, since $Y\widehat{\otimes}W$ is the completion of $Y \otimes W$ in the norm $\| \|_{Y\widehat{\otimes}W}$, and similarly for $Y\widehat{\otimes}Z$. □

Corollary 3.2. *W^* is 1-injective if and only if $Y\widehat{\otimes}W \subseteq Z\widehat{\otimes}W$ isometrically whenever $Y \subseteq Z$.* □

Example 3.3. Let $W = L^1(S, \mathcal{F}, \mu)$ for a measure space $(S, \mathcal{F}, \mu)$. Then $Y\widehat{\otimes}W = L^1(S, \mathcal{F}, \mu; Y)$, the space of Bochner integrable Y-valued functions on $(S, \mathcal{F}, \mu)$, se [16, Section 2.3]. Thus, if $Y \subseteq Z$ isometrically, then

$$Y\widehat{\otimes}W = L^1(S, \mathcal{F}, \mu; Y) \subseteq L^1(S, \mathcal{F}, \mu; Z) = Z\widehat{\otimes}W$$

isometrically, so Theorem 3.1(iv) is satisfied with $\lambda = 1$. Thus Theorem 3.1 (or Corollary 3.2) shows that $W^* = L^1(S, \mathcal{F}, \mu)^*$ is 1-injective.

In particular, if μ is σ-finite, we obtain the following result [16, p. 30].

Theorem 3.4. *If μ is a σ-finite measure on some measurable space, then $L^\infty(\mu) = L^1(\mu)^*$ is 1-injective.*

Proof. $L^\infty(\mu) = L^1(\mu)^*$, which is 1-injective by Example 3.3. □

Corollary 3.5. *If μ is a σ-finite measure on some measurable space, then the maximal ideal space of $L^\infty(\mu)$ is extremally disconnected.*

Proof. By Theorem 3.4 and Corollary 2.14. □

Example 3.6. Theorem 3.4 and Corollary 3.5 do not extend to $L^\infty(\mu)$ for all (non-σ-finite) μ, as shown by the following example.

Let $X = L^\infty([0,1], \mathcal{B}, \mu)$, where $\mathcal{B}$ is the Borel σ-field and μ is the counting measure. Thus there are no null sets (except $\emptyset$), so X consists of all bounded Borel measurable functions on $[0,1]$ with $\|f\| = \sup_x |f(x)|$.

The Gelfand transform $f \mapsto \hat{f}$ is an algebra isomorfism $X \to C(K)$ for some compact Hausdorff space K. The idempotents in X are $\mathbf{1}_E$, $E \in \mathcal{B}$, and the idempotents in $C(K)$ are $\mathbf{1}_F$, $F \subseteq K$ is open and closed. Hence the Gelfand transform gives a bijection $E \mapsto \widehat{E}$ of $\mathcal{B}$ onto the collection of open and closed subsets of K. (Thus, $\widehat{\mathbf{1}_E} = \mathbf{1}_{\widehat{E}}$.)

Note that

$$E \subseteq F \iff \mathbf{1}_E \mathbf{1}_F = \mathbf{1}_E \iff \mathbf{1}_{\widehat{E}} \mathbf{1}_{\widehat{F}} = \mathbf{1}_{\widehat{E}} \iff \widehat{E} \subseteq \widehat{F}, \tag{3.1}$$

and similarly

$$E \cap F = \emptyset \iff \mathbf{1}_E \mathbf{1}_F = 0 \iff \mathbf{1}_{\widehat{E}} \mathbf{1}_{\widehat{F}} = 0 \iff \widehat{E} \cap \widehat{F} = \emptyset. \tag{3.2}$$

For $x \in [0,1]$, let

$$U_x := \widehat{\{x\}} \subseteq K. \tag{3.3}$$

Note that if $x \neq y$, then U_x and U_y are disjoint by (3.2),

Furthermore, for any set $A \subseteq [0,1]$, let

$$U_A := \bigcup_{x \in A} U_x. \tag{3.4}$$

Thus, U_A is an open subset of K.

