RIEMANNIAN GEOMETRY: SOME EXAMPLES, INCLUDING MAP PROJECTIONS

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Abstract. Some standard formulas are collected on curvature in Riemannian geometry, using coordinates. (There are, as far as I know, no new results). Many examples are given, in particular for manifolds with constant curvature, including many well-known map projections of the sphere and several well-known representations of hyperbolic space, but also some lesser known.

1. Introduction

We collect general formulas on curvature in Riemannian geometry and give some examples, with emphasis on manifolds with constant curvature, in particular some standard map projections of the sphere (Section 6) and some standard representations of hyperbolic space (Section 7).

We work with coordinates, and therefore represent tensors as indexed quantities $A^{i_1,\dots,i_p}_{j_1,\dots,j_q}$ with arbitrary numbers $p \geqslant 0$ and $q \geqslant 0$ of indices (*contravariant* and *covariant*, respectively); we say that this tensor is of type (p,q), or a (p,q)-*tensor*. Some formulas and calculations are much more elegant in coordinate-free formulations, see e.g. [4], but our emphasis is on explicit calculations.

We assume knowledge of basic tensor calculus, including covariant and contravariant tensors, contractions and raising and lowering of indices.

1.1. **Basic notation.** We consider an n-dimensional Riemannian manifold M and a coordinate system $(x_1,\dots,x_n)$, i.e. a diffeomorphic map of an open subset of M onto an open subset of $\mathbb{R}^n$; we assume $n \geqslant 2$. All indices thus take the values $1,\dots,n$. We use the Einstein summation convention: $a^i{}_i$ means $\sum_{i=1}^n a^i{}_i$.

The metric tensor is g_{ij}, and the inverse matrix is denoted g^{ij}:

$$\big(g^{ij}\big) = \big(g_{ij}\big)^{-1} \tag{1.1}$$

or in the usual notation with coordinates

$$g^{ik} g_{kj} = \delta^i_j, \tag{1.2}$$

where δ^i_j denotes the Kronecker delta. (We may also write δ_{ij} when more convenient, which often is the case when considering an example with a given

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coordinate system.) Recall that the metric tensor is symmetric: $g_{ij} = g_{ji}$ and note that contracting the metric tensor or δ yields the dimension:

$$g^{ij}g_{ij} = \delta^i_i = n. \tag{1.3}$$

We denote the determinant of a 2-tensor $A = (A_{ij})$ by $|A|$. In particular,

$$|g| := \det(g_{ij}). \tag{1.4}$$

It is easily seen, using (1.2) and expansions of the determinant, that

$$\frac{\partial |g|}{\partial g_{ij}} = |g|g^{ij}. \tag{1.5}$$

The invariant measure μ on M is given by

$$\mathrm{d}\mu = |g|^{1/2}\,\mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{1.6}$$

(This measure can be defined in an invariant way as the n-dimensional Hausdorff measure on M.)

The volume scale is thus $|g|^{-1/2}$.

1.2. **Distortion.** The *condition number* $\varkappa \geqslant 1$ of the metric tensor g_{ij} (at a given point) is the ratio between the largest and smallest eigenvalue of g_{ij}. Thus $\varkappa = 1$ if and only if g_{ij} is conformal. It is seen, by taking an ON basis diagonalizing g_{ij}, that the map maps infinitesimally small spheres to ellipsoids with the ratio of largest and smallest axes $\varkappa^{1/2}$, and conversely, infinitesimally small spheres on the map correspond to ellipsoids with this ratio of largest and smallest axes. We define

$$\varepsilon := \sqrt{1 - 1/\varkappa}. \tag{1.7}$$

Thus $0 \leqslant \varepsilon < 1$, and $\varepsilon = 0$ if and only if g_{ij} is conformal. If $n = 2$, then ε is the eccentricity of the ellipses that correspond to (infinitesimally) small circles by either the coordinate map or its inverse.

If $n = 2$, then the two eigenvalues of the matrix g_{ij} are, with $\tau := \mathrm{Tr}(g_{ij}) = g_{11} + g_{22}$,

$$\frac{\tau \pm \sqrt{\tau^2 - 4|g|}}{2} = \frac{g_{11} + g_{22} \pm \sqrt{(g_{11} + g_{22})^2 - 4|g|}}{2} \tag{1.8}$$

and thus

$$\varkappa = \frac{g_{11} + g_{22} + \sqrt{(g_{11} + g_{22})^2 - 4|g|}}{g_{11} + g_{22} - \sqrt{(g_{11} + g_{22})^2 - 4|g|}} = \frac{g_{11} + g_{22} + \sqrt{(g_{11} - g_{22})^2 + 4g_{12}^2}}{g_{11} + g_{22} - \sqrt{(g_{11} - g_{22})^2 + 4g_{12}^2}} \tag{1.9}$$

and, by (1.7),

$$\varepsilon^2 = \frac{2\sqrt{(g_{11} + g_{22})^2 - 4|g|}}{g_{11} + g_{22} + \sqrt{(g_{11} + g_{22})^2 - 4|g|}} = \frac{2\sqrt{(g_{11} - g_{22})^2 + 4g_{12}^2}}{g_{11} + g_{22} + \sqrt{(g_{11} - g_{22})^2 + 4g_{12}^2}}. \tag{1.10}$$

1.3. **Derivatives.** We denote partial derivatives simply by indices after a comma:

$$A^{pq\dots}_{kl\dots,i} := \frac{\partial}{\partial x_i} A^{pq\dots}_{kl\dots} \tag{1.11}$$

and similarly for higher derivatives.

The Levi-Civita connection has the components

$$\Gamma^m_{ij} = g^{mk}\Gamma_{kij} \tag{1.12}$$

where

$$\Gamma_{kij} = \tfrac{1}{2}\big(g_{ki,j} + g_{kj,i} - g_{ij,k}\big). \tag{1.13}$$

(Γ_{kij} is often denote $[ij,k]$. Γ_{kij} and Γ^k_{ij} are also called the Christoffel symbols of the first and second kind.) Note that Γ^k_{ij} is symmetric in i and j:

$$\Gamma^k_{ij} = \Gamma^k_{ji}. \tag{1.14}$$

(This says that the Levi-Civita connection that is used in a Riemannian manifold is torsion-free.)

The covariant derivative of a tensor field is denoted by indices after a semicolon. Recall that for a function (scalar) f, the covariant derivative equals the usual partial derivative in (1.11):

$$f_{;i} = f_{,i}. \tag{1.15}$$

For a contravariant vector field A^k we have

$$A^k_{;i} = A^k_{,i} + \Gamma^k_{ji}A^j \tag{1.16}$$

and for a covariant vector field A_k we have

$$A_{k;i} = A_{k,i} - \Gamma^j_{ki}A_j \tag{1.17}$$

and for tensor fields of higher order we similarly add to the partial derivative one term as in (1.16) for each contravariant index and subtract one term as in (1.17) for each covariant index, for example

$$A^i_{jkl;m} = A^i_{jkl,m} + \Gamma^i_{pm}A^p_{jkl} - \Gamma^p_{jm}A^i_{pkl} - \Gamma^p_{km}A^i_{jpl} - \Gamma^p_{lm}A^i_{jkp}. \tag{1.18}$$

Recall that the Levi-Civita connection has the property that the covariant derivative of the metric tensor vanishes:

$$g_{ij;k} = 0; \tag{1.19}$$

indeed, this together with (1.14) is easily seen to be equivalent to (1.13).

Higher covariant derivatives are defined by iteration, for example

$$f_{;ij} := (f_{;i})_{;j} = f_{,ij} - \Gamma^k_{ij}f_{,k}. \tag{1.20}$$

Recall that the covariant derivative of a tensor field is a tensor field, i.e., invariant in the right way under changes of coordinates, while the partial derivative is not (except for a scalar field, where the two derivatives are the same, see (1.15)).

A simple calculation, using integration by parts in coordinate neighbourhoods together with (1.6), (1.5), (1.16) and (1.19), shows that if f is a smooth function and A^i is a vector field, and at least one of them has compact support, then

$$\int_M f_{,i} A^i \,\mathrm{d}\mu = -\int_M f A^i_{;i} \,\mathrm{d}\mu. \tag{1.21}$$

If A_{ij} and B_{ij} are symmetric covariant 2-tensors, we define their *Kulkarni–Nomizu* product to be the covariant 4-tensor

$$A_{ij} \odot B_{kl} := A_{ik}B_{jl} + A_{jl}B_{ik} - A_{il}B_{jk} - A_{jk}B_{il}. \tag{1.22}$$

We may also use the notation $(A \odot B)_{ijkl}$. (The notation $A_{ij} \odot B_{kl}$ is slightly improper, but will be convenient. The notation $(A \odot B)_{ijkl}$ is formally better.) Note that $A \odot B = B \odot A$.

Remark 1.1. The Kulkarni–Nomizu product in (1.22) is the bilinear extension of the map

$$(A_iC_j, B_kD_l) \mapsto A_iC_j \odot B_kD_l = (A_i \wedge B_j)(C_k \wedge D_l) \tag{1.23}$$

where $A_i \wedge B_j := A_iB_j - A_jB_i$ is the exterior product.

1.4. **The Laplace–Beltrami operator.** The *Laplace–Beltrami operator* Δ is a second-order differential operator defined on (smooth) functions on M as the contraction of the second covariant derivative:

$$\Delta F := g^{ij} F_{;ij} = g^{ij} F_{,ij} - g^{ij}\Gamma^k_{ij} F_{,k}. \tag{1.24}$$

A function F is *harmonic* if $\Delta F = 0$.

For an extension to differential forms, see e.g. [19, Chapter 6].

The integration by parts formula (1.21) implies that if F and G are smooth functions and at least one of them has compact support, then

$$\int_M F\Delta G \,\mathrm{d}\mu = -\int_M g^{ij} F_{,i} G_{,j} \,\mathrm{d}\mu = \int_M G\Delta F \,\mathrm{d}\mu. \tag{1.25}$$

In particular, if M is compact, then this holds for all smooth F and G and shows that $-\Delta$ is a non-negative operator. (For this reason, the Laplace–Beltrami operator is sometimes defined with the opposite sign.) See further [19].

1.5. **Curves.** Let $\gamma : (a, b) \to M$ be a smooth curve, i.e., a smooth map of an open interval into the Riemannian manifold M. Let D_t denote covariant differentiation along γ. For a tensor field $A^{pq\cdots}_{kl\dots}$ it can be defined by

$$D_t A^{pq\cdots}_{kl\dots}(\gamma(t)) = A^{pq\cdots}_{kl\dots;i}(\gamma(t))\dot\gamma^i(t); \tag{1.26}$$

for example, for a tensor of type (1,1), cf. (1.16)–(1.18),

$$\begin{aligned} D_t A^p_k(\gamma(t)) &= \bigl(A^p_{k,i} + \Gamma^p_{ji} A^j_k - \Gamma^j_{ki} A^p_j\bigr)\dot\gamma^i(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} A^p_k(\gamma(t)) + \Gamma^p_{ji} A^j_k \dot\gamma^i - \Gamma^j_{ki} A^p_j \dot\gamma^i. \end{aligned} \tag{1.27}$$

However, this depends only on the values of the tensor field along the curve, and thus it can be defined for any tensor $A^{pq\cdots}_{kl\dots}(t)$ defined along the curve.

For each $t \in (a,b)$, $D_t\gamma(t) = \dot\gamma(t)$ is the tangent vector at $\gamma(t)$ with components $\dot\gamma^i(t) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma^i(t)$; this is called the *velocity vector*, and the *velocity* is its length

$$\|\dot\gamma\| := \langle\dot\gamma,\dot\gamma\rangle^{1/2} = \bigl(g_{ij}\dot\gamma^i\dot\gamma^j\bigr)^{1/2}. \tag{1.28}$$

The *acceleration* is the vector $D_t^2\gamma(t)$ which has components, cf. (1.26)–(1.27),

$$D_t^2\gamma^i = D_t\dot\gamma^i = \ddot\gamma^i + \Gamma^i_{jk}\dot\gamma^j\dot\gamma^k, \tag{1.29}$$

where $\Gamma^i_{jk} = \Gamma^i_{jk}(\gamma)$.

Since the covariate derivative $Dg = 0$ for the Levi-Civita connection, see (1.19),

$$D_t\bigl(g_{ij}\dot\gamma^i\dot\gamma^j\bigr) = g_{ij}D_t\dot\gamma^i\dot\gamma^j + g_{ij}\dot\gamma^iD_t\dot\gamma^j = 2\langle D_t\dot\gamma,\dot\gamma\rangle = 2\langle D_t^2\gamma,\dot\gamma\rangle. \tag{1.30}$$

Consequently, γ has constant velocity, i.e., $\langle\dot\gamma,\dot\gamma\rangle = g_{ij}\dot\gamma^i\dot\gamma^j$ is constant, if and only if the acceleration is orthogonal to the velocity vector.

Let, assuming $\dot\gamma \neq 0$,

$$P_ND_t^2\gamma := D_t^2\gamma - \frac{\langle D_t^2\gamma,\dot\gamma\rangle}{\langle\dot\gamma,\dot\gamma\rangle}\dot\gamma \tag{1.31}$$

be the normal component of the acceleration, i.e., the component orthogonal to the velocity. The *(geodesic) curvature* κ of the curve is

$$\kappa = \kappa(t) = \|P_ND_t^2\gamma\|/\|D_t\gamma\|^2; \tag{1.32}$$

this is invariant under reparametrization of γ (including time-reversal). We thus may assume that γ is parametrized by arc length, i.e., $\|D_t\gamma\| = 1$. In this case, the acceleration and velocity vectors are orthogonal as said above, so $P_ND_t^2\gamma = D_t^2\gamma$ and thus

$$\kappa = \|D_t^2\gamma\| \qquad \text{if } \|\dot\gamma\| = 1. \tag{1.33}$$

We usually consider the curvature as the non-negative number κ, but we can also give it a direction and consider the *curvature vector*, which is the tangent vector

$$\vec\kappa := P_ND_t^2\gamma/\|D_t\gamma\|^2; \tag{1.34}$$

this too is invariant under reparametrization (including time-reversal). Note that

$$\|\vec\kappa\| = \kappa \tag{1.35}$$

and that (by definition) $\vec\kappa$ is orthogonal to the velocity vector $\dot\gamma$.

Remark 1.2. In an oriented 2-dimensional manifold, we can alternatively consider the curvature κ with sign, with $\kappa > 0$ if $\vec\kappa$ is to the left of $\dot\gamma$. (Note that the sign changes under time-reversal.) See further Appendix C for curves in the complex plane.

1.6. **Geodesics.** A *geodesic* is a curve $\gamma(t)$ with acceleration $D_t^2\gamma = 0$, i.e., by (1.29), a curve $\gamma(t)$ such that

$$\ddot{\gamma}^i + \Gamma^i_{jk}(\gamma)\dot{\gamma}^j\dot{\gamma}^k = 0. \tag{1.36}$$

By (1.30), a geodesic has constant velocity.

If the connection has the special form

$$\Gamma^k_{ij} = \delta^k_i a_j + \delta^k_j a_i \tag{1.37}$$

for some functions a_i, then (1.36) becomes

$$\ddot{\gamma}^i = -2a_j(\gamma)\dot{\gamma}^j\dot{\gamma}^i \tag{1.38}$$

which shows that $\ddot{\gamma}$ is parallel to $\dot{\gamma}$; consequently each geodesic is a straight line in the coordinates x^i. The converse holds too, so straight lines are geodesics if and only if the connection is of the form (1.37), see Theorem D.1.

2. The curvature tensors

2.1. **The Riemann curvature tensor.** Covariant differentiation is not commutative. For a contravariant vector field A^i we have

$$A^i_{;jk} - A^i_{;kj} = R^i{}_{lkj}A^l, \tag{2.1}$$

where the tensor $R^i{}_{lkj}$, the *Riemann curvature tensor*, is given by

$$R^i{}_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{ks}\Gamma^s_{jl} - \Gamma^i_{ls}\Gamma^s_{jk}. \tag{2.2}$$

The covariant form $R_{ijkl} = g_{im}R^m{}_{jkl}$ of the curvature tensor is given by

$$R_{ijkl} = \tfrac{1}{2}\big(g_{il,jk} + g_{jk,il} - g_{ik,jl} - g_{jl,ik}\big) + g_{pq}\big(\Gamma^p_{il}\Gamma^q_{jk} - \Gamma^p_{ik}\Gamma^q_{jl}\big). \tag{2.3}$$

For a covariant vector field A_i we have the analoguous commutation relation

$$A_{i;jk} - A_{i;kj} = -R^l{}_{ikj}A_l, \tag{2.4}$$

and similarly for tensors of higher order (with one term for each index).

Remark 2.1. In (2.1) and (2.4), it might seem more natural to reverse the order of the last two indices in the curvature tensor, which means replacing R^i_{jkl} in (2.2) by $R^i_{jlk} = -R^i_{jkl}$. This convention also occurs in the literature, but the one used here seems to be more common. (The difference is thus in the sign of R^i_{jkl}. The Ricci tensor R_{ij} defined below is defined to be the same in any case.)

Note the symmetries

$$R_{ijkl} = -R_{ijlk} = -R_{jikl} = R_{jilk} \tag{2.5}$$

and

$$R_{ijkl} = R_{klij}. \tag{2.6}$$

In particular, with two coinciding indices,

$$R_{iikl} = R_{ijkk} = 0. \tag{2.7}$$

Moreover, (2.3) also implies the *Bianchi identity*

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0. \tag{2.8}$$

Note also the *second Bianchi identity*

$$R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} = 0. \tag{2.9}$$

Four distinct indices i, j, k, l can be permuted in 24 ways, and (2.5)–(2.6) show that only three of the resulting R_{ijkl} are essentially distinct; each of the 24 equals $\pm R_{ijkl}$, $\pm R_{iklj}$ or $\pm R_{iljk}$. Moreover, two of these three values determine the third by the Bianchi identity. Hence, of these 24 different values, there are only 2 linearly independent. See further Appendix A.