Suppose that $\overline{U_A}$ is open; then $\overline{U_A}$ is open and closed so $\overline{U_A} = \widehat{E}$ for some Borel set $E \subseteq [0,1]$. If $x \in A$, then $\widehat{\{x\}} = U_{\{x\}} \subseteq U_A \subseteq \overline{U_A} = \widehat{E}$ by (3.1), and thus $\{x\} \subseteq E$ by (3.1) again, i.e., $x \in E$. On the other hand, if $x \notin A$, then U_x is an open set disjoint from U_A, and thus $\widehat{\{x\}} \cap \widehat{E} = U_x \cap \overline{U_A} = \emptyset$; hence (3.2) yields $\{x\} \cap E = \emptyset$, i.e., $x \notin E$. Consequently, $E = A$, which means $A = E \in \mathcal{B}$.

In other words, if $A \notin \mathcal{B}$, then U_A is an open subset of K but $\overline{U_A}$ is not open. Hence K is not extremally disconnected and thus, by Theorem 2.10, $C(K)$ is not 1-injective. Consequently, $L^\infty([0,1], \mathcal{B}, \mu)$ is not 1-injective.

Example 3.3 can be extended somewhat as follows.

Definition 3.7. A Banach space X is an $\mathcal{L}_{1,\lambda}$-*space* if every finite-dimensional subspace M of X is contained in a finite-dimensional subspace N such that the Banach-Mazur distance between N and ℓ_1^n (where $n = \dim N$) is at most λ, i.e., there exists an isomorphism $T : N \to \ell_1^n$ such that $\|T\| \, \|T^{-1}\| \leqslant \lambda$.

A Banach space is an $\mathcal{L}_{1,\lambda+}$-*space* if it is an $\mathcal{L}_{1,\lambda+\varepsilon}$-space for every $\varepsilon > 0$.

A Banach space is an $\mathcal{L}_1$-*space* if it is an $\mathcal{L}_{1,\lambda}$-space for some $\lambda < \infty$,

It is easily seen that if W is an $\mathcal{L}_{1,\lambda}$-space, and $Y \subseteq Z$ (isometrically), then $\|u\|_{Y\widehat{\otimes}W} \leqslant \lambda\|u\|_{Z\widehat{\otimes}W}$ for every $u \in Y \otimes W$, see [16, Section 2.4]. (The idea is that that the tensor norm $\|u\|_{Z\widehat{\otimes}W}$ can be approximated by the norm in $Z\widehat{\otimes}M$ for a suitable finite-dimensional subspace $M \subseteq W$.) Consequently, Theorem 3.1(iii) $\Longrightarrow$ (i) yields the following.

Theorem 3.8. *If W is an $\mathcal{L}_{1,\lambda}$-space, or more generally an $\mathcal{L}_{1,\lambda+}$-space, then W^* is λ-injective.* □

Corollary 3.9. (i) *If W is an $\mathcal{L}_1$-space, then W^* is ∞-injective.*
(ii) *If W is an $\mathcal{L}_{1,1+}$-space, then W^* is 1-injective.* □

Example 3.10. If K is a compact Hausdorff space, then $C(K)^* = M_{\mathsf{r}}(K)$, the space of regular real-valued (or complex-valued) Borel measures on K, is an $\mathcal{L}_{1,1+}$-space [16, p. 32]. (This is easily verified, since every finite-dimensional subspace of $M_{\mathsf{r}}(K) \subseteq M(K)$ can be seen isometrically as a subspace of $L^1(K,\nu)$ for some finite Borel measure ν by the Radon–Nikodým theorem, and $L^1(K,\nu)$ is an $\mathcal{L}_{1,1+}$-space, see [16, Proposition 2.21].)

Consequently, the bidual $C(K)^{**}$ is 1-injective for every compact K.

4. More on $C(K)$

We give a few simple results on the Banach spaces $C(K)$, where K is a compact Hausdorff space. See e.g. [15] for further results.

Theorem 4.1. *$C(K)$ is separable if and only if K is metrizable.* □

See [2, Theorem V.6.6] for a proof.