2.2. **The Ricci tensor.** The *Ricci tensor* is defined as the contraction

$$\begin{aligned} R_{ij} &:= R^k{}_{ikj} = g^{kl} R_{kilj} = g^{kl} R_{ikjl} \\ &= \Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{ks}\Gamma^s_{ij} - \Gamma^k_{js}\Gamma^s_{ik}. \end{aligned} \tag{2.10}$$

The Ricci tensor is symmetric,

$$R_{ij} = R_{ji}, \tag{2.11}$$

as a consequence of (2.6).

The second Bianchi identity (2.9) implies

$$R^i{}_{jkl;i} = g^{im} R_{ijkl;m} = R_{jl;k} - R_{jk;l}. \tag{2.12}$$

2.3. **Scalar curvature.** The *scalar curvature* is the contraction (trace) of the Ricci tensor

$$R := R^i{}_i = g^{ij} R_{ij} = g^{ij} g^{kl} R_{ikjl}. \tag{2.13}$$

The second Bianchi identity (2.9), or (2.12), implies

$$R_{,i} = R_{;i} = 2R^k{}_{i;k} = 2g^{jk} R_{ij;k}. \tag{2.14}$$

2.4. **The Weyl tensor.** When $n \geqslant 3$, the curvature tensor can be decomposed as

$$R_{ijkl} = \frac{1}{2n(n-1)} R\, g_{ij} \odot g_{kl} + \frac{1}{n-2}\Big(R_{ij} - \frac{1}{n} R g_{ij}\Big) \odot g_{kl} + W_{ijkl}, \tag{2.15}$$

where

$$W_{ijkl} := R_{ijkl} - \frac{1}{n-2} R_{ij} \odot g_{kl} + \frac{1}{2(n-1)(n-2)} R\, g_{ij} \odot g_{kl} \tag{2.16}$$

is the *Weyl tensor*. The two first terms on the right-hand side of (2.15) can be seen as the parts of R_{ijkl} determined by the scalar curvature R and the traceless part of the Ricci curvature $R_{ij} - \frac{1}{n} R g_{ij}$; together they thus yield the part of R_{ijkl} determined by the Ricci tensor, and the Weyl tensor is the remainder; see Appendix A.

The Weyl tensor satisfies the same symmetry relations (2.5)–(2.6) and (2.8) as the Riemann curvature tensor, see Appendix A. Furthermore, the contraction vanishes:

$$W^k{}_{ikj} = g^{kl} W_{kilj} = g^{kl} W_{ikjl} = 0, \tag{2.17}$$

see (A.20)–(A.23).

When $n = 2$, the Weyl tensor is trivially defined to be 0. Moreover, then $R_{ij} - \frac{1}{n} R g_{ij} = 0$, and (2.15) still holds in the form

$$R_{ijkl} = \frac{1}{2n(n-1)} R\, g_{ij} \odot g_{kl} + W_{ijkl} = \frac{1}{4} R g_{ij} \odot g_{kl}, \tag{2.18}$$

see (A.28).

Also when $n = 3$, the Weyl tensor W_{ijkl} vanishes identically; (2.15) then becomes, see (A.27),

$$R_{ijkl} = \frac{1}{12} R\, g_{ij} \odot g_{kl} + \Bigl(R_{ij} - \frac{1}{3} R g_{ij}\Bigr) \odot g_{kl} = \Bigl(R_{ij} - \frac{1}{4} R g_{ij}\Bigr) \odot g_{kl}. \tag{2.19}$$

2.5. **The traceless Ricci tensor.** The *traceless Ricci tensor* is defined as

$$T_{ij} := R_{ij} - \frac{1}{n} R g_{ij}; \tag{2.20}$$

thus the contraction

$$T^i_i := g^{ij} T_{ij} = 0, \tag{2.21}$$

which explains its name. Note that $T_{ij} = 0$ if and only if R_{ij} is a multiple of g_{ij}. See further Appendix A; T_{ij} is the traceless component denoted $R^{(2)}_{ij}$ in (A.8) and (A.10).

2.6. **The Schouten tensor.** The *Schouten tensor* is the symmetric tensor defined, for $n \geqslant 3$, by

$$S_{ij} := \frac{1}{n-2} R_{ij} - \frac{R}{2(n-1)(n-2)} g_{ij} = \frac{1}{n-2} \Bigl(R_{ij} - \frac{1}{2(n-1)} R g_{ij}\Bigr); \tag{2.22}$$

thus (2.16) can be written as

$$W_{ijkl} = R_{ijkl} - S_{ij} \odot g_{kl}. \tag{2.23}$$

Equivalently,

$$R_{ijkl} = W_{ijkl} + S_{ij} \odot g_{kl}, \tag{2.24}$$

and by (A.15), this is the unique decomposition of R_{ijkl} of this type with tensors W_{ijkl} and S_{ij} such that $W^i{}_{jil} = 0$ and $S_{ij} = S_{ji}$.

The contraction is

$$S^i_i = \frac{1}{2(n-1)} R. \tag{2.25}$$

2.7. **The Einstein tensor.** The *Einstein tensor* is defined as

$$G_{ij} = R_{ij} - \tfrac{1}{2} R g_{ij} = R_{ij} - \tfrac{1}{2} g_{ij} g^{kl} R_{kl} \tag{2.26}$$

or equivalently

$$G^{ij} = R^{ij} - \tfrac{1}{2} R g^{ij}. \tag{2.27}$$

The Einstein tensor G_{ij} is symmetric, and it follows from (2.14) that its divergence vanishes:

$$G^{ij}{}_{;j} = 0. \tag{2.28}$$

The contraction is

$$G^i_i = -\frac{n-2}{2}R. \tag{2.29}$$

Note that for $n \geqslant 3$, the traceless Ricci tensor (2.20), the Schouten tensor (2.22) and the Einstein tensor (2.26) are three different linear combinations of R_{ij} and R. (If $n = 2$, then $G_{ij} = T_{ij}$ and S_{ij} is not defined.)

2.8. **The Cotton tensor.** The *Cotton tensor* is defined, for $n \geqslant 3$, as

$$C_{ijk} := S_{ij;k} - S_{ik;j}. \tag{2.30}$$

The reason for this definition is that by a simple calculation using (2.16), (2.12) and (2.14),

$$\begin{aligned}(n-2)W^i{}_{jkl;i} &:= (n-2)g^{im}W_{ijkl;m} \\ &= (n-2)R^i{}_{jkl;i} - \bigl(R^i_{k;i}g_{jl} - R^i_{l;i}g_{jk} + R_{jl;k} - R_{jk;l}\bigr) + \frac{1}{n-1}\bigl(R_{;k}g_{jl} - R_{;l}g_{jk}\bigr) \\ &= (n-3)\bigl(R_{jl;k} - R_{jk;l}\bigr) - \frac{n-3}{2(n-1)}\bigl(R_{;k}g_{jl} - R_{;l}g_{jk}\bigr) \end{aligned} \tag{2.31}$$

and thus

$$W^i{}_{jkl;i} = (n-3)\bigl(S_{jl;k} - S_{jk;l}\bigr) = -(n-3)C_{jkl}. \tag{2.32}$$

2.9. **Sectional curvature.** Let σ be a two-dimensional subspace of the tangent space M_x at some point x, and let U and V be two tangent vectors spanning σ (i.e., any two linearly independent vectors in σ). Then the *sectional curvature* at σ is defined as

$$K(\sigma) := K(U,V) := \frac{R_{ijkl}U^iV^jU^kV^l}{\|U \wedge V\|^2} \tag{2.33}$$

where we define the norm of the 2-form $U \wedge V = U^iV^j - U^jV^i$ by

$$\begin{aligned}\|U \wedge V\|^2 &:= \tfrac{1}{2}g_{ik}g_{jl}(U^iV^j - U^jV^i)(U^kV^l - U^lV^k) \\ &= \tfrac{1}{2}(g_{ij} \odot g_{kl})U^iV^jU^kV^l \\ &= \langle U,U\rangle\langle V,V\rangle - \langle U,V\rangle^2. \end{aligned} \tag{2.34}$$

It is easily seen that the fractions in (2.33) are the same for any pair of vectors U and V spanning σ; thus $K(U,V)$ depends only on σ which justifies the definition of $K(\sigma)$. By (2.34) we can also write

$$K(\sigma) := K(U,V) = \frac{R_{ijkl}U^iV^jU^kV^l}{\frac{1}{2}(g_{ij} \odot g_{kl})U^iV^jU^kV^l}. \tag{2.35}$$

A Riemannian manifold M is *isotropic* if $K(\sigma) = K(x)$ is the same for all two-dimensional subspaces σ of M_x, for every $x \in M$. This is equivalent to, see [4, Lemmas 3.3 and 3.4],

$$R_{ijkl} = K(x)\tfrac{1}{2}\, g_{ij} \odot g_{kl} = K(x)\bigl(g_{ik}g_{jl} - g_{il}g_{jk}\bigr). \tag{2.36}$$

By (2.15) (and (2.18) when $n = 2$), see further Appendix A, this is also equivalent to
$$W_{ijkl} = 0 \quad \text{and} \quad R_{ij} = \tfrac{1}{n} R g_{ij}. \tag{2.37}$$
For an isotropic manifold we have, with $K = K(x)$, as a consequence of (2.36),
$$R_{ij} = (n-1)K g_{ij}, \tag{2.38}$$
$$R = n(n-1)K. \tag{2.39}$$
Moreover, by (2.20), (2.22) and (2.26),
$$T_{ij} = 0, \tag{2.40}$$
$$S_{ij} = \tfrac{1}{2} K g_{ij} \qquad (n \geqslant 3), \tag{2.41}$$
$$G_{ij} = -\frac{(n-1)(n-2)}{2} K g_{ij}. \tag{2.42}$$

A Riemannian manifold M is of *constant (sectional) curvature* if $K(\sigma) = K$ is the same for all two-dimensional subspaces σ of M_x for all points $x \in M$, i.e., if M is isotropic and $K(x)$ is a constant function. In other words, by (2.36), M is of constant curvature if and only if
$$R_{ijkl} = \tfrac{1}{2} K \, g_{ij} \odot g_{kl} = K\big(g_{ik} g_{jl} - g_{il} g_{jk}\big) \tag{2.43}$$
for some constant K; in this case (2.38)–(2.39) hold, and in particular $K = R/(n(n-1))$.

Every two-dimensional manifold is isotropic by definition, or by (2.18), but not necessarily of constant curvature.

On the other hand, if M is isotropic then $R_{ij} = \frac{1}{n} R g_{ij}$ so (2.14) yields
$$R_{;i} = 2 g^{jk} R_{ij;k} = \tfrac{2}{n} g^{jk} R_{;k} \, g_{ij} = \tfrac{2}{n} R_{;i}. \tag{2.44}$$
Hence, if furthermore $n \geqslant 3$, then $R_{,i} = R_{;i} = 0$ and thus R is locally constant. Thus a connected manifold with dimension $\geqslant 3$ is isotropic if and only if it is of constant curvature. (In other words, a connected manifold is isotropic if and only if it has dimension 2 or has constant curvature.)

It follows from (2.30) and (2.41) that for a manifold with constant curvature and $n \geqslant 3$, $C_{ijk} = 0$. Furthermore, $W_{ijkl} = 0$ by (2.37).

3. **Special cases**

3.1. **The two-dimensional case.** If $n = 2$, the only non-zero components of R_{ijkl} are, by (2.5)–(2.6),
$$R_{1212} = R_{2121} = -R_{1221} = -R_{2112}. \tag{3.1}$$
It follows from (2.10) that, with $\hat{R} := R_{1212}$,
$$R_{11} = \hat{R} g^{22}, \qquad R_{22} = \hat{R} g^{11}, \qquad R_{12} = R_{21} = -\hat{R} g^{12} \tag{3.2}$$
and thus by (1.1)
$$R_{ij} = \hat{R} |g|^{-1} g_{ij} \tag{3.3}$$

and

$$R = 2|g|^{-1}\hat{R}. \tag{3.4}$$

This can also be written

$$R_{1212} = R_{2121} = -R_{1221} = -R_{2112} = \tfrac{1}{2}|g|R \tag{3.5}$$

or, equivalently,

$$R_{ijkl} = \tfrac{1}{2}R\big(g_{ik}g_{jl} - g_{il}g_{jk}\big) = \tfrac{1}{4}R\, g_{ij} \odot g_{kl}, \tag{3.6}$$

and

$$R_{ij} = \tfrac{1}{2}Rg_{ij}. \tag{3.7}$$

(See (A.28)–(A.29) for alternative proofs.)

By (3.6) and (2.36), the metric is isotropic with sectional curvature

$$K = \tfrac{1}{2}R. \tag{3.8}$$

The Weyl tensor vanishes by definition: $W_{ijkl} = 0$.

Also the Einstein tensor vanishes: $G_{ij} = 0$ by (3.7) and (2.26).

The Schouten and Cotton tensors are not defined.

For the special case of a smooth surface in $\mathbb{R}^3$, the *Gaussian curvature* at a point p can be defined as the product of the two principal curvatures at p (or as the determinant of the shape operator $S : M_p \to M_p$ given by $S(v) = -\frac{\partial}{\partial v}N$, where N is the unit normal field). The Gaussian curvature equals the sectional curvature K.

3.2. **Flat metric.** A Riemannian manifold M is *flat* if it locally is isometric to Euclidean space, i.e., if it can be covered by coordinate systems with $g_{ij} = \delta_{ij}$. It follows immediately by (1.13) and (1.12) that the connection coefficients Γ_{kij} and Γ_{ij}^k vanish; hence by (2.2), the curvature tensor vanishes: $R_{ijkl} = 0$. Conversely, it is well-known that the converse also holds (see e.g. [4, Corollary 8.2.2]):

Theorem 3.1. *A manifold M is flat if and only if $R_{ijkl} = 0$.* □

By (2.43), M is flat if and only if it is of constant curvature 0.

It follows immediately from the definitions above that in a flat manifold, also the Ricci tensor, Weyl tensor, Einstein tensor and scalar curvature vanish, as well as, when $n \geqslant 3$, the Schouten and Cotton tensors. Conversely, we see for example from (2.15) (and (2.18) when $n = 2$) that M is flat if and only if the Weyl and Ricci tensors vanish; in particular, if $n = 3$ then M is flat if and only if the Ricci tensor vanishes. Similarly, by (2.24), if $n \geqslant 3$, then M is flat if and only if the Weyl and Schouten tensors vanish.

3.3. **Conformal metric.** Consider a conformal Riemannian metric

$$\mathrm{d}s = w(x)|\mathrm{d}x| = e^{\varphi(x)}|\mathrm{d}x|. \tag{3.9}$$

In other words,

$$g_{ij} = w(x)^2\delta_{ij} = e^{2\varphi(x)}\delta_{ij}. \tag{3.10}$$

Then

$$g^{ij} = w(x)^{-2}\delta_{ij} = e^{-2\varphi(x)}\delta_{ij}. \tag{3.11}$$

The invariant measure (1.6) is

$$\mathrm{d}\mu = w(x)^n \,\mathrm{d}x_1 \cdots \mathrm{d}x_n = e^{n\varphi(x)}\,\mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{3.12}$$

We have, cf. [4, Section 8.3],

$$\Gamma_{kij} = [ij,k] = ww_{,i}\delta_{jk} + ww_{,j}\delta_{ik} - ww_{,k}\delta_{ij}. \tag{3.13}$$

and thus

$$\Gamma^k_{ij} = \varphi_{,i}\delta_{jk} + \varphi_{,j}\delta_{ik} - \varphi_{,k}\delta_{ij}. \tag{3.14}$$

It follows by a calculation that, with $\|\nabla\varphi\|^2 = \sum_{i=1}^n \varphi_{,i}^2$,

$$\begin{aligned} R_{ijkl} &= e^{2\varphi}\Bigl((\varphi_{,i}\varphi_{,j} - \varphi_{,ij}) \odot \delta_{kl} - \|\nabla\varphi\|^2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\Bigr) \\ &= e^{2\varphi}\Bigl(\varphi_{,i}\varphi_{,j} - \varphi_{,ij} - \tfrac12\|\nabla\varphi\|^2\delta_{ij}\Bigr) \odot \delta_{kl}, \end{aligned} \tag{3.15}$$

and thus, by another calculation, with $\Delta\varphi := \sum_{i=1}^n \varphi_{,ii}$,

$$R_{ij} = (n-2)\bigl(\varphi_{,i}\varphi_{,j} - \varphi_{,ij} - \|\nabla\varphi\|^2\delta_{ij}\bigr) - \Delta\varphi\,\delta_{ij} \tag{3.16}$$

and

$$R = -e^{-2\varphi}\Bigl((n-1)(n-2)\|\nabla\varphi\|^2 + 2(n-1)\Delta\varphi\Bigr). \tag{3.17}$$

The Weyl tensor thus vanishes, by (2.16), or by (A.20),

$$W_{ijkl} = 0. \tag{3.18}$$

The Einstein tensor is

$$G_{ij} = (n-2)\Bigl(\varphi_{,i}\varphi_{,j} - \varphi_{,ij} + \tfrac12(n-3)\|\nabla\varphi\|^2\delta_{ij} + \Delta\varphi\,\delta_{ij}\Bigr). \tag{3.19}$$

For $n \geqslant 3$, the Schouten tensor is by (3.15)

$$S_{ij} = \varphi_{,i}\varphi_{,j} - \varphi_{;ij} - \tfrac12\|\nabla\varphi\|^2 g_{ij} \tag{3.20}$$

and a calculation shows that the Cotton tensor vanishes:

$$C_{ijk} = 0. \tag{3.21}$$

The Laplace–Beltrami operator Δ is given by

$$\begin{aligned} \Delta F &= e^{-2\varphi}\left(\sum_{i=1}^n F_{,ii} + (n-2)\sum_{i=1}^n \varphi_{,i}F_{,i}\right) \quad (3.22)\\ &= w^{-n}\sum_{i=1}^n (w^{n-2}F_{,i})_{,i}. \quad (3.23) \end{aligned}$$

A Riemannian manifold M is (locally) *conformally flat* if it locally is conformally equivalent to Euclidean space, i.e., if it can be covered by coordinate systems with metrics as in (3.10). (Such coordinates are called *isothermal*.) A manifold of dimension 2 is always conformally flat. (By Theorem 3.1, (3.6) and (3.43) below, this is equivalent to the existence of local solutions to $\Delta\varphi = R/2$. See e.g. [6, P.4.8] together with [19, Theorem 6.8] for a proof. This holds also under much weaker regularity, see e.g. Korn [9], Lichtenstein [12], Chern [2] and Hartman and Wintner [5].) If $n \geqslant 3$, (3.18) and (3.21) show that if M is conformally flat, then $W_{ijkl} = 0$ and $C_{ijk} = 0$. The Weyl–Schouten theorem says that the converse holds too; moreover, if $n = 3$ then $W_{ijkl} = 0$ automatically, and if $n \geqslant 4$, then $W_{ijkl} = 0$ in M implies $C_{ijk} = 0$ by (2.32). Hence we have the following characterisation. (For a proof, see e.g. [6]. The necessity of $W_{ijkl} = 0$ when $n \geqslant 4$ was shown by Weyl [18] and the sufficiency by Schouten [14]; the case $n = 3$ was treated earlier by Cotton [3].)

Theorem 3.2 (Weyl–Schouten)**.** *M is conformally flat if and only if*

- $n = 2$,
- $n = 3$ *and* $C_{ijk} = 0$, *or*
- $n \geqslant 4$ *and* $W_{ijkl} = 0$. □

Note that for $n = 3$ the condition is a third-order system of partial differential equations in g_{ij}, but for $n \geqslant 4$ only a second-order system.

3.3.1. *The case $n = 2$.* For a conformal Riemannian metric in the special case $n = 2$, when (3.6)–(3.7) hold, (3.15)–(3.17) simplify to

$$R_{ijkl} = -\tfrac{1}{2} e^{2\varphi} \Delta\varphi \, \delta_{ij} \odot \delta_{kl}, \tag{3.24}$$

$$R_{ij} = -\Delta\varphi \, \delta_{ij}, \tag{3.25}$$

$$R = -2e^{-2\varphi} \Delta\varphi. \tag{3.26}$$

The sectional curvature is, by (2.36) or (3.8),

$$K = R/2 = -e^{-2\varphi} \Delta\varphi. \tag{3.27}$$

The Laplace–Beltrami operator is by (3.22)

$$\Delta F = e^{-2\varphi} \sum_{i=1}^{n} F_{,ii}. \tag{3.28}$$

In particular, the coordinate functions x^j are harmonic, since $x^j_{,ii} = 0$.

3.4. **Conformally equivalent metrics.** More generally, consider two conformally equivalent Riemannian metrics g_{ij} and

$$\tilde{g}_{ij} = w(x)^2 g_{ij} = e^{2\varphi(x)} g_{ij}. \tag{3.29}$$

We denote quantities related to $\tilde{g}_{ij}$ by a tilde. Note that covariant derivatives and norms below are calculated for g_{ij}.

We have

$$\tilde{g}^{ij} = w(x)^{-2} g_{ij} = e^{-2\varphi(x)} g_{ij}. \tag{3.30}$$

Calculations show that

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \varphi_{,i}\delta^k_j + \varphi_{,j}\delta^k_i - g^{kl}\varphi_{,l}\, g_{ij} \tag{3.31}$$

and, with $\|\nabla\varphi\|^2 := g^{ij}\varphi_{,i}\varphi_{,j}$ and $\Delta\varphi := g^{ij}\varphi_{;ij}$ as in (1.24),

$$\begin{aligned}\widetilde{R}_{ijkl} &= e^{2\varphi}\Bigl(R_{ijkl} + (\varphi_{,i}\varphi_{,j} - \varphi_{;ij}) \odot g_{kl} - \tfrac{1}{2}\|\nabla\varphi\|^2 g_{ij} \odot g_{kl}\Bigr) \\ &= e^{2\varphi}\Bigl(R_{ijkl} + \bigl(\varphi_{,i}\varphi_{,j} - \varphi_{;ij} - \tfrac{1}{2}\|\nabla\varphi\|^2 g_{ij}\bigr) \odot g_{kl}\Bigr), \qquad (3.32)\end{aligned}$$

$$\widetilde{R}_{ij} = R_{ij} + (n-2)\bigl(\varphi_{,i}\varphi_{,j} - \varphi_{;ij} - \|\nabla\varphi\|^2 g_{ij}\bigr) - \Delta\varphi\, g_{ij}, \qquad (3.33)$$

$$\widetilde{R} = e^{-2\varphi}\Bigl(R - (n-1)(n-2)\|\nabla\varphi\|^2 - 2(n-1)\Delta\varphi\Bigr); \qquad (3.34)$$

for $n > 2$, the latter formula can also be written

$$\widetilde{R} = e^{-2\varphi}\Bigl(R - \frac{4(n-1)}{n-2} e^{-(n-2)\varphi/2} \Delta e^{(n-2)\varphi/2}\Bigr). \qquad (3.35)$$

The Weyl tensor is, by (2.16), or by (A.20),

$$\widetilde{W}_{ijkl} = e^{2\varphi} W_{ijkl}. \qquad (3.36)$$

This becomes nicer for the $(3,1)$-tensor W^i_{jkl} which is completely invariant:

$$\widetilde{W}^i_{jkl} = W^i_{jkl}. \qquad (3.37)$$

The Weyl tensor W^i_{jkl} is thus sometimes called the *conformal curvature tensor*.

The Schouten tensor (for $n \geqslant 3$) transforms as, e.g. by (3.32) and (2.24),

$$\widetilde{S}_{ij} = S_{ij} + \varphi_{,i}\varphi_{,j} - \varphi_{;ij} - \tfrac{1}{2}\|\nabla\varphi\|^2 g_{ij}. \qquad (3.38)$$

A calculation shows that the Cotton tensor (for $n \geqslant 3$) transforms as

$$\widetilde{C}_{ijk} = C_{ijk} + W^l{}_{ijk}\varphi_{,l}. \qquad (3.39)$$

In particular, if $n = 3$, then the Weyl tensor vanishes and $\widetilde{C}_{ijk} = C_{ijk}$, i.e., the Cotton tensor is conformally invariant.

The Laplace–Beltrami operator Δ transforms as

$$\widetilde{\Delta} F = e^{-2\varphi}\left(\Delta F + (n-2) g^{ij}\varphi_{,i} F_{,j}\right). \qquad (3.40)$$

3.4.1. *The case* $n = 2$. In the special case $n = 2$, when (3.6)–(3.7) hold, (3.32)–(3.34) simplify to

$$\widetilde{R}_{ijkl} = e^{2\varphi}\Bigl(R_{ijkl} - \tfrac{1}{2}\Delta\varphi\, g_{ij} \odot g_{kl}\Bigr) \qquad (3.41)$$

$$\widetilde{R}_{ij} = R_{ij} - \Delta\varphi\, g_{ij}, \qquad (3.42)$$

$$\widetilde{R} = e^{-2\varphi}\Bigl(R - 2\Delta\varphi\Bigr). \qquad (3.43)$$

3.5. **Scaling.** As a special case of (3.29), consider scaling all lengths by a constant factor $a > 0$:

$$\tilde{g}_{ij} = a^2 g_{ij}. \tag{3.44}$$

This is (3.29) with $w(x) = a$ and $\varphi(x) = \log a$ constant, and thus all derivatives $\varphi_{,i} = 0$.

It follows from (3.30)–(3.39) that

$$\tilde{g}^{ij} = a^{-2} g_{ij}, \tag{3.45}$$

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij}, \tag{3.46}$$

$$\widetilde{R}_{ijkl} = a^2 R_{ijkl}, \tag{3.47}$$

$$\widetilde{R}_{ij} = R_{ij}, \tag{3.48}$$

$$\widetilde{R} = a^{-2} R, \tag{3.49}$$

$$\widetilde{W}_{ijkl} = a^2 W_{ijkl}, \tag{3.50}$$

$$\widetilde{S}_{ij} = \widetilde{S}_{ij} \qquad (n \geqslant 3), \tag{3.51}$$

$$\widetilde{C}_{ijk} = \widetilde{C}_{ijk} \qquad (n \geqslant 3). \tag{3.52}$$

In particular, it follows from (2.43) that if g_{ij} has constant curvature K, then $\tilde{g}_{ij}$ has constant curvature $a^{-2}K$.

4. **Examples, general**

Example 4.1. Let, for x in a suitable domain in $\mathbb{R}^n$,

$$\mathrm{d}s = \frac{a\,|\mathrm{d}x|}{1 + c|x|^2}, \tag{4.1}$$

i.e.,

$$g_{ij} = \frac{a^2}{(1 + c|x|^2)^2}\delta_{ij}. \tag{4.2}$$

We assume $a > 0$ and $c \in \mathbb{R}$; if $c \geqslant 0$ the metric is defined in $\mathbb{R}^n$, but if $c < 0$ only for $|x| < |c|^{-1/2}$. (It is also possible to take $a < 0$, $c < 0$ and $|x| > |c|^{-1/2}$. The results below hold in this case too; see Remark 4.4.)

This is a conformal metric (3.10) with $w(x) = a/(1 + c|x|^2)$ and $\varphi(x) = \log a - \log(1 + c|x|^2)$. The invariant measure (1.6) is

$$\mathrm{d}\mu = w(x)^n \,\mathrm{d}x_1 \cdots \mathrm{d}x_n = \frac{a^n}{(1 + c|x|^2)^n}\,\mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{4.3}$$

We have

$$g^{ij} = \frac{(1 + c|x|^2)^2}{a^2}\delta_{ij} \tag{4.4}$$

and

$$\varphi_{,i} = -\frac{2cx_i}{1 + c|x|^2} \tag{4.5}$$

and thus by (3.14)

$$\Gamma_{ij}^{k} = -\frac{2c}{1+c|x|^2}\big(x_i\delta_{jk} + x_j\delta_{ik} - x_k\delta_{ij}\big). \tag{4.6}$$

Furthermore,

$$\varphi_{,ij} = -\frac{2c\delta_{ij}}{1+c|x|^2} + \frac{4c^2x_ix_j}{(1+c|x|^2)^2} = -\frac{2c}{1+c|x|^2}\delta_{ij} + \varphi_{,i}\varphi_{,j}, \tag{4.7}$$

$$\|\nabla\varphi\|^2 = \frac{4c^2|x|^2}{(1+c|x|^2)^2}, \tag{4.8}$$

and (3.15) yields

$$\begin{aligned} R_{ijkl} &= \frac{a^2}{(1+c|x|^2)^2}\left(\frac{2c}{1+c|x|^2} - \frac{2c^2|x|^2}{(1+c|x|^2)^2}\right)\delta_{ij}\odot\delta_{kl} \\ &= \frac{2ca^2}{(1+c|x|^2)^4}\delta_{ij}\odot\delta_{kl} = 2ca^{-2}g_{ij}\odot g_{kl}. \end{aligned} \tag{4.9}$$

The metric thus has constant curvature, see (2.43), which also shows that the sectional curvature is

$$K = 4ca^{-2}, \tag{4.10}$$

and hence, by (2.38)–(2.39), or by (3.16)–(3.17),

$$R_{ij} = 4(n-1)ca^{-2}g_{ij} = \frac{4(n-1)c}{(1+c|x|^2)^2}\delta_{ij}, \tag{4.11}$$

$$R = 4n(n-1)ca^{-2}. \tag{4.12}$$

See further Examples 6.1 and 7.3–7.4.

Remark 4.2. Of course, different values of $a > 0$ in Example 4.1 differ only by a simple scaling of the metric, see Section 3.5.

Remark 4.3. Denote the metric (4.2) by $g_{ij}^{(a,c)}$. If $t > 0$, then the dilation $\psi(x) = tx$ maps the metric $g_{ij}^{(a,c)}$ to $t^{-2}a^2(1+ct^{-2}|x|^2)^{-2}\delta_{ij}$, i.e. $g_{ij}^{(t^{-1}a,t^{-2}c)}$. Note that these metrics have the same sectional curvature $4ca^{-2}$ by (4.10) (as they must when there is an isometry).

Hence, if $c \neq 0$, we may by choosing $t = |c|^{1/2}$ obtain an isometry with $g_{ij}^{(a_1,c/|c|)}$ for some a_1, i.e., we may by a dilation assume that $c \in \{0, \pm 1\}$.

The case $c = 0$ is flat and yields a multiple of the standard Euclidean metric (a special case of Example 5.1).

The case $c = 1$ yields a positive curvature $K = 4a^{-2}$. (This is by a dilation $x \mapsto (a/2)x$ equivalent to Example 6.1 with $\rho = a/2$.)

The case $c = -1$ yields a negative curvature $K = -4a^{-2}$. (This is Example 7.4.)

Remark 4.4. As said in Example 4.1, it is possible to take $a < 0$ in (4.1)–(4.2) if $c < 0$ and $|x| > |c|^{-1/2}$; we thus have

$$\mathrm{d}s = \frac{|a||\mathrm{d}x|}{|c||x|^2 - 1}. \tag{4.13}$$

The map $x \mapsto |c|^{-1}x/|x|^2$, i.e., reflection in the sphere $|x| = |c|^{-1/2}$, maps this to the metric $g_{ij}^{(|a|,c)}(y) = a^2(1 + c|y|^2)^{-2}\delta_{ij}$ in $0 < |y| < |c|^{-1/2}$, so this case is by an isometry equivalent to the case $a > 0$, $c < 0$ (with the point 0 removed).

Similarly, if $c > 0$, then the map $x \mapsto c^{-1}x/|x|^2$, i.e., reflection in the sphere $|x| = c^{-1/2}$, maps the metric (4.1)–(4.2) to itself, so it is an isometry of $\mathbb{R}^n \setminus 0$ with the metric $g_{ij}^{(a,c)}$ onto itself.

Example 4.5 (Polar coordinates, general)**.** Let $n = 2$ and let M be a subset of $\mathbb{R}^2 = \{(r, \theta)\}$ with the metric

$$|\mathrm{d}s|^2 = |\mathrm{d}r|^2 + w(r)^2|\mathrm{d}\theta|^2 \tag{4.14}$$

for some smooth function $w(r) > 0$, i.e.,

$$g_{11} = 1, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = w(r)^2, \tag{4.15}$$

or in matrix form

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & w(r)^2 \end{pmatrix}. \tag{4.16}$$

(We thus denote the coordinates by $r = x_1$ and $\theta = x_2$.)

We have

$$(g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & w(r)^{-2} \end{pmatrix}, \tag{4.17}$$

i.e.,

$$g^{11} = 1, \qquad g^{12} = g^{21} = 0, \qquad g^{22} = w(r)^{-2}. \tag{4.18}$$

The invariant measure is

$$\mathrm{d}\mu = w(r)\,\mathrm{d}r\,\mathrm{d}\theta. \tag{4.19}$$

The only non-zero derivatives $g_{ij,k}$ and $g_{ij,kl}$ are $g_{22,1} = 2ww'$ and $g_{22,11} = 2ww'' + 2(w')^2$, with $w = w(r)$, and it follows from (1.13) that

$$\Gamma_{122} = -ww', \tag{4.20}$$

$$\Gamma_{212} = \Gamma_{221} = ww', \tag{4.21}$$

with the remaining components 0, which by (1.12) yields

$$\Gamma^1_{22} = -ww', \tag{4.22}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = w'/w, \tag{4.23}$$

again with all other components 0.

By (2.3), this yields

$$R_{1212} = -(ww'' + (w')^2) + w^2(w'/w)^2 = -ww'', \tag{4.24}$$

and thus, e.g. by (3.3)–(3.4) and (3.8)

$$R_{ij} = -\frac{w''}{w} g_{ij}, \tag{4.25}$$

$$R = -2\frac{w''}{w} \tag{4.26}$$

$$K = -\frac{w''}{w}. \tag{4.27}$$

See also the generalization Example 4.6.

It is obvious, by translation invariance, that the lines with r constant have constant curvature. More precisely, if $\gamma(t) = (r, t)$ for some fixed r, then $\dot\gamma = (0, 1)$ and $\ddot\gamma = (0, 0)$, and thus by (1.29)

$$D_t^2 \gamma^i = \Gamma^i_{jk} \dot\gamma^j \dot\gamma^k = \Gamma^i_{22}, \tag{4.28}$$

i.e., by (4.22) and $\Gamma^2_{22} = 0$,

$$D_t^2 \gamma = (\Gamma^1_{22}, \Gamma^2_{22}) = \bigl(-w(r)w'(r), 0\bigr). \tag{4.29}$$

Hence, by (1.31),

$$P_N D_t^2 \gamma = D_t^2 \gamma = \bigl(-w(r)w'(r), 0\bigr). \tag{4.30}$$

Consequently, the curvature vector (1.34) of γ is

$$\vec\kappa = -\frac{\bigl(w(r)w'(r), 0\bigr)}{w(r)^2} = \Bigl(-\frac{w'(r)}{w(r)}, 0\Bigr) \tag{4.31}$$

and the curvature is

$$\kappa = \frac{|w'(r)|}{w(r)}. \tag{4.32}$$

It follows easily from (4.27) that the cases of a metric (4.14)–(4.16) with constant curvature are, up to translation and reflection in r and dilation in θ, and with some constant $a > 0$:

(i) $w(r) = a$ $(K = 0)$, see Example 5.1;
(ii) $w(r) = r$ $(K = 0)$, see Example 5.2;
(iii) $w(r) = \sin(ar)$ $(K = a^2)$, see Examples 6.7–6.9;
(iv) $w(r) = e^{ar}$ $(K = -a^2)$, see Example 7.2;
(v) $w(r) = \sinh(ar)$ $(K = -a^2)$, see Example 7.8;
(vi) $w(r) = \cosh(ar)$ $(K = -a^2)$, see Example 7.15.