Theorem 4.2. *The dual space $C(K)^*$ is separable if and only if K is countable.*

Proof. If K is countable, then every subset of K is a Borel set and $C(K)^* = M(K) = \ell^1(K)$ which is separable.

On the other hand, for any K, the point evaluations δ_x, $x \in K$, form a discrete subset of $C(K)^*$, so if K is uncountable, then $C(K)^*$ is not separable. □

As a corollary to these two theorems we see that every countable compact space is metrizable, since X is separable whenever X^* is.

We have the following isomorphism theorems. Note the difference between isometries and isomorphisms.

Theorem 4.3 (Banach–Stone)**.** *If K_1 and K_2 are compact Hausdorff spaces such that $C(K_1)$ and $C(K_2)$ are isometric Banach spaces, then K_1 and K_2 are homeomorphic.* □

Theorem 4.4 (Miljutin)**.** *All spaces $C(K)$ where K is an uncountable compact metric space are isomorphic as Banach spaces. (Equivalently, they are all isomorphic to $C[0,1]$.)* □

For the proof of Theorem 4.4 see Miljutin [11], Pełczyński [13] or Rosenthal [15].

The spaces $C(K)$ with K countable are not isomorphic to $C[0,1]$ by Theorem 4.2. These spaces are classified up to isomorphisms by Bessaga and Pełczyński [1], see alse [13] and [15].

Remark 4.5. Every countable compact set is homeomorphic to $[0,\gamma]$ for some countable ordinal γ, i.e., the space of all ordinals $\leqslant \gamma$ with the order topology. (This space is always compact, see [5, Example 3.1.27 and Problem 3.12.3].) In fact, every such space is homeomorphic to $[0,\omega^\alpha n]$ for some countable ordinal α and integer $n \geqslant 1$, and α and n are uniquely determined (by the fact that the α:th derived set is finite and non-empty with exactly n points), see [10].

Consequently, if K is compact and countable, then $C(K)$ is isometric to $C[0,\omega^\alpha n]$ for some (unique) $\alpha < \aleph_1$ and $n \geqslant 1$.

Example 4.6. c_0 is isomorphic to $c = C(\overline{\mathbb{N}})$. By Theorem 4.2, c_0 is not isomorphic to $C[0,1]$.

Remark 4.7. The corresponding non-separable spaces ℓ^∞ and $L^\infty[0,1]$ *are* isomorphic. (Their preduals ℓ^1 and $L^1[0,1]$ are not. This can be seen because $L^1[0,1]$ contains a subspace isomorphic to ℓ^2, for example by Khinchine's inequalities the closed subspace spanned by the Rademacher functions, see e.g. [3, p. 105]; thus $L^1[0,1]$ does not have the Schur property that ℓ^1 has [3, p. 85]. Conversely, ℓ^1 does not contain any subspace isomorphic to ℓ^2, because ℓ^1 has the Schur property.

Remark 4.8. It is easily seen that if K is any infinite compact metric space, then $C(K)$ contains a subspace isometric to c, and thus a subspace isometric to c_0. (Take a convergent sequence $x_n \to x_\infty$ in K with $d(x_n,x_\infty) \searrow 0$, and find continuous functions $f_n : K \to [0,1]$ with disjoint supports and $f_n(x_m) = \delta_{nm}$.)

5. Embeddings into $C(K)$.

Theorem 5.1 (Banach–Mazur)**.** *Every separable Banach space is isometric to a closed subspace of* $C[0,1]$.

Sketch of proof. If X is separable, then the dual unit ball $K{=}B(X^*)$ is a compact metric space, and X can be regarded as a subspace of $C(K)$.

There exists a surjective continuous map of the Cantor set $D = \{0,1\}^\infty$ onto any compact metric space, and thus onto K, which gives an embedding of $C(K)$ as a subspace of $C(D)$. Finally, the Cantor set D has a ttraditional embedding as a subset of $[0,1]$, and $C(D)$ may be embedded into $C[0,1]$ by extending each function linearly across each interval in the complement of the Cantor set. □

Here $[0,1]$ can be replaced by various other spaces, for example (as seen in the proof) the Cantor set, as well as any other compact space that maps continuously onto $[0,1]$.