Example 4.6 (Spherical coordinates, general)**.** Example 4.5 can be generalized to arbitrary dimension as follows. Let $\tilde M$ be an $(n-1)$-dimensional Riemannian manifold and let M be $J \times \tilde M$ for an open interval $J \subseteq \mathbb{R}$, with the metric

$$|\mathrm{d}s|^2 = |\mathrm{d}r|^2 + w(r)^2 |\mathrm{d}\tilde s|^2, \tag{4.33}$$

where $|\mathrm{d}\tilde s|$ is the distance in $\tilde M$. We introduce coordinates $\tilde x = (x_1, \dots, x_{n-1})$ in (a subset of) $\tilde M$ and use the coordinates $(r, \tilde x) = (x_0, \dots, x_{n-1})$ (with an unusual numbering) in M; we thus have $r = x_0$. We use $\alpha, \beta, \gamma, \delta$ to denote indices in $1, \dots, n-1$.

Denote the metric tensor in $\tilde{M}$ by $\tilde{g}_{\alpha\beta}$; then the metric tensor in M is given by

$$g_{00} = 1, \qquad g_{\alpha\beta} = w(x_0)^2 \tilde{g}_{\alpha\beta}(\tilde{x}), \qquad g_{0\alpha} = g_{\alpha 0} = 0, \tag{4.34}$$

or in block matrix form

$$\big(g_{ij}\big) = \begin{pmatrix} 1 & 0 \\ 0 & w(x_0)^2 g_{\alpha\beta}(\tilde{x}) \end{pmatrix}. \tag{4.35}$$

It follows from (1.13) that, with $w = w(x_0)$ and $\tilde{g}_{\alpha\beta} = \tilde{g}_{\alpha\beta}(\tilde{x})$,

$$\Gamma_{000} = \Gamma_{00\alpha} = \Gamma_{0\alpha 0} = \Gamma_{\alpha 00} = 0, \tag{4.36}$$

$$\Gamma_{\alpha\beta 0} = \Gamma_{\alpha 0\beta} = ww'\tilde{g}_{\alpha\beta}, \tag{4.37}$$

$$\Gamma_{0\alpha\beta} = -ww'\tilde{g}_{\alpha\beta}, \tag{4.38}$$

$$\Gamma_{\alpha\beta\gamma} = w^2 \tilde{\Gamma}_{\alpha\beta\gamma} \tag{4.39}$$

and thus by (1.12),

$$\Gamma^0_{00} = \Gamma^0_{0\alpha} = \Gamma^0_{\alpha 0} = \Gamma^\alpha_{00} = 0, \tag{4.40}$$

$$\Gamma^\alpha_{\beta 0} = \Gamma^\alpha_{0\beta} = \frac{w'}{w}\delta^\alpha_\beta, \tag{4.41}$$

$$\Gamma^0_{\alpha\beta} = -ww'\tilde{g}_{\alpha\beta} = -\frac{w'}{w} g_{\alpha\beta}, \tag{4.42}$$

$$\Gamma^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma}. \tag{4.43}$$

This and (2.3) yield

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= w^2 \widetilde{R}_{\alpha\beta\gamma\delta} + g_{00}\big(\Gamma^0_{\alpha\delta}\Gamma^0_{\beta\gamma} - \Gamma^0_{\alpha\gamma}\Gamma^0_{\beta\delta}\big) \\ &= w^2 \widetilde{R}_{\alpha\beta\gamma\delta} - (w'/w)^2\big(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\big), \end{aligned} \tag{4.44}$$

$$\begin{aligned} R_{\alpha\beta\gamma 0} &= \tfrac{1}{2}\big(g_{\beta\gamma,\alpha 0} - g_{\alpha\gamma,\beta 0}\big) + g_{pq}\frac{w'}{w}\big(\delta^p_\alpha \Gamma^q_{\beta\gamma} - \delta^q_\beta \Gamma^p_{\alpha\gamma}\big) \\ &= ww'\big(\tilde{g}_{\beta\gamma,\alpha} - \tilde{g}_{\alpha\gamma,\beta}\big) + ww'\big(\tilde{\Gamma}_{\alpha\beta\gamma} - \tilde{\Gamma}_{\beta\alpha\gamma}\big) \\ &= 0, \end{aligned} \tag{4.45}$$

$$\begin{aligned} R_{\alpha 0\gamma 0} &= -\tfrac{1}{2} g_{\alpha\gamma,00} + g_{pq}\Big(\frac{w'}{w}\Big)^2 \delta^p_\alpha \delta^q_\gamma = -\big(ww'' + (w')^2\big)\tilde{g}_{\alpha\gamma} + (w')^2 \tilde{g}_{\alpha\gamma} \\ &= -\frac{w''}{w} g_{\alpha\gamma} = -\frac{w''}{w}\big(g_{\alpha\gamma}g_{00} - g_{\alpha 0}g_{0\gamma}\big), \end{aligned} \tag{4.46}$$

$$R_{\alpha 000} = R_{0000} = 0. \tag{4.47}$$

(The remaining components are given by the symmetry rules (2.5)–(2.6).) By contraction, this further yields the Ricci tensor, see (2.10),

$$\begin{aligned} R_{\alpha\gamma} &= \widetilde{R}_{\alpha\gamma} + R_{\alpha 0\gamma 0} - \Big(\frac{w'}{w}\Big)^2 (n-2) g_{\alpha\gamma} \\ &= \widetilde{R}_{\alpha\gamma} - \Big((n-2)\Big(\frac{w'}{w}\Big)^2 + \frac{w''}{w}\Big) g_{\alpha\gamma}, \end{aligned} \tag{4.48}$$

$$R_{\alpha 0} = 0, \tag{4.49}$$

$$R_{00} = g^{\alpha\gamma} R_{\alpha 0 \gamma 0} = -(n-1)\frac{w''}{w} \tag{4.50}$$

and the scalar curvature, see (2.13),

$$R = w^{-2}\big(\widetilde{R} - (n-1)(n-2)(w')^2 - 2(n-1)ww''\big). \tag{4.51}$$

The Weyl tensor (2.16) vanishes if $n \leqslant 3$, as always; for $n \geqslant 4$,

$$W_{\alpha\beta\gamma\delta} = w^2 \widetilde{W}_{\alpha\beta\gamma\delta} + \frac{w^2}{(n-2)(n-3)}\Big(\widetilde{R}_{\alpha\beta} - \frac{\widetilde{R}}{n-1}\tilde{g}_{\alpha\beta}\Big) \odot \tilde{g}_{\gamma\delta}, \tag{4.52}$$

$$W_{\alpha 0 \gamma 0} = -\frac{1}{n-2}\Big(\widetilde{R}_{\alpha\gamma} - \frac{\widetilde{R}}{n-1}\tilde{g}_{\alpha\gamma}\Big); \tag{4.53}$$

the remaining components follow by symmetry or vanish.

Now specialize to the case when $\tilde{M}$ has constant curvature, $\widetilde{R}_{\alpha\beta\gamma\delta} = \tilde{K}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})$, see (2.43). Then (4.44) yields

$$R_{\alpha\beta\gamma\delta} = \Big(\frac{\tilde{K} - (w')^2}{w^2}\Big)\big(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\big) \tag{4.54}$$

and a comparison with (4.45)–(4.46) shows that M has constant curvature if and only if

$$\tilde{K} - (w')^2 = -ww'', \tag{4.55}$$

and then the scalar curvature is

$$K = \frac{\tilde{K} - (w')^2}{w^2} = -\frac{w''}{w}. \tag{4.56}$$

We note that (4.55) also can be written

$$\Big(\frac{w'}{w}\Big)' = (\log w)'' = -\frac{\tilde{K}}{w^2}. \tag{4.57}$$

It follows from (4.56) that the cases with constant curvature (and constant curvature of $\tilde{M}$) are given by the same $w(r)$ as for $n = 2$, see the list at the end of Example 4.5, where moreover, by (4.55), the curvature $\tilde{K}$ of $\tilde{M}$ equals 0, 1, a^2, 0, a^2, $-a^2$, respectively, in the six cases given there.

Example 4.7. Let $n = 2$ and let the metric be of the special form

$$(g_{ij}) = \begin{pmatrix} g_{11}(x_1) & 0 \\ 0 & g_{22}(x_1) \end{pmatrix}, \tag{4.58}$$

i.e. diagonal and depending on x_1 only. (This generalizes Example 4.5, where we also assume $g_1 1 = 1$.) By (1.13), the only non-zero components Γ_{kij} are

$$\Gamma_{111} = \tfrac{1}{2}g_{11,1}, \qquad \Gamma_{122} = -\tfrac{1}{2}g_{22,1}, \qquad \Gamma_{212} = \Gamma_{221} = \tfrac{1}{2}g_{22,1}. \tag{4.59}$$

and by (1.12), the only non-zero components Γ^k_{ij} are thus

$$\Gamma^1_{11} = \frac{g_{11,1}}{2g_{11}}, \qquad \Gamma^1_{22} = -\frac{g_{22,1}}{2g_{11}}, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{g_{22,1}}{2g_{22}}. \tag{4.60}$$

Consequently, (2.3) yields

$$R_{1212} = -\tfrac{1}{2}g_{22,11} + \frac{(g_{22,1})^2}{4g_{22}} + \frac{g_{11,1}g_{22,1}}{4g_{11}} = -\tfrac{1}{2}|g|^{1/2}\big(g_{22,1}\,|g|^{-1/2}\big)_{,1}. \quad (4.61)$$

which by (3.1)–(3.8) yields all components R_{ijkl}, R_{ij} and R as well as $K = R/2$; in particular, by (3.5) and (3.8),

$$K = \frac{1}{2}R = \frac{R_{1212}}{g_{11}g_{22}} = -\tfrac{1}{2}|g|^{-1/2}\big(g_{22,1}\,|g|^{-1/2}\big)_{,1}. \quad (4.62)$$

As in Example 4.5, (Euclidean) lines with constant x_1 are curves with constant curvature. More precisely, if $\gamma(t) = (x_1, t)$, then $\dot{\gamma} = (0, 1)$ and, by (1.29) and (1.31),

$$P_N D_t^2\gamma = D_t^2\gamma = (\Gamma^1_{22}, \Gamma^2_{22}) = \Big(-\frac{g_{22,1}}{2g_{11}}, 0\Big). \quad (4.63)$$

Hence, the curvature vector (1.34) of γ is

$$\vec{\kappa} = \frac{P_N D_t^2\gamma}{g_{22}} = \Big(-\frac{g_{22,1}}{2g_{11}g_{22}}, 0\Big) \quad (4.64)$$

and the curvature of γ is

$$\kappa = \frac{|g_{22,1}|}{2g_{11}^{1/2}g_{22}}. \quad (4.65)$$

Example 4.8. Consider an non-isotropic scaling of the metric (4.58) to

$$(\tilde{g}_{ij}) = \begin{pmatrix} a^2 g_{11}(x_1) & 0 \\ 0 & b^2 g_{22}(x_1) \end{pmatrix}, \quad (4.66)$$

with $a, b > 0$. This is also of the form (4.58), with $\tilde{g}_{11} = a^2 g_{11}$ and $\tilde{g}_{22} = b^2 g_{22}$, so we can use the results in Example 4.7. In particular, (4.60) yields

$$\tilde{\Gamma}^1_{11} = \Gamma^1_{11}, \qquad \tilde{\Gamma}^1_{22} = \frac{b^2}{a^2}\Gamma^1_{22} \qquad \tilde{\Gamma}^2_{12} = \tilde{\Gamma}^2_{21} = \Gamma^2_{12}, \quad (4.67)$$

while (4.61) and (4.62) yield

$$\widetilde{R}_{1212} = b^2 R_{1212} \quad (4.68)$$

and

$$\tilde{K} = a^{-2}K. \quad (4.69)$$

Note that if $a = 1$, then the sectional curvature K remains the same for any choice of b.

Example 4.9. Consider the special case $g_{11} = g_{22}$ of the metric (4.58), i.e., the conformal metric

$$(\tilde{g}_{ij}) = \begin{pmatrix} g_{11}(x_1) & 0 \\ 0 & g_{11}(x_1) \end{pmatrix} = \big(e^{2\varphi(x_1)}\delta_{ij}\big), \quad (4.70)$$

where φ is a function of x_1 only. This is also of the form (3.10). By Example 4.7 or (3.14) and (3.24)–(3.27),

$$\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = \frac{g_{11,1}}{2g_{11}} = \varphi' \tag{4.71}$$

with all other components $\Gamma^k_{ij} = 0$, and

$$R_{1212} = -\tfrac{1}{2}g_{11,11} + \frac{(g_{11,1})^2}{2g_{11}} = -g_{11}\varphi'', \tag{4.72}$$

which by (3.1)–(3.8) yields all components R_{ijkl}, R_{ij} and R as well as

$$K = \frac{1}{2}R = \frac{R_{1212}}{g_{11}^2} = -\frac{\varphi''}{g_{11}} = -e^{-2\varphi}\varphi''. \tag{4.73}$$

Example 4.10. Let $n = 2$ and let the metric be of the form

$$|\mathrm{d}s|^2 = |\mathrm{d}x_1|^2 + |f(x)\,\mathrm{d}x_1 + \mathrm{d}x_2|^2 \tag{4.74}$$

for some smooth function f, i.e.,

$$(g_{ij}) = \begin{pmatrix} 1+f^2(x) & f(x) \\ f(x) & 1 \end{pmatrix}. \tag{4.75}$$

Note that the determinant

$$|g| = 1, \tag{4.76}$$

so the invariant measure (1.6) is the Lebesgue measure

$$\mu = \mathrm{d}x_1\,\mathrm{d}x_2. \tag{4.77}$$

The inverse to (4.75) is

$$(g^{ij}) = \begin{pmatrix} 1 & -f(x) \\ -f(x) & 1+f^2(x) \end{pmatrix}. \tag{4.78}$$

By (1.13) and (4.75), the Christoffel symbols Γ_{kij} are

$$\Gamma_{111} = ff_{,1} \qquad \Gamma_{112} = \Gamma_{121} = ff_{,2} \qquad \Gamma_{122} = f_{,2} \tag{4.79}$$

$$\Gamma_{211} = f_{,1} - ff_{,2} \qquad \Gamma_{212} = \Gamma_{221} = 0 \qquad \Gamma_{222} = 0 \tag{4.80}$$

and by (1.12) and (4.78), the connection components Γ^k_{ij} are

$$\Gamma^1_{11} = f^2f_{,2} \qquad \Gamma^1_{12} = \Gamma^1_{21} = ff_{,2} \qquad \Gamma^1_{22} = f_{,2} \tag{4.81}$$

$$\Gamma^2_{11} = f_{,1} - f(1+f^2)f_{,2} \qquad \Gamma^2_{12} = \Gamma^2_{21} = -f^2f_{,2} \qquad \Gamma^2_{22} = -ff_{,2} \tag{4.82}$$

Consequently, (2.3) yields, after some calculations and massive cancellations,

$$R_{1212} = f_{,12} - ff_{,22} - f_{,2}^2, \tag{4.83}$$

which by (3.1)–(3.8) yields all components R_{ijkl}, R_{ij} and R as well as $K = R/2$; in particular, by (3.5), (3.8), and (4.76)

$$K = \frac{1}{2}R = R_{1212} = f_{,12} - ff_{,22} - f_{,2}^2. \tag{4.84}$$

See Examples 6.31 and 7.22 for examples with constant curvature K.

5. **Examples, flat**

A manifold M is *flat* if it is locally isometric to Euclidean space, or, equivalently the curvature tensor vanishes, $R_{ijkl} = 0$, see Theorem 3.1. (Then also the Ricci tensor, Weyl tensor, Einstein tensor, scalar curvature and sectional curvature vanish, as well as, when $n \geqslant 3$, the Schouten and Cotton tensors.)

Example 5.1 (Constant metric)**.** Let (g_{ij}) be a constant matrix. Then all derivatives $g_{ij,k} = 0$, and thus all $\Gamma^k_{ij} = 0$ and the curvature tensor vanishes by (2.2).

In fact, the manifold is isometric to a subset of Euclidean space with the standard metric $|\mathrm{d}x|^2$ (i.e., $g_{ij} = \delta_{ij}$, so (g_{ij}) is the identity matrix) by a linear map $x \mapsto Ax$, where A is a matrix with $A^2 = (g_{ij})$. (A can be chosen symmetric and positive as the square root, in operator sense, of (g_{ij}).)

For the standard metric, the invariant measure (1.6) is the usual Lebesgue measure.

Example 5.2. The usual polar coordinates in the plane are obtained by Example 4.5 with $w(r) = r$, $r > 0$. More precisely, if $\theta_0 \in \mathbb{R}$ and $M = (0,\infty) \times (\theta_0, \theta_0 + 2\pi)$, then

$$(r, \theta) \mapsto (r\cos\theta, r\sin\theta) \tag{5.1}$$

is an isometry of M onto $\mathbb{R}^2 \setminus \ell$, where ℓ is the ray $\{(r\cos\theta_0, r\sin\theta_0) : r \geqslant 0\}$. (Identifying $\mathbb{R}^2 = \mathbb{C}$, the isometry can be written $(r,\theta) \to re^{\mathrm{i}\theta}$.)

The metric tensor is

$$g_{11} = 1, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = r^2, \tag{5.2}$$

or in matrix form

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{5.3}$$

The connection coefficients are, by (4.22)–(4.23),

$$\Gamma^1_{22} = -r, \tag{5.4}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = 1/r, \tag{5.5}$$

with all other components 0. The invariant measure is

$$\mathrm{d}\mu = r\,\mathrm{d}r\,\mathrm{d}\theta. \tag{5.6}$$

We see again from (4.24)–(4.27) that M is flat, since $w'' = 0$.

The Laplace(–Beltrami) operator is by (1.24) and (5.2)–(5.5),

$$\Delta F = F_{,rr} + r^{-2}F_{,\theta\theta} + r^{-1}F_{,r} = r^{-1}\frac{\partial}{\partial r}\Big(r\frac{\partial F}{\partial r}\Big) + r^{-2}\frac{\partial^2 F}{\partial\theta^2}. \tag{5.7}$$

Example 5.3 (Spherical coordinates in $\mathbb{R}^3$)**.** Consider the general spherical coordinates in Example 4.6 with $n = 3$, $\tilde{M} = \mathbb{S}^2$ (the unit sphere in $\mathbb{R}^3$), $J = (0,\infty)$, the coordinates (φ,θ) (polar angle and longitude) on $\mathbb{S}^2$, see (6.73) below, and the identification $(r,\xi) \mapsto r\xi$ of $J \times \tilde{M}$ with $\mathbb{R}^3 \setminus \{0\}$;

this gives the *spherical coordinates* (r,φ,θ) in $\mathbb{R}^3$, with the map to the corresponding Cartesian coordinates given by[1]

$$(r,\varphi,\theta)\mapsto(r\sin\varphi\,\cos\theta,\ r\sin\varphi\,\sin\theta,\ r\cos\varphi). \tag{5.8}$$

The inverse map is

$$(x,y,z)\mapsto\left(\sqrt{x^2+y^2+z^2},\ \arccos\frac{z}{\sqrt{x^2+y^2+z^2}},\ \arctan\frac{y}{x}\right). \tag{5.9}$$

(Of course, this is not a map defined on all of $\mathbb{R}^3$; we have to exclude the z-axis (where $\varphi=0$ or π) as well as a half-plane with $\theta=\theta_0$ for some θ_0 in order to obtain a diffeomeorphism. We omit the well-known details.)

A small calculation shows that (5.8) implies that the metric tensor is given by the diagonal matrix

$$\bigl(g_{ij}\bigr)=\begin{pmatrix}1&0&0\\0&r^2&0\\0&0&r^2\sin^2\varphi\end{pmatrix}, \tag{5.10}$$

i.e.,

$$g_{rr}=1,\qquad g_{\varphi\varphi}=r^2,\qquad g_{\theta\theta}=r^2\sin^2\varphi, \tag{5.11}$$

with all other (non-diagonal) components 0. (This can also be seen from (4.35) with $w(r)=r$ and (6.74).)

The invariant measure (1.6) is by (5.10)

$$\mathrm{d}\mu=r^2\sin\varphi\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta. \tag{5.12}$$

By (4.40)–(4.43) and (6.76)–(6.77),

$$\Gamma^{\varphi}_{\varphi r}=\Gamma^{\varphi}_{r\varphi}=\Gamma^{\theta}_{\theta r}=\Gamma^{\theta}_{r\theta}=\frac{1}{r}, \tag{5.13}$$

$$\Gamma^{r}_{\varphi\varphi}=-r, \tag{5.14}$$

$$\Gamma^{r}_{\theta\theta}=-r\sin^2\varphi, \tag{5.15}$$

$$\Gamma^{\varphi}_{\theta\theta}=-\sin\varphi\cos\varphi, \tag{5.16}$$

$$\Gamma^{\theta}_{\varphi\theta}=\Gamma^{\theta}_{\theta\varphi}=\cot\varphi, \tag{5.17}$$

with all other components 0.

Since $\mathbb{S}^2$ has constant curvature $\tilde{K}=1$, and $w(r)=r$, we see that (4.55) is satisfied, and, by (4.56), that the metric (5.10) has constant curvature $K=0$, i.e., it is flat. This also follows immediately from the fact that the space is isometric to (a subset of) the Euclidean space $\mathbb{R}^3$ by (5.8).

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[1]Notation varies; for example, physicists often interchange φ and θ. Moreover, we may replace φ by the latitude $\phi=\frac{\pi}{2}-\varphi$, with correponding minor changes below, see Example 6.8.

The Laplace(–Beltrami) operator is by (1.24) and (5.13)–(5.17)

$$\begin{aligned}\Delta F &= F_{,rr} + \frac{1}{r^2}F_{,\varphi\varphi} + \frac{1}{r^2\sin^2\varphi}F_{,\theta\theta} + \frac{2}{r}F_{,r} + \frac{\cot\varphi}{r^2}F_{,\varphi}\\ &= r^{-1}\frac{\partial^2}{\partial r^2}(rF) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\Bigl(\sin\varphi\frac{\partial}{\partial\varphi}F\Bigr) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}F. \qquad (5.18)\end{aligned}$$

6. **Examples, sphere**

Let $\mathbb{S}^n_\rho$ be the sphere of radius $\rho > 0$ in $\mathbb{R}^{n+1}$, centred at the origin 0:

$$\mathbb{S}^n_\rho = \Bigl\{(\xi_1,\dots,\xi_{n+1}) : \sum_{i=1}^{n+1}\xi_i^2 = \rho^2\Bigr\}. \qquad (6.1)$$

$\mathbb{S}^n_\rho$ is an n-dimensional submanifold of $\mathbb{R}^{n+1}$, and we equip it with the induced Riemannian metric. The geodesics are *great circles*, i.e., the intersections of $\mathbb{S}^n_\rho$ and 2-dimensional planes through the origin. The distance $d_{\mathbb{S}^n_\rho}(\xi,\eta)$ between two points $\xi,\eta \in \mathbb{S}^n_\rho$ is ρ times the angle between the radii from 0 to the points; it is given by

$$\cos\left(\frac{d_{\mathbb{S}^n_\rho}(\xi,\eta)}{\rho}\right) = \frac{\langle\xi,\eta\rangle}{\rho^2}. \qquad (6.2)$$

Note that this differs from the Euclidean distance $|\xi-\eta|$ in $\mathbb{R}^{n+1}$; we have

$$\frac{|\xi-\eta|}{\rho} = 2\sin\left(\frac{d_{\mathbb{S}^n_\rho}(\xi,\eta)}{2\rho}\right). \qquad (6.3)$$

The maximum distance in $\mathbb{S}^n_\rho$ is $\pi\rho$ (the distance between two antipodal points).

Recall that the traditional coordinates on the earth are *latitude* and *longitude*, see Example 6.8, that a *meridian* is the set of points with a given longitude and a *parallel* (*parallel of latitude*) is the set of points with a given latitude. The meridians are great semicircles, extending from pole to pole, and are thus geodesics. The *equator* is the parallel with latitude 0; it is a great circle, and thus a geodesic; the other parallels are *small circles*, i.e., intersections of the sphere with planes not passing through the centre, they are thus not geodesics. A meridian and a parallel are always orthogonal to each other.

We consider in this section various projections, mapping (parts of) $\mathbb{S}^n_\rho$ to (parts of) $\mathbb{R}^n$, and the Riemannian metrics they induce in $\mathbb{R}^n$. (No homeomorphism with an open subset of $\mathbb{R}^n$ can be defined on all of $\mathbb{S}^n_\rho$ since $\mathbb{S}^n_\rho$ is compact. We often omit discussions of the domain or range of the maps, leaving the rather obvious details to the reader.) We begin with some projections for general $n \geqslant 2$, but many further examples are for the case $n=2$ only. (Some of them too can be generalized to arbitrary n; we leave that to the eader.) In particular, we consider (for $n = 2$) several important and well-known map projections that have been used by cartographers for a long time (and also some less important ones). See Snyder [15, 16] for many more details of these and many other map projections, including versions for ellipsoids (not considered here) as well as various practical aspects.[2] For example, several map projections below can be regarded as projections onto a plane tangent at the North Pole; by composing the map with a rotation of the sphere, we obtain another map which can be seen as projection onto a tangent plane at some other point.[3] Similarly, the cylindrical projections below (see Example 6.10) have *transverse* versions, where the sphere first is rotated so that a given meridian becomes (half) the equator. In both cases, the new projection will obviously have the same Riemannian metric, and thus the same metrical properties as the original map projection, but it will be different when expressed in, e.g., latitude and longitude. Such variations are important for practical use, but are usually ignored here (although mentioned a few times).

Usually we take for simplicity $\rho = 1$ and consider the unit sphere $\mathbb{S}^n := \mathbb{S}^n_1$ (leaving generalizations to the sphere $\mathbb{S}^n_\rho$ with arbitrary radius ρ to the reader), but we keep a general ρ in the first examples to see the homogeneity properties of different quantities; spheres of different radii ρ_1 and ρ_2 obviously correspond by the map $\xi \mapsto (\rho_2/\rho_1)\xi$, and their Riemannian metrics are the same up to a scaling by ρ_2/ρ_1, see Section 3.5.

We write the points on $\mathbb{S}^n_\rho$ as $\xi = (\xi', \xi_{n+1})$, with $\xi' \in \mathbb{R}^n$.

Example 6.1 (Stereographic projection of a sphere)**.** The *stereographic projection* Φ_P from a point P on the sphere projects the sphere from the point P to a given hyperplane orthogonal to the line through 0 and P, i.e., it maps each point $\xi \neq P$ to the intersection of the line through P and ξ and the given hyperplane.[4] The stereographic projection from the South

[2] We include in footnotes some historical comments taken from Snyder [15, 16].

[3] The two projections are said to have different *aspects*; in this case polar and oblique or equatorial, respectively [16, p. 16]. The same applies to more general azimuthal projections, see Example 6.16.

[4] "The Stereographic projection was probably known in its polar form to the Egyptians, while Hipparchus was apparently the first Greek to use it. He is generally considered its inventor. Ptolemy referred to it as "Planisphaerum," a name used into the 16th century. The name "Stereographic" was assigned to it by François d' Aiguillon in 1613. The polar Stereographic was exclusively used for star maps until perhaps 1507, when the earliest-known use for a map of the world was made by Walther Ludd (Gaultier Lud) of St. Dié, Lorraine. The oblique aspect was used by Theon of Alexandria in the fourth century for maps of the sky, but it was not proposed for geographical maps until Stabius and Werner discussed it together with their cordiform (heart-shaped) projections in the early 16th century. The earliest-known world maps were included in a 1583 atlas by Jacques de Vaulx (c. 1555–97). The two hemispheres were centered on Paris and its opposite point, respectively. The equatorial Stereographic originated with the Arabs, and was used by the Arab astronomer Ibn-el-Zarkali (1029–87) of Toledo for an astrolabe. It became a basis for world maps in the early 16th century, with the earliest-known examples by Jean Roze (or Rotz), a Norman, in 1542. After Rumold (the son of Gerardus) Mercator's use of the equatorial Stereographic for the world maps of the atlas of 1595, it became very popular among cartographers." [15, p. 154]

Pole $S := (0, -\rho)$ onto $\mathbb{R}^n$ maps a point $(\xi', \xi_{n+1}) \in \mathbb{S}^n_\rho$ to

$$\Phi_S(\xi', \xi_{n+1}) = \frac{\rho}{\rho + \xi_{n+1}} \xi' \in \mathbb{R}^n; \tag{6.4}$$

this is a bijection of $\mathbb{S}^n_\rho \setminus \{S\}$ onto $\mathbb{R}^n$ (and is extended to a homeomorphism of $\mathbb{S}^n_\rho$ onto the one-point compactification $\mathbb{R}^n \cup \{\infty\}$ by defining $\Phi_S(S) = \infty$). The inverse map is

$$\Phi_S^{-1}(x) = \Bigl(\frac{2\rho^2}{\rho^2 + |x|^2} x,\ \rho \frac{\rho^2 - |x|^2}{\rho^2 + |x|^2}\Bigr). \tag{6.5}$$

In the coordinates $x = \Phi_S(\xi)$, the Riemannian metric induced on $\mathbb{S}^n_\rho$ by $\mathbb{R}^{n+1}$ is, after a small calculation,

$$|\mathrm{d}s|^2 = |\mathrm{d}\Phi_S^{-1} \cdot \mathrm{d}x|^2 = \frac{4\rho^4}{(\rho^2 + |x|^2)^2} |\mathrm{d}x|^2, \tag{6.6}$$

i.e.,

$$g_{ij} = \frac{4\rho^4}{(\rho^2 + |x|^2)^2} \delta_{ij} = \frac{4}{(1 + |x|^2/\rho^2)^2} \delta_{ij}. \tag{6.7}$$

This is a conformal metric of the form (4.2) with $a = 2$ and $c = 1/\rho^2$; thus (3.10) holds with $f(x) = 2/(1 + |x|^2/\rho^2)$ and $\varphi(x) = \log 2 - \log(1 + |x|^2/\rho^2)$. Since the metric is conformal, $\kappa = 1$ and $\varepsilon = 0$, see Section 1.2. The area scale is

$$|g|^{-1/2} = \frac{(1 + |x|^2/\rho^2)^2}{4}. \tag{6.8}$$

We have, specializing the formulas in Example 4.1,

$$g^{ij} = \frac{(1 + |x|^2/\rho^2)^2}{4} \delta_{ij} \tag{6.9}$$

and

$$\Gamma^k_{ij} = -\frac{2}{\rho^2 + |x|^2} \bigl(x_i \delta_{jk} + x_j \delta_{ik} - x_k \delta_{ij}\bigr); \tag{6.10}$$

the metric has by (4.10) constant sectional curvature

$$K = 1/\rho^2, \tag{6.11}$$

and thus, see (2.36)–(2.39),

$$\begin{aligned} R_{ijkl} &= \frac{1}{2\rho^2} g_{ij} \odot g_{kl} = \frac{1}{\rho^2}(g_{ik} g_{jl} - g_{il} g_{jk}) \\ &= \frac{8\rho^{-2}}{(1 + |x|^2/\rho^2)^4} \delta_{ij} \odot \delta_{kl} \end{aligned} \tag{6.12}$$

and

$$R_{ij} = \frac{n-1}{\rho^2} g_{ij} = \frac{4(n-1)\rho^2}{(\rho^2 + |x|^2)^2} \delta_{ij}, \tag{6.13}$$

$$R = n(n-1)\rho^{-2}. \tag{6.14}$$

See also [16, p. 169–170] for modern use.

The invariant measure (1.6) is by (4.3)

$$\mathrm{d}\mu = \frac{2^n}{(1+|x|^2/\rho^2)^n}\,\mathrm{d}x_1\cdots\mathrm{d}x_n. \tag{6.15}$$

The distance between two points $x, y \in \mathbb{R}^n$ is by (6.2) given by

$$\cos\Bigl(\frac{d(x,y)}{\rho}\Bigr) = \cos\Bigl(\frac{d_{\mathbb{S}^n_\rho}(\Phi_S^{-1}(x),\Phi_S^{-1}(y))}{\rho}\Bigr) = \frac{\langle \Phi_S^{-1}(x),\Phi_S^{-1}(y)\rangle}{\rho^2}$$
$$= 1 - \frac{2\rho^2|x-y|^2}{(\rho^2+|x|^2)(\rho^2+|y|^2)}. \tag{6.16}$$

Note that, as said above, the maximum distance is $\pi\rho$ (the distance between two antipodal points).

In particular,

$$\cos\Bigl(\frac{d(x,0)}{\rho}\Bigr) = 1 - \frac{2|x|^2}{\rho^2+|x|^2} = \frac{1-|x|^2/\rho^2}{1+|x|^2/\rho^2}, \tag{6.17}$$

which by standard trigonometric formulas leads to

$$\sin\Bigl(\frac{d(x,0)}{\rho}\Bigr) = \frac{2|x|/\rho}{1+|x|^2/\rho^2}, \tag{6.18}$$
$$\tan\Bigl(\frac{d(x,0)}{\rho}\Bigr) = \frac{2|x|/\rho}{1-|x|^2/\rho^2}, \tag{6.19}$$
$$\cos\Bigl(\frac{d(x,0)}{2\rho}\Bigr) = \frac{1}{\sqrt{1+|x|^2/\rho^2}}, \tag{6.20}$$
$$\sin\Bigl(\frac{d(x,0)}{2\rho}\Bigr) = \frac{|x|/\rho}{\sqrt{1+|x|^2/\rho^2}}, \tag{6.21}$$
$$\tan\Bigl(\frac{d(x,0)}{2\rho}\Bigr) = \frac{|x|}{\rho}. \tag{6.22}$$

Note that a point on the sphere with polar angle φ (and thus latitude $\frac{\pi}{2}-\varphi$, see Examples 6.7–6.8 below) has distance $\rho\varphi$ to the North Pole N; thus the corresponding point $x \in \mathbb{R}^n$ has $d(x,0) = \rho\varphi$ (since N corresponds to $0 \in \mathbb{R}^n$), and thus (6.22) yields

$$|x| = \rho\tan\frac{\varphi}{2}, \tag{6.23}$$

which also easily is seen geometrically.

It is easily seen from (6.16) that an $n-1$-dimensional sphere A (a circle when $n=2$) on $\mathbb{S}^n_\rho$ is mapped to a Euclidean $n-1$-dimensional sphere or hyperplane (the latter when A goes through S).

Note also that it follows from (6.4) that two antipodal points ξ and $-\xi$ on $\mathbb{S}^n_\rho$ are mapped to two points $\Phi_S(\xi)$ and $\Phi_S(-\xi)$ on the same line through 0, but on opposite sides, and satisfying $|\Phi_S(\xi)|\cdot|\Phi_S(-\xi)| = \rho^2$. In other words, $\Phi_S(\xi)$ and $-\Phi_S(-\xi)$ are the reflections of each other in the sphere

$|x| = \rho$. In the case $n = 2$, regarding Φ_S as a map to the complex plane $\mathbb{C}$, or rather to $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$, this says

$$\Phi_S(-\xi) = -\frac{\rho^2}{\overline{\Phi_S(\xi)}}. \tag{6.24}$$

Example 6.2 (Stereographic projection from the North Pole)**.** The stereographic projection from the North Pole $N := (0, \rho)$ of $\mathbb{S}^n_\rho$ (with $\mathbb{S}^n_\rho$ as in Example 6.1) maps a point $(\xi', \xi_{n+1}) \in \mathbb{S}^n_\rho$ to

$$\Phi_N(\xi', \xi_{n+1}) = \frac{\rho}{\rho - \xi_{n+1}}\xi' = \Phi_S(\xi', -\xi_{n+1}) \in \mathbb{R}^n; \tag{6.25}$$

this is a bijection of $\mathbb{S}^n_\rho \setminus \{S\}$ onto $\mathbb{R}^n$ with inverse

$$\Phi_S^{-1}(x) = \Bigl(\frac{2\rho^2}{\rho^2 + |x|^2}x,\ \rho\frac{|x|^2 - \rho^2}{|x|^2 + \rho^2}\Bigr). \tag{6.26}$$

Φ_N is another isometry of $\mathbb{S}^n_\rho$ minus a point onto $\mathbb{R}^n$ with the metric (6.7). (Note that Φ_S and Φ_N together are two coordinate systems covering $\mathbb{S}_\rho$; thus they together define the manifold structure.)