Remark 5.2. We cannot replace $[0,1]$ by any countable compact space in Theorem 5.1; in particular, we cannot replace $C[0,1]$ by $c_0 \cong c = C(\overline{\mathbb{N}})$. In fact, if $X \subseteq C(K)$ with K countable, then X^* is a quotient space of $C(K)^*$, and thus X^* has to be separable by Theorem 4.2; hence $C[0,1]$ cannot be embedded in $C(K)$ (not even isomorphically).

6. Separable injectivity

We note the following interesting example in the subcategory of separable Banach spaces; see e.g. [3, Theorem VII.4] or [9, Theorem 3.11.12] for a proof.

We say that a Banach space X is *separably λ-injective* if Definition 2.1 is satisfied for all *separable* Banach spaces Y and Z with $Y \subseteq Z$.

Theorem 6.1 (Sobczyk)**.** *c_0 is separably 2-injective.* □

It is easily seen that Theorem 2.7 holds also if we only consider separable spaces; thus Theorem 6.1 is equivalent to:

Theorem 6.2. *If c_0 is a subspace (isometrically) of a separable Banach space Z, then there is a projection $Z \to c_0$ of norm at most 2.*

Remark 6.3. c_0 is *not* separably 1-injective; the constant 2 is best possible in Theorems 6.1–6.2. In fact, if we take $Z = c$ in Theorem 6.2, then every projection $c \to c_0$ has norm at least 2, as shown by Taylor [17]. (Note that in this case, the subspace c_0 has codimension 1.)

Remark 6.4. There is no bounded projection $\ell^\infty \to c_0$. Thus c_0 is *not* injective in the category of all Banach spaces. (Phillips [14]; see also Phillips's lemma [3, p. 83]. In fact, every bounded operator $\ell^\infty \to c_0$ is weakly compact [3, Exercise VII.4].) This follows also by Theorem 2.16.

Remark 6.5. If K is an infinite compact metric space (so $C(K)$ is separable), then $C(K)$ is separably injective if and only if $C(K)$ is isomorphic to c_0, which holds if and only if K is homeomorphic to $[0, \omega^m n]$ for some integers $m, n \geqslant 1$, cf. Remark 4.5. (Equivalently, the derived set $K^{(\omega)} = \emptyset$.)

In particular, $C[0,1]$ is not separably injective. A concrete witness is the embedding of $C[0,1]$ into $C(D)$, where $D := \{0,1\}^\infty$ is the Cantor cube, induced by the surjection $\varphi : D \to [0,1]$ given by $\varphi((x_i)_1^\infty) = \sum_{i=1}^\infty x_i 2^{-i}$; this embeds $C[0,1]$ as an uncomplemented subspace of $C(D)$.

For proofs, see [15, Section 3C].

Appendix A. Projective compact spaces

Theorem 2.10 is in some sense dual to the following result by Gleason for compact topological spaces. See [9, §7] for a proof.

A compact Hausdorff space K is *projective* (in the category of compact Hausdorff spaces) if whenever S and T are compact Hausdorff spaces, $f : S \to T$ is an onto continuous map, and $g : K \to T$ is any continuous map, then g can be lifted to a map $G : K \to S$, i.e., a map G such that $g = fG$.

Theorem A.1. *A compact Hausdorff space is projective if and only if it is extremally disconnected.* □

Furtermore, standard (category theoretical) arguments yield the following, see [9, §7]:

We say that a compact Hausdorff space K is *free* if it is (homeomorphic to) the Stone–Čech compactification βS of a discrete space S. Equivalently, there exists a subset $S \subset K$ (necessarily the set of all isolated points), such that any map from S into a compact Hausdorff space has a unique continuous extension to K.

Theorem A.2. *A compact Hausdorff space K is projective if and only if it is a retract of a free compact Hausdorff space. (I.e., there exists a free compact space L such that $K \subseteq L$ and there exists a continuous map $r : L \to K$ that is the identity on K.)* □

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