By (6.25) and (6.23), if a point with polar angle φ is mapped to $x \in \mathbb{R}^n$, then

$$|x| = \rho \tan\frac{\pi - \varphi}{2} = \rho \cot\frac{\varphi}{2}. \tag{6.27}$$

It is easily seen that if $\xi \neq N, S$ and $\Phi_S(\xi) = x$, then $\Phi_N(\xi) = \rho^2 x/|x|^2$, i.e., $\Phi_S(x)$ and $\Phi_N(x)$ are related by reflection in the sphere $\{x : |x| = \rho\}$. Hence this reflection is an isometry of $\mathbb{R}^n$ with the metric (6.7), which also easily is verified directly as said in Remark 4.4.

Example 6.3 (Alternative stereographic projection of a sphere)**.** The stereographic projection Φ_S in Example 6.1 is geometrically the stereographic projection from the South Pole onto the hyperplane $\xi_{n+1} = 0$ in $\mathbb{R}^{n+1}$. An alternative is to project onto the plane $\xi_{n+1} = \rho$ tangent to the North Pole. This maps $(\xi', \xi_{n+1}) \in \mathbb{S}^n_\rho$ to

$$2\Phi_S(\xi', \xi_{n+1}) = \frac{2\rho}{\rho + \xi_{n+1}}\xi' \in \mathbb{R}^n, \tag{6.28}$$

and yields the Riemannian metric

$$|\mathrm{d}s|^2 = \frac{\rho^4}{(\rho^2 + |x|^2/4)^2}|\mathrm{d}x|^2, \tag{6.29}$$

i.e.,

$$g_{ij} = \frac{\rho^4}{(\rho^2 + |x|^2/4)^2}\delta_{ij} = \frac{1}{(1 + |x|^2/4\rho^2)^2}\delta_{ij}, \tag{6.30}$$

which is a conformal metric of the form (4.2) with $a = 1$ and $c = 1/4\rho^2$. (Compared to (6.7), this has the advantage of reducing to δ_{ij} at $x = 0$, but is otherwise often less convenient.)

We have by (4.10), of course, the same constant sectional curvature

$$K = 1/\rho^2, \tag{6.31}$$

and thus by (2.36)–(2.39), cf. (6.12)–(6.14),

$$\begin{aligned} R_{ijkl} &= \frac{1}{2\rho^2} g_{ij} \odot g_{kl} = \frac{1}{\rho^2}(g_{ik}g_{jl} - g_{il}g_{jk}) \\ &= \frac{\rho^{-2}}{2(1+|x|^2/4\rho^2)^4} \delta_{ij} \odot \delta_{kl} \end{aligned} \tag{6.32}$$

and

$$R_{ij} = \frac{n-1}{\rho^2} g_{ij} = \frac{(n-1)\rho^2}{(\rho^2+|x|^2/4)^2}\delta_{ij}, \tag{6.33}$$

$$R = n(n-1)\rho^{-2}. \tag{6.34}$$

The invariant measure (1.6) is by (4.3)

$$\mathrm{d}\mu = (1+|x|^2/4\rho^2)^{-n}\,\mathrm{d}x_1\cdots\mathrm{d}x_n. \tag{6.35}$$

Example 6.4 (Stereographic projections from other points)**.** As said in Example 6.1, we can take the stereographic projection Φ_P from any point P on the sphere, mapping onto the hyperplane $L_P \cong \mathbb{R}^n$ through 0 orthogonal to P (or, as in Example 6.3, the hyperplane tangent at the antipodal point $-P$); Example 6.2 is just one example. This can also be described as first rotating the sphere and then projecting as in Example 6.1. The resulting metric on $\mathbb{R}^n$ is the same and given by (6.6)–(6.7).

In the case $n = 2$ and (for simplicity) $\rho = 1$, we may identify L_P with $\mathbb{C}$ (in an orientation preserving way), and then each such projection Φ_P equals $h \circ \Phi_S$, where Φ_S is the standard stereographic projection in Example 6.1 and $h : \mathbb{C}^* \to \mathbb{C}^*$ is some analytic Möbius transformation

$$z \mapsto h(z) = \frac{az+b}{cz+d} \tag{6.36}$$

with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$ (we may assume $ad - bc = 1$), such that

$$|a| = |d|, \qquad |b| = |c|, \qquad a\bar{b} = -c\bar{d}. \tag{6.37}$$

Conversely, each $h \circ \Phi_S$, where h is a Möbius map (6.36) satisfying (6.37), equals Φ_P for some P and some identification of L_P and $\mathbb{C}$. Moreover, if we assume (as we may) $ad - bc = 1$, then

$$|a| = |d| = \sin\frac{\varphi}{2}, \qquad |b| = |c| = \cos\frac{\varphi}{2} \tag{6.38}$$

where φ is the polar angle of P, i.e., the distance from P to N.

If we use an orientation-reversing identification between L_P and $\mathbb{C}$, then Φ_P instead equals the complex conjugate $\overline{h \circ \Phi_S}$ for some h as above.

Note that the identification between L_P and $\mathbb{C}$ is not unique; different identifications are obtained by rotating the complex plane, i.e., multiplications by a complex constant ω with $|\omega| = 1$, which corresponds to multiplying h by ω, and to multiplying a and b in (6.36) by ω.

The set of Möbius transformations (6.36) satisfying (6.37) equals the set of Möbius transformations that preserve the relation $z_1\bar{z}_2 = -1$ for pairs of points $(z_1, z_2) \in \mathbb{C}$; this is not surprising since this relation by (6.24) characterizes pairs of antipodes in any of the stereographic projections of $\mathbb{S}^2$. (Thus (6.37) defines a subgroup of the group $PSL(2,\mathbb{C})$ of all Möbius transformations that is isomorphic to the group $SO(3)$ of rotations of $\mathbb{S}^2$.)

See also Example 6.37.

We give explicit examples for the 6 intersection points of $\mathbb{S}^2$ and the coordinate axes; these are the North Pole $N = (0,0,1)$ and South Pole $S = (0,0,-1)$ and the four points on the equator with longitudes 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ (or $-\frac{\pi}{2}$), which we denote by O, E, Z, W. (E and W are the "East Pole" and "West Pole".) For Φ_N and Φ_S we choose the natural identification of $L_N = L_S = \mathbb{R}^2$ and $\mathbb{C}$; note that this is orientation-preserving for S but not for N; for the points P on the equator we choose the orientation-preserving identification with north along the positive imaginary axis, and east along the positive real axis. The 6 stereographic projections from these points then maps these 6 points to complex numbers as shown in Table 1, where the last column shows Φ_P as a Möbius transformation of Φ_S (and a conjugation for Φ_N); this column is easily verified from the others, since a Möbius transformation is uniquely determined by the images of three points. (For convenience, we have here not normalized the Möbius transformations (6.36) to have $ad - bc = 1$.)

	N	S	E	W	O	Z	
Φ_N	∞	0	i	$-$i	1	-1	$1/\overline{\Phi_S}$
Φ_S	0	∞	i	$-$i	1	-1	Φ_S
Φ_E	i	$-$i	∞	0	1	-1	$-\mathrm{i}(\Phi_S + \mathrm{i})/(\Phi_S - \mathrm{i})$
Φ_W	i	$-$i	0	∞	-1	1	$-\mathrm{i}(\Phi_S - \mathrm{i})/(\Phi_S + \mathrm{i})$
Φ_O	i	$-$i	-1	1	∞	0	$-\mathrm{i}(\Phi_S + 1)/(\Phi_S - 1)$
Φ_Z	i	$-$i	1	-1	0	∞	$-\mathrm{i}(\Phi_S - 1)/(\Phi_S + 1)$

Table 1. Some stereographic projections

For future use we give some explicit formulas for Φ_Z and the latitude and longitude (ϕ, θ). The Cartesian coordinates (x_1, x_2, x_3) of the point are $\bigl(\cos\phi\,\cos\theta,\ \cos\phi\,\sin\theta,\ \sin\phi\bigr)$, see (6.79), and thus

$$\Phi_Z = \frac{x_2 + \mathrm{i}x_3}{1 + x_1} = \frac{\cos\phi\,\sin\theta + \mathrm{i}\sin\phi}{1 + \cos\phi\,\cos\theta}. \tag{6.39}$$

It follows that

$$x_1 = \frac{1 - |\Phi_Z|^2}{1 + |\Phi_Z|^2}, \tag{6.40}$$

$$x_2 = (1 + x_1)\operatorname{Re}\Phi_Z = \frac{\Phi_Z + \overline{\Phi}_Z}{1 + |\Phi_Z|^2}, \tag{6.41}$$

$$x_3 = (1 + x_1)\operatorname{Im}\Phi_Z = -\mathrm{i}\frac{\Phi_Z - \overline{\Phi}_Z}{1 + |\Phi_Z|^2}. \tag{6.42}$$

$$\tag{6.43}$$

Since $\sin\phi = x_3$, $\cos\phi\cos\theta = x_1$ and $\cos\phi\sin\theta = x_2$, elementary calculations yield

$$\sin\phi = -\mathrm{i}\frac{\Phi_Z - \overline{\Phi}_Z}{1 + |\Phi_Z|^2}, \tag{6.44}$$

$$\cos\phi = \frac{|1 + \Phi_Z^2|}{1 + |\Phi_Z|^2}, \tag{6.45}$$

$$\cos\theta = \frac{1 - |\Phi_Z|^2}{|1 + \Phi_Z^2|}, \tag{6.46}$$

$$\sin\theta = \frac{\Phi_Z + \overline{\Phi}_Z}{|1 + \Phi_Z^2|}. \tag{6.47}$$

Example 6.5 (Gnomonic (central) projection)**.** The *gnomonic projection* projects the sphere $\mathbb{S}^n$ from the centre 0 to a tangent plane. Obviously, such a map projection cannot distinguish between antipodes, so it is limited to a hemisphere.[5]

We choose to project the upper hemisphere $\{\xi \in \mathbb{S}^n : \xi_{n+1} > 0\}$ onto the tangent plane at the North Pole $\{(x, 1) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$. We thus map a point $(\xi', \xi_{n+1}) \in \mathbb{S}^n$ with $\xi_{n+1} > 0$ to

$$\Psi(\xi', \xi_{n+1}) = \frac{\xi'}{\xi_{n+1}} \in \mathbb{R}^n. \tag{6.48}$$

The inverse map is

$$\Psi^{-1}(x) = \frac{(x, 1)}{\sqrt{1 + |x|^2}}. \tag{6.49}$$

A small calculation shows that the metric is

$$|\mathrm{d}s|^2 = \frac{|\mathrm{d}x|^2}{1 + |x|^2} - \frac{|\langle \mathrm{d}x, x\rangle|^2}{(1 + |x|^2)^2} = \frac{|\mathrm{d}x|^2 + |x \wedge \mathrm{d}x|^2}{(1 + |x|^2)^2}, \tag{6.50}$$

i.e.,

$$g_{ij} = \frac{\delta_{ij}}{1 + |x|^2} - \frac{x_i x_j}{(1 + |x|^2)^2}. \tag{6.51}$$

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[5]The gnomonic projection is also one of the oldest used in cartography. It is useful for long-distance navigation, e.g. for intercontinental flights, since it shows geodesics (the shortest paths) as straight lines (this includes the equator and all meridians) [16, p. 169].

"It was used by Thales (636?–546?B.C.) of Miletus for star maps. Called "horologium" (sundial or clock) in early times, it was given the name "gnomonic" in the 19th century. It has also been called the Gnomic and the Central projection. The name Gnomonic is derived from the fact that the meridians radiate from the pole (or are spaced, on the equatorial aspect) just as the corresponding hour markings on a sundial for the same central latitude. The gnomon of the sundial is the elevated straightedge pointed toward the pole and casting its shadow on the various hour markings as the sun moves across the sky." [15, p. 164]

The metric is thus not conformal. The inverse matrix is, see Lemma E.1,

$$g^{ij} = (1 + |x|^2)(\delta_{ij} + x_i x_j). \tag{6.52}$$

The determinant is, again using Lemma E.1,

$$|g| = (1 + |x|^2)^{-n-1}, \tag{6.53}$$

and the invariant measure (1.6) is thus

$$\mathrm{d}\mu = (1 + |x|^2)^{-(n+1)/2} \, \mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{6.54}$$

The volume scale is thus $(1 + |x|^2)^{(n+1)/2}$.

By (6.50)–(6.51), the largest and smallest eigenvalues of g_{ij} are $(1+|x|^2)^{-1}$ and $(1 + |x|^2)^{-2}$, respectively, and thus the condition number is

$$\varkappa = 1 + |x|^2. \tag{6.55}$$

By (1.7), the eccentricity is

$$\varepsilon = \frac{|x|}{\sqrt{1 + |x|^2}}. \tag{6.56}$$

Calculations yield

$$\Gamma_{kij} = -\frac{x_i\delta_{jk} + x_j\delta_{ik}}{(1 + |x|^2)^2} + \frac{2x_i x_j x_k}{(1 + |x|^2)^3} = -\frac{x_i g_{jk} + x_j g_{ik}}{1 + |x|^2}, \tag{6.57}$$

$$\Gamma^k_{ij} = -\frac{x_i\delta_{jk} + x_j\delta_{ik}}{1 + |x|^2} \tag{6.58}$$

The connection is of the form (1.37), and thus the geodesics are Euclidean lines; this is also obvious geometrically, since the geodesics on the sphere are intersections of the sphere with 2-dimensional planes through the centre, which project to the intersection of the plane with the plane $\{(x, 1) : x \in \mathbb{R}^n\}$, which is a line in that plane. The equation (1.36) for a geodesic becomes

$$\ddot{\gamma}^i = \frac{2\sum_j \gamma^j \dot{\gamma}^j}{1 + |\gamma|^2}\dot{\gamma}^i = \frac{\mathrm{d}\log(1 + |\gamma|^2)}{\mathrm{d}t}\dot{\gamma}^i. \tag{6.59}$$

A geodesic can be parametrized (with unit speed) as

$$\gamma(t) = x + \tan(t) \cdot v \tag{6.60}$$

where $x = \gamma(0) \in \mathbb{R}^n$, $v = \dot{\gamma}(0) \in \mathbb{R}^n$ with $x \perp v$ and $|v|^2 = 1 + |x|^2$.

The gnomonic projection evidently maps a circle on the sphere to a conic section, i.e., a circle, an ellipse, a parabola or (one branch of) a hyperbola, or (in the case of a geodesic) a straight line. (Of course, only the part of the circle in the upper hemisphere is mapped.)

Example 6.6 (Orthogonal (orthographic) projection)**.** The *orthogonal projection* of the sphere $\mathbb{S}^n$ onto $\mathbb{R}^n$ is in cartography called the *orthographic* *projection*.[6] The orthogonal (orthographic) projection is thus the map

$$(\xi', \xi_{n+1}) \mapsto \xi'. \tag{6.61}$$

This is a diffeomorphism in each of the two hemispheres $\xi_{n+1} > 0$ and $\xi_{n+1} < 0$, mapping each of them onto the open unit ball $\{x : |x| < 1\}$ in $\mathbb{R}^n$; if we choose to regard it as a map from the upper hemisphere, its inverse is

$$x \mapsto \bigl(x, \sqrt{1-|x|^2}\bigr), \qquad |x| < 1. \tag{6.62}$$

A small calculation shows that the metric is

$$|\mathrm{d}s|^2 = |\mathrm{d}x|^2 + \frac{|\langle \mathrm{d}x, x\rangle|^2}{1-|x|^2} \tag{6.63}$$

i.e.,

$$g_{ij} = \delta_{ij} + \frac{x_i x_j}{1-|x|^2}. \tag{6.64}$$

The metric is thus not conformal. By Lemma E.1, the determinant is

$$|g| = 1 + \frac{|x|^2}{1-|x|^2} = \frac{1}{1-|x|^2}, \tag{6.65}$$

and the inverse matrix is

$$g^{ij} = \delta_{ij} - x_i x_j. \tag{6.66}$$

The invariant measure (1.6) is thus

$$\mathrm{d}\mu = (1-|x|^2)^{-1/2}\,\mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{6.67}$$

I.e., the volume scale is $(1-|x|^2)^{1/2}$.

By (6.63)–(6.64), the largest and smallest eigenvalues of g_{ij} are $(1-|x|^2)^{-1}$ and 1, respectively, and thus the condition number is

$$\varkappa = (1-|x|^2)^{-1}. \tag{6.68}$$

[6] This is how the sphere looks from a great distance, for example (approximatively) the earth seen (or photographed) from the moon; conversely, this is how we see the surface of the moon.

"To the layman, the best known perspective azimuthal projection is the Orthographic, although it is the least useful for measurements. While its distortion in shape and area is quite severe near the edges, and only one hemisphere may be shown on a single map, the eye is much more willing to forgive this distortion than to forgive that of the Mercator projection because the Orthographic projection makes the map look very much like a globe appears, especially in the oblique aspect.

The Egyptians were probably aware of the Orthographic projection, and Hipparchus of Greece (2nd century B.C.) used the equatorial aspect for astronomical calculations. Its early name was "analemma," a name also used by Ptolemy, but it was replaced by "orthographic" in 1613 by François d' Aiguillon of Antwerp. While it was also used by Indians and Arabs for astronomical purposes, it is not known to have been used for world maps older than 16th-century works by Albrecht Dürer (1471–1528), the German artist and cartographer, who prepared polar and equatorial versions." [15, p. 145]

"The Roman architect and engineer Marcus Vitruvius Pollio, ca. 14 B.C., used it to construct sundials and to compute sun positions." Vitruvius also, apparently, originated the term orthographic. [16, p. 17]

See also [16, p. 170] for modern use.

By (1.7), the eccentricity is

$$\varepsilon = |x|. \tag{6.69}$$

Calculations yield

$$\Gamma_{kij} = \frac{x_k\delta_{ij}}{1-|x|^2} + \frac{x_i x_j x_k}{(1-|x|^2)^2} = \frac{x_k g_{ij}}{1-|x|^2}, \tag{6.70}$$

$$\Gamma^k_{ij} = x_k g_{ij}. \tag{6.71}$$

Example 6.7 (Polar coordinates on a sphere)**.** Consider Example 4.5 with $w(r) = \sin r$, $r \in (0,\pi)$. Then $w''/w = -1$, and (4.27) shows that M has constant curvature 1.

In fact, if $M = (0,\pi) \times (\theta_0, \theta_0 + 2\pi)$, for some $\theta_0 \in \mathbb{R}$, then

$$(r,\theta) \mapsto (\sin r\cos\theta, \sin r\sin\theta, \cos r) \tag{6.72}$$

is an isometry of M onto $\mathbb{S}^2\setminus\ell$, where ℓ is the meridian $\{(\sin r\cos\theta_0, \sin r\sin\theta_0, \cos r) : r \geqslant 0\}$. r is the distance to the North Pole $N = (0,0,1)$, so this is polar coordinates on the sphere $\mathbb{S}^2$ with the North Pole as centre. In this context, r is usually denoted φ; $\varphi = r$ is called the *polar angle* (or *zenith angle* or *colatitude*); it is the angle from the North Pole N, seen from the centre of the sphere. The map (6.72) to $\mathbb{S}^2$ is thus

$$(\varphi,\theta) \mapsto (\sin\varphi\,\cos\theta,\, \sin\varphi\,\sin\theta,\, \cos\varphi). \tag{6.73}$$

The metric tensor is, using the notation (φ,θ),

$$g_{11} = 1, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = \sin^2\varphi, \tag{6.74}$$

or in matrix form

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\varphi \end{pmatrix}. \tag{6.75}$$

The connection coefficients are, by (4.22)–(4.23),

$$\Gamma^1_{22} = -\sin\varphi\cos\varphi, \tag{6.76}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \cot\varphi, \tag{6.77}$$

with all other components 0. The invariant measure is

$$\mathrm{d}\mu = \sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta. \tag{6.78}$$

Example 6.8 (Latitude and longitude)**.** *Latitude* and *longitude*, the traditional coordinate system on a sphere,[7] is the pair of coordinates (ϕ,θ) obtained from the coordinates (φ,θ) in Example 6.7 by $\phi := \pi/2 - \varphi$; in other words, the point (ϕ,θ) has Cartesian coordinates

$$\big(\cos\phi\,\cos\theta,\, \cos\phi\,\sin\theta,\, \sin\phi\big). \tag{6.79}$$

[7] First used in a formalized way by Hipparchus from Rhodes (ca. 190–after 126 B.C.) [16, p. 4].

We take $\phi \in [-\pi/2, \pi/2]$. Traditionally, either $\theta \in [-\pi, \pi]$ (with negative values West and positive values East) or $\theta \in [0, 2\pi)$;[8] for a coordinate map in differential geometry sense we take $\phi \in (-\pi/2, \pi/2)$ and $\theta \in (\theta_0, \theta_0 + 2\pi)$, omitting a meridian.[9]

The metric tensor is, e.g. by (6.74),

$$g_{11} = 1, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = \cos^2 \phi, \tag{6.80}$$

or in matrix form

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 \phi \end{pmatrix}. \tag{6.81}$$

This is another instance of Example 4.5, now with (using ϕ instead of r) $w(\phi) = \cos \phi$, $\phi \in (-\pi/2, \pi/2)$. We again have $w''/w = -1$, and (4.27) shows again that the sphere $\mathbb{S}^2$ has constant curvature 1.

The connection coefficients are, by (4.22)–(4.23),

$$\Gamma^1_{22} = \sin \phi \, \cos \phi, \tag{6.82}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = -\tan \phi, \tag{6.83}$$

with all other components 0. The invariant measure is

$$\mathrm{d}\mu = \cos \phi \, \mathrm{d}\phi \, \mathrm{d}\theta. \tag{6.84}$$

Example 6.9 (Equirectangular projection)**.** This is simply the coordinates $(\phi, a\theta)$, where $a > 0$ is a constant. This is thus obtained from latitude and longitude by scaling one of the coordinates. Meridians and parallels (spaced at, say, 1°) form two perpendicular families of parallel lines, each of them uniformly spaced, which thus together form rectangles of equal size.[10]

The special case $a = 1$ is thus just (ϕ, φ) as in Example 6.8; in this case, meridians and parallels form squares. As a map projection, this special case is called *plate carrée* or *plane chart.* [16, p. 5].[11]

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[8] More precisely, $[-180°, 180°]$ or $[0°, 360°)$, since traditionally degrees are used. (Astronomers sometimes also use $[0^h, 24^h)$.) In geography, $[-180°, 180°]$ is standard, while astronomers use $[0°, 360°)$ [1, pp. 11 and 203].

0° is currently at the meridian through Greenwich; various other standard meridians have been used on older maps. On the celestial sphere, 0° is at the vernal equinox (first point of Aries).

Similarly, latitude is traditionally given in $[-90°, 90°]$, with positive values North and negative values South.

[9] Coordinates are traditionally given in the order (ϕ, θ), but maps are plotted with the coordinates in the order (θ, ϕ), i.e., with the longitude along the x-axis. We ignore this trivial difference. In other words, we regard maps as plotted with the x-axis vertical (North) and the y-axis horizontal (East). (The same applies to several projections discussed below.)

[10] This explains the name *equirectangular* or *rectangular* projection. This projection was invented by Marinus of Tyre about 100 A.D., and was widely used until the 17th century; it has also some modern use [16, pp. 4–5, 158]

[11] The transverse form of plate carée is called *Cassini's projection.* It was invented in 1745 (in the ellipsoidal version) by César François Cassini de Thury (1714–1784), and has

The metric tensor is by (6.80),

$$g_{11} = 1, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = a^{-2}\cos^2\phi, \tag{6.85}$$

or in matrix form

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & a^{-2}\cos^2\phi \end{pmatrix}. \tag{6.86}$$

Note that if $a = \cos\varphi_0$ for some $\varphi_0 \in [0, \pi/2)$ (and thus $0 < a \leqslant 1$), then $g_{ij} = \delta_{ij}$ when $\varphi = \pm\varphi_0$, so the map is locally conformal and isometric at latitude $\varphi = \pm\varphi_0$; these parallels are called *standard parallels*. In particular, with $a = 1$ as in Example 6.8, the equator is the standard parallel.

The metric (6.85) is another instance of Example 4.5, now with (using ϕ instead of r) $w(\phi) = a\cos\phi$, $\phi \in (-\pi/2, \pi/2)$. We again have $w''/w = -1$, and (4.27) shows again that the sphere $\mathbb{S}^2$ has constant curvature 1.

The connection coefficients are, by (4.22)–(4.23),

$$\Gamma^1_{22} = a^2 \sin\phi\,\cos\phi, \tag{6.87}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = -\tan\phi, \tag{6.88}$$

with all other components 0. The invariant measure is

$$\mathrm{d}\mu = a\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\theta. \tag{6.89}$$

The area scale is thus $a^{-1}\cos^{-1}\phi$.

By (6.85), the condition number is

$$\varkappa = \max\bigl(a^{-2}\cos^2\varphi, a^2\cos^{-2}\varphi\bigr). \tag{6.90}$$

If $a = \cos\varphi_0$, we see again that $\varkappa = 1$, and thus $\varepsilon = 0$, if and only if $\varphi = \pm\varphi_0$.

Example 6.10 (Cylindrical projections)**.** Let (ϕ, θ) be the latitude and longitude as in Example 6.8 and consider the transformation $\zeta = \zeta(\phi)$ of the latitude, where ζ is a smooth function with (smooth) inverse $\phi(\zeta)$. This yields the coordinates

$$(\zeta, \theta) = (\zeta(\phi), \theta) \tag{6.91}$$

on $\mathbb{S}^2$. A map projection of this type is called a *cylindrical projection*. The metric tensor is by (6.80)

$$g_{11} = (\phi'(\zeta))^2, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = \cos^2(\phi(\zeta)). \tag{6.92}$$

This is a metric of the type in Example 4.7.

The meridians are equispaced vertical lines and the parallels are horizontal lines (but not necessarily equispaced). In particular, meridians and parallels are orthogonal to each other; however, the projection in general

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been used in several countries, in particular during the 19th century [16, pp. 74–76, 97, 159].

	$\zeta(\phi)$	$\phi(\zeta)$
plate carée (Examples 6.8–6.9)	ϕ	ζ
equirectangular (Example 6.9)	$a^{-1}\phi$	$a\zeta$
Mercator (Example 6.11)	$\log\tan(\frac{\phi}{2}+\frac{\pi}{4})$	$2\arctan(e^{\zeta})-\frac{\pi}{2}$
central (Example 6.12)	$\tan(\phi)$	$\arctan(\zeta)$
equal-area (Example 6.13)	$\sin(\phi)$	$\arcsin(\zeta)$
modified equal-area (Example 6.14)	$a\sin(\phi)$	$\arcsin(\zeta/a)$
Gall stereographic (Example 6.15)	$a\tan(\phi/2)$	$2\arctan(\zeta/a)$

Table 2. Some cylindrical projections

is not conformal. In fact, (6.92) shows that the (local) scales along meridians and parallels are typically different, and that the projection is locally conformal at a point if and only if

$$\phi'(\zeta) = \cos(\phi(\zeta)), \tag{6.93}$$

or equivalently

$$\zeta'(\phi) = \frac{1}{\phi'(\zeta(\phi))} = \frac{1}{\cos\phi}. \tag{6.94}$$

This depends on the latitude but not on the longitude. Parallels where (6.94) holds are called *standard parallels*.

The plate carée (latitude and longitude) in Examples 6.8–6.9 is a trivial example of a cylindric projection, and so is the general equirectangular projection in Example 6.9 (up to an unimportant scale factor a). Several other cylindrical projections are discussed in the examples below, see Table 2. (In all these examples, $\zeta(\phi)$ is an odd function, which puts the equator at $\zeta = 0$ and gives a mirror symmetry between the northern and southern hemispheres; this is a very natural condition, but it is formally not needed.) Further examples are given in [16, pp. 178–179, 183–184].

Example 6.11 (Mercator's projection)**.** In particular, a cylindrical projection (6.91) is conformal if and only if (6.94) holds for all ϕ, which has the solution (up to an additive constant)

$$\zeta(\phi) = \log\tan\Bigl(\frac{\phi}{2}+\frac{\pi}{4}\Bigr). \tag{6.95}$$

Consequently, the coordinates

$$(\zeta,\theta) = \biggl(\log\tan\Bigl(\frac{\phi}{2}+\frac{\pi}{4}\Bigr),\,\theta\biggr) = \Bigl(-\log\tan\frac{\varphi}{2},\,\theta\Bigr), \tag{6.96}$$

with φ the polar angle in Example 6.7, yield a conformal coordinate system; this (or rather (θ,ζ)) is the classical *Mercator's projection*.[12] The projection

[12] "While the projection was apparently used by Erhard Etzlaub (1462–1532) of Nuremberg on a small map on the cover of some sundials constructed in 1511 and 1513, the principle remained obscure until Gerardus Mercator (1512–94) independently developed it and presented it in 1569 on a large world map of 21 sections totaling about 1.3 by 2 m.

composed with a rotation of the sphere, so that the equator of the projection is a meridian on the sphere, is called the *transverse Mercator projection* or the *Gauss conformal projection.*[13]

If we regard the stereographic projection Φ_N from the North Pole in Example 6.2 as a mapping into the complex plane, then a point with coordinates (φ, θ) on the sphere is by (6.27) mapped to $\Phi_N(\varphi, \theta) = \cot \frac{\varphi}{2} \cdot e^{\mathrm{i}\theta}$ and thus (6.96) can be written

$$\zeta + \mathrm{i}\theta = \log \Phi_N(\varphi, \theta). \tag{6.97}$$

In other words, Mercator's projection equals the (complex) logarithm of the stereographic projection. (This yields another proof that the projection is conformal, since the stereographic projection is.) With the stereographic projection Φ_S in Example 6.1, the same holds except that the sign of ζ has to be reversed.

The change of coordinates (6.96) can be expressed in different ways. We have, for example,

$$\sin \phi = \cos \varphi = \tanh \zeta, \tag{6.98}$$

His 1569 map is entitled "Nova et Aucta Orbis Terrae Descriptio ad Usum Navigantium Emendate Accommodata (A new and enlarged description of the Earth with corrections for use in navigation)." He described in Latin the nature of the projection in a large panel covering much of his portrayal of North America:

In this mapping of the world we have [desired] to spread out the surface of the globe into a plane that the places shall everywhere be properly located, not only with respect to their true direction and distance, one from another, but also in accordance with their due longitude and latitude; and further, that the shape of the lands, as they appear on the globe, shall be preserved as far as possible. For this there was needed a new arrangement and placing of meridians, so that they shall become parallels, for the maps hitherto produced by geographers are, on account of the curving and the bending of the meridians, unsuitable for navigation Taking all this into consideration, we have somewhat increased the degrees of latitude toward each pole, in proportion to the increase of the parallels beyond the ratio they really have to the equator" [15, p. 38]

The main practical feature for navigation is that meridians are parallel and that sailing routes keeping a constant direction with respect to North are shown as straight lines. These lines are called *loxodromes* or *rhumb lines*; they are not geodesics, i.e. the shortest paths (unless on a meridian or the equator), but they are convenient for navigation. On the other hand, distances and the area are much distorted at high latitudes as shown by (6.102)–(6.103) and (6.99). See [15, p. 39–41].

Mercator's projection is thus the only projection really suitable for navigation (at least before the GPS), and it its thus used for nautical charts. It is also often used for other purposes, such as world maps, where it really is a bad choice; it may be attractive for wall charts because it is rectangular, but it has large distortions at high (and even not so high) latitudes. See also [16, p. 156–157].

[13]It was really invented by Johann Heinrich Lambert in 1772 (together with six other map projections, see [16, p. 76] and Examples 6.13, 6.18, 6.26, 6.27); the version for ellipsoids was analyzed by Carl Friedrich Gauss in 1822, and further by L. Krüger in 1912; hence it is (in the ellipsoidal version) also called the *Gauss–Krüger projection.* It is one of the most widely used map projections, since it is conformal but has small distortions on an area that does not extend too far from the central meridian. [15, p. 48, 51], [16, p. 159–161]

$$\cos\phi = \sin\varphi = \frac{1}{\cosh\zeta}, \tag{6.99}$$

$$\tan\phi = \cot\varphi = \sinh\zeta, \tag{6.100}$$

$$\tan\frac{\phi}{2} = \tanh\frac{\zeta}{2}. \tag{6.101}$$

Mercator's projection has by (6.92) and (6.99) the conformal metric (with $\zeta = x_1$)

$$g_{ij} = \frac{1}{\cosh^2\zeta}\,\delta_{ij} = \cos^2\!\phi\,\delta_{ij}. \tag{6.102}$$

(It is easily verified directly that (3.27) indeed yields the constant sectional curvature $K = 1$ for this metric.) The invariant measure (1.6) is

$$\mathrm{d}\mu = \cosh^{-2}\zeta\,\mathrm{d}\zeta\,\mathrm{d}\theta. \tag{6.103}$$

By (3.14) (with $\varphi(\zeta,\theta) = -\log\cosh\zeta$),

$$\Gamma^k_{ij} = -\bigl(\delta_{i1}\delta_{jk} + \delta_{j1}\delta_{ik} - \delta_{k1}\delta_{ij}\bigr)\tanh\zeta, \tag{6.104}$$

i.e., the non-zero coefficients are

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = -\Gamma^1_{22} = -\tanh\zeta. \tag{6.105}$$

The Laplace–Beltrami operator is, by (3.28),

$$\Delta F = \cosh^2\zeta\bigl(F_{,\zeta\zeta} + F_{,\theta\theta}\bigr). \tag{6.106}$$

Example 6.12 (Central cylindrical projection)**.** By first taking the central projection (from 0) of the sphere $\mathbb{S}^2$ onto a vertical cylinder tangent to the sphere, and then unwrapping the cylinder, we obtain a map projection, the *central cylindrical projection*. In terms of latitude and longitude (ϕ,θ), see Example 6.8, the coordinates are, by elementary geometry,

$$(\zeta,\theta) = (\tan\phi,\,\theta). \tag{6.107}$$

This is thus another example of the general type (6.91), with $\zeta(\phi) = \tan\phi$ and thus $\phi(\zeta) = \arctan\zeta$, and (6.92) shows, using $\cos^2\phi = 1/(1+\zeta^2)$, that the metric is given by

$$g_{11} = (1+\zeta^2)^{-2} = \cos^4\phi, \tag{6.108}$$

$$g_{12} = g_{21} = 0, \tag{6.109}$$

$$g_{22} = (1+\zeta^2)^{-1} = \cos^2\phi. \tag{6.110}$$

The invariant measure (1.6) is

$$\mathrm{d}\mu = (1+\zeta^2)^{-3/2}\,\mathrm{d}\zeta\,\mathrm{d}\theta. \tag{6.111}$$

In other words, the area scale is $(1+\zeta^2)^{3/2} = \cos^{-3}\phi$.

This projection is often confused with, and mistakenly described as, Mercator's projection, but we see from (6.108)–(6.110) that it is *not* conformal. By (6.108)–(6.110), the condition number is

$$\varkappa = 1 + \zeta^2 = \cos^{-2}\phi. \tag{6.112}$$

By (1.7), the eccentricity is

$$\varepsilon = \frac{|\zeta|}{\sqrt{1+\zeta^2}} = |\sin\phi|. \tag{6.113}$$

The metric (6.108)–(6.110) is an example of Example 4.7, and it follows from (4.60) that the non-zero components Γ^k_{ij} of the connection are

$$\Gamma^1_{11} = -\frac{2\zeta}{1+\zeta^2}, \qquad \Gamma^1_{22} = \zeta, \qquad \Gamma^2_{12} = \Gamma^2_{21} = -\frac{\zeta}{1+\zeta^2}. \tag{6.114}$$

Example 6.13 (Lambert cylindrical equal-area projection)**.** By first projecting the sphere $\mathbb{S}^2$ radially from the z-axis (keeping z fixed) to a tangent vertical cylinder, and then unwrapping the cylinder, we obtain the *Lambert cylindrical equal-area projection*.[14] (Do not confuse with Example 6.12, which is quite different.) In terms of latitude and longitude (ϕ, θ), see Example 6.8, the coordinates are, by elementary geometry, cf. (6.107),

$$(\zeta, \theta) = (\sin\phi,\, \theta). \tag{6.115}$$

This is thus another example of the general type (6.91), with $\zeta(\phi) = \sin\phi$ and thus $\phi(\zeta) = \arcsin\zeta$, and (6.92) shows, using $\cos^2\phi = 1 - \zeta^2$, that the metric is given by (note that we have $|\zeta| = |\sin\phi| < 1$ for this map projection)

$$g_{11} = (1-\zeta^2)^{-1} = \cos^{-2}\phi, \tag{6.116}$$

$$g_{12} = g_{21} = 0, \tag{6.117}$$

$$g_{22} = 1 - \zeta^2 = \cos^2\phi. \tag{6.118}$$

Consequently,

$$|g| = g_{11}g_{22} = 1, \tag{6.119}$$

and the invariant measure (1.6) is

$$\mathrm{d}\mu = \mathrm{d}\zeta\,\mathrm{d}\theta, \tag{6.120}$$

i.e., the coordinates (6.115) are area-preserving. (This is a well-known geometric fact.)

By (6.116)–(6.118), the condition number is

$$\varkappa = (1-\zeta^2)^{-2} = \cos^{-4}\phi. \tag{6.121}$$

By (1.7), the eccentricity is

$$\varepsilon = \sqrt{1-(1-\zeta^2)^2} = |\zeta|\sqrt{2-\zeta^2}. \tag{6.122}$$

The metric (6.116)–(6.118) is an example of Example 4.7, and it follows from (4.60) that the non-zero components Γ^k_{ij} of the connection are

$$\Gamma^1_{11} = \frac{\zeta}{1-\zeta^2}, \qquad \Gamma^1_{22} = \zeta(1-\zeta^2), \qquad \Gamma^2_{12} = \Gamma^2_{21} = -\frac{\zeta}{1-\zeta^2}. \tag{6.123}$$

[14] Invented by Johann Heinrich Lambert 1772. [15, p. 76]

Example 6.14 (Modified cylindrical equal-area projection)**.** The cylindrical equal-area projection (6.115) is conformal on the equator, $\phi = 0$, i.e., for $\zeta = 0$, since then $g_{11} = g_{22}$, see (6.116)–(6.118). The projection can be modified by a non-isotropic scaling to $(a \sin\varphi, b\theta)$ with $a, b > 0$; we choose (without loss of generality up to a linear scaling) $a = 1$, and take $b = \cos^2\phi_0 \in (0, 1]$ for some $\phi_0 \in [0, \pi/2)$. The coordinates are thus

$$(\zeta, b\theta) = (\sin\phi,\, b\theta) = (\sin\phi,\, \theta\cos^2\phi_0). \tag{6.124}$$

The metric tensor is, cf. (6.116)–(6.118), for $|\zeta| = |\sin\phi| < 1$,

$$g_{11} = \frac{1}{1-\zeta^2} = \cos^{-2}\phi, \tag{6.125}$$

$$g_{12} = g_{21} = 0, \tag{6.126}$$

$$g_{22} = b^{-2}(1-\zeta^2) = \frac{1-\zeta^2}{\cos^4\phi_0} = \frac{\cos^2\phi}{\cos^4\phi_0}. \tag{6.127}$$

Hence the metric is conformal at the points where $\zeta^2 = 1 - b = \sin^2\phi_0$, i.e., at latitude $\phi = \pm\phi_0$. In other words, the projection (6.124) has standard parallels $\pm\phi_0$.

The case $\phi_0 = 0$ ($b = 1$) yields the Lambert cylindrical equal-area projection in Example 6.13. The case $\phi_0 = 45°$ ($b = 1/2$) is called the *Gall (orthographic) projection* or *Peters projection* or (as a compromise) *Gall–Peters projection* (it was invented by Gall 1855 and reinvented by Peters 1973 with exaggerated claims and publicity). The case $\phi_0 = 30°$ ($b = 3/4$) is called the *Behrmann projection*.[15]

Geometrically, this projection can be described by first taking the radial projection from the z-axis to a vertical cylinder with radius $b = \cos^2\phi$, and then unwrapping the cylinder; cf. Example 6.13.

By (6.125)–(6.127),

$$|g| = g_{11}g_{22} = b^{-2} = (\cos\phi_0)^{-4} \tag{6.128}$$

[15] "The earliest such modification is from Scotland: James Gall's Orthographic Cylindrical, not the same as his preferred Stereographic Cylindrical, both of which were originated in 1855, has standard parallels of 45° N. and S. (Gall, 1885). Walther Behrmann (1910) of Germany chose 30°, based on certain overall distortion criteria. Very similar later projections were offered by Trystan Edwards of England in 1953 and Arno Peters of Germany in 1967; they were presented as revolutionary and original concepts, rather than as modifications of these prior projections with standard parallels at about 37° and 45°–47°, respectively." [15, p. 76]

"There is extreme shape and scale distortion 90° from the central line, or at the poles on the normal aspect. These are the points which have infinite area and linear scale on the various aspects of the Mercator projection. This distortion, even on the modifications described above, is so great that there has been little use of any of the forms for world maps by professional cartographers, and many of them have strongly criticized the intensive promotion in the noncartographic community which has accompanied the presentation of one of the recent modifications." [15, p. 77]

See also [16, p. 164–166] for a discussion of the controversy around Peters' claims.

is constant and the invariant measure (1.6) is

$$d\mu = b^{-1}\, d\zeta\, d\theta = (\cos\phi_0)^{-2}\, d\zeta\, d\theta, \tag{6.129}$$

i.e., the coordinates (6.124) are area-preserving up to a constant factor. (We obtain equal area by multiplying the coordinates (6.124) by $1/\cos\phi_0$, i.e., for the coordinates

$$\big(\sin\phi/\cos\phi_0,\ \theta\cos\phi_0\big), \tag{6.130}$$

but we prefer to use (6.124).)

The transformation from (6.116)–(6.118) to (6.125)–(6.127) is an example of Example 4.8, and it follows from (4.60) or (4.67) that the non-zero components Γ^k_{ij} of the connection are

$$\Gamma^1_{11} = \frac{\zeta}{1-\zeta^2}, \qquad \Gamma^1_{22} = b^{-2}\zeta(1-\zeta^2), \qquad \Gamma^2_{12} = \Gamma^2_{21} = -\frac{\zeta}{1-\zeta^2}. \tag{6.131}$$

The sectional curvature $K = -1$ by (4.69).

This projection (or rather $(b\theta, \sin\phi)$) maps the sphere to a rectangle with aspect ratio width/heigth $= b\pi$.

Example 6.15 (The Gall stereographic projection)**.** In the *Gall stereographic projection*, the sphere (except for the poles) is projected onto a cylinder, which then is unwrapped (as in Examples 6.12–6.14); each meridian is projected using a perspective projection from the point on the equator opposite to the meridian.[16] Furthermore, the cylinder is chosen to intersect the sphere at two standard parallels $\pm\phi_0$, and has thus radius $\cos\phi_0$.[17]

In terms of latitude and longitude (ϕ, θ), see Example 6.8, the coordinates are, by elementary geometry,

$$(\zeta, \theta) = \Big((1+\cos\phi_0)\tan\frac{\phi}{2},\ \cos\phi_0\cdot\theta\Big). \tag{6.132}$$

If we divide by the scale factor $\cos\varphi_0$ (thus making the scale correct along the equator instead of along the standard parallel, this becomes another example of the general type (6.91), with

$$\zeta(\phi) = a\tan\frac{\phi}{2}, \tag{6.133}$$

where

$$a = \frac{1+\cos\phi_0}{\cos\phi_0}. \tag{6.134}$$

Thus $\phi(\zeta) = 2\arctan(\zeta/a)$, and (6.92) shows that the metric is given by

$$g_{11} = \Big(\frac{2a}{a^2+\zeta^2}\Big)^2 = \Big(\frac{1+\cos\phi}{a}\Big)^2, \tag{6.135}$$

$$g_{12} = g_{21} = 0, \tag{6.136}$$

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[16] The projection is thus not a perspective projection from some fixed point.

[17] This projection was proposed by Gall in 1855, with standard parallels $\phi_0 = 45^\circ$; versions with standard parallels 55° and 30° have later been used in the Soviet Union. [16, pp. 108–110, 163–164]

$$g_{22} = \left(\frac{a^2 - \zeta^2}{a^2 + \zeta^2}\right)^2 = \cos^2 \phi. \tag{6.137}$$

Consequently, the projection is conformal at points where $(1 + \cos \phi)/a = \cos \phi$, i.e., at $\phi = \pm\phi_0$.

It follows from (6.135)–(6.137) that the invariant measure (1.6) is

$$\mathrm{d}\mu = a^{-1}(1 + \cos \phi) \cos \phi \, \mathrm{d}\zeta \, \mathrm{d}\theta, \tag{6.138}$$

so the projection is not area-preserving.

Example 6.16 (Azimuthal projections)**.** By regarding the coordinates (6.91) as polar coordinates in the plane, we obtain an *azimuthal projection*, where the longitude equals the angle from the x-axis. We write the polar coordinates as $(r(\varphi), \theta)$ with $\varphi = \pi/2 - \phi$ the polar angle on the sphere, see Example 6.7; the (Cartesian) coordinates are thus

$$\big(r(\varphi) \cos \theta, \ r(\varphi) \sin \theta\big), \tag{6.139}$$

where $r = r(\varphi)$ is some smooth function, defined in some interval $[0, \varphi_0)$. (We consider here only azimuthal projections centred at the North Pole; azimuthal projections centred at other points can be defined by composition with a rotation of the sphere, see Footnote 3. Furthermore, we consider only $r(\varphi)$ with $r(0) = 0$ and $r'(0) > 0$, and assume that $r(\varphi)$ can be extended to a smooth odd function in $(-\varphi_0, \varphi_0)$; then the map (6.139) is smooth also at the North Pole, which is mapped to 0.)

Let $\varphi(r)$ be the inverse function to $r(\varphi)$; then the inverse map to (6.139) is

$$(x, y) \mapsto \Big(\sin \varphi(r) \frac{x}{r}, \ \sin \varphi(r) \frac{y}{r}, \ \cos \varphi(r)\Big) \qquad \text{with } r := \sqrt{x^2 + y^2}. \tag{6.140}$$

A calculation shows that the metric is given by, still with $r := \sqrt{x^2 + y^2}$,

$$g_{11} = \frac{x^2}{r^2} \varphi'(r)^2 + \frac{y^2}{r^4} \sin^2 \varphi(r), \tag{6.141}$$

$$g_{12} = g_{21} = \frac{xy}{r^2} \varphi'(r)^2 - \frac{xy}{r^4} \sin^2 \varphi(r), \tag{6.142}$$

$$g_{22} = \frac{y^2}{r^2} \varphi'(r)^2 + \frac{x^2}{r^4} \sin^2 \varphi(r). \tag{6.143}$$

Furthermore, (6.141)–(6.143) yield

$$|g| = \frac{\varphi'(r)^2 \sin^2 \varphi(r)}{r^2} \tag{6.144}$$

and thus the invariant measure μ in (1.6) is

$$\mathrm{d}\mu = \frac{\varphi'(r) \sin \varphi(r)}{r} \, \mathrm{d}x \, \mathrm{d}y. \tag{6.145}$$

By (1.8), the two eigenvalues of g_{ij} are

$$\varphi'(r)^2 \quad \text{and} \quad \frac{\sin^2 \varphi(r)}{r^2}. \tag{6.146}$$

	$r(\varphi)$	$\varphi(r)$
stereographic (Example 6.1)	$\tan(\varphi/2)$	$2\arctan(r)$
stereographic (Example 6.3)	$2\tan(\varphi/2)$	$2\arctan(r/2)$
gnomonic (Example 6.5)	$\tan(\varphi)$	$\arctan(r)$
orthographic (Example 6.6)	$\sin(\varphi)$	$\arcsin(r)$
equidistant (Example 6.17)	φ	r
equal-area (Example 6.18)	$2\sin(\varphi/2)$	$2\arcsin(r/2)$
Breusing harmonic (Example 6.21)	$4\tan(\varphi/4)$	$4\arctan(r/4)$

Table 3. Some azimuthal projections

Note that the stereographic projection in Example 6.1 is the special case $r = \tan(\varphi/2)$, see (6.23), and thus $\varphi(r) = 2\arctan r$. Similarly, the alternative stereographic projection in Example 6.3, the gnomonic projection in Example 6.5 and the orthographic projection in Example 6.6 are special cases, see Table 3; it is easily verified that (6.141)–(6.143) yield (6.7), (6.30), (6.51) and (6.64) for these cases. We give further examples in the following examples.

Example 6.17 (Azimuthal equidistant projection)**.** By taking $r = \varphi$ in Example 6.16, the distances from the centre point (the North Pole) are preserved; this projection thus gives correct directions and distances from a fixed point, and it is therefore called the *azimuthal equidistant projection*. (Of course, distances between other points are not preserved.)[18]

We have $0 \leqslant r < \pi$; the sphere minus the South Pole is mapped diffeomorphically onto the open disc with radius π. (The South Pole has $\varphi = \pi$, but θ is undetermined, and we cannot define the map to include the South Pole in any continuous manner.)

The metric is by (6.141)–(6.143) given by, with $r := \sqrt{x^2 + y^2}$,

$$g_{11} = \frac{x^2}{r^2} + \frac{y^2 \sin^2 r}{r^4}, \tag{6.147}$$

$$g_{12} = g_{21} = \frac{xy(r^2 - \sin^2 r)}{r^4}, \tag{6.148}$$

[18]One example is the United Nations flag and emblem, showing a map in azimuthal equidistant projection centred on the North Pole, extending to 60° south and inscribed in a wreath. [17] See [16, p. 170–172] for other modern examples.

"The Azimuthal Equidistant was apparently used centuries before the 15th-century surge in scientific mapmaking. It is believed that Egyptians used the polar aspect for star charts, but the oldest existing celestial map on the projection was prepared in 1426 by Conrad of Dyffenbach. It was also used in principle for small areas by mariners from earliest times in order to chart coasts, using distances and directions obtained at sea. The first clear examples of the use of the Azimuthal Equidistant for polar maps of the Earth are those included by Gerardus Mercator as insets on his 1569 world map, which introduced his famous cylindrical projection. As Northern and Southern Hemispheres, the projection appeared in a manuscript of about 1510 by the Swiss Henricus Loritus, usually called Glareanus (1488–1563), and by several others in the next few decades." [15, p. 191]

$$g_{22} = \frac{y^2}{r^2} + \frac{x^2 \sin^2 r}{r^4}, \tag{6.149}$$

and the invariant measure μ is by (6.145)

$$\mathrm{d}\mu = \frac{\sin r}{r}\,\mathrm{d}x\,\mathrm{d}y. \tag{6.150}$$

By (6.146), the eigenvalues of g_{ij} are 1 and $\sin^2 r/r^2$, and thus the condition number is

$$\varkappa = \Bigl(\frac{r}{\sin r}\Bigr)^2; \tag{6.151}$$

the eccentricity is by (1.7)

$$\varepsilon = \sqrt{1 - \Bigl(\frac{\sin r}{r}\Bigr)^2}. \tag{6.152}$$

Example 6.18 (Lambert azimuthal equal-area projection)**.** The *Lambert azimuthal equal-area projection*[19] is obtained by choosing $r(\varphi)$ in Example 6.16 such that the invariant measure $\mathrm{d}\mu = \mathrm{d}x\,\mathrm{d}y$; in other words, the map is area-preserving.

By (6.145), the condition for this is

$$1 = \frac{\varphi'(r)\sin\varphi(r)}{r} = \frac{\sin\varphi}{r'(\varphi)r(\varphi)}; \tag{6.153}$$

we thus need

$$(r(\varphi)^2)' = 2r'(\varphi)r(\varphi) = 2\sin\varphi, \tag{6.154}$$

with the solution

$$r(\varphi)^2 = 2 - 2\cos\varphi = 4\sin^2(\varphi/2). \tag{6.155}$$

(This is also obvious geometrically, since the spherical cap with radius φ has area $2\pi(1-\cos\varphi)$ and it is mapped onto the disc with radius $r(\varphi)$ having area $\pi r(\varphi)^2$; these two areas have to be equal.) In other words, we choose

$$r(\varphi) = 2\sin(\varphi/2). \tag{6.156}$$

We thus have $0 \leqslant r < 2$; the sphere minus the South Pole is mapped diffeomorphically onto the open disc with radius 2 (which has the area 4π, just as the sphere). (The South Pole would have $r = 2\sin(\pi/2) = 2$, but we cannot define the map to include the South Pole in any continuous manner.)

The metric is by (6.141)–(6.143) given by, with $r := \sqrt{x^2+y^2} < 2$,

$$g_{11} = \frac{x^2}{r^2(1-r^2/4)} + \frac{y^2(1-r^2/4)}{r^2} = \frac{1-(1-r^2/8)y^2/2}{1-r^2/4}, \tag{6.157}$$

$$g_{12} = g_{21} = \frac{xy}{r^2(1-r^2/4)} - \frac{xy(1-r^2/4)}{r^2} = \frac{xy(1-r^2/8)}{2(1-r^2/4)}, \tag{6.158}$$

$$g_{22} = \frac{y^2}{r^2(1-r^2/4)} + \frac{x^2(1-r^2/4)}{r^2} = \frac{1-(1-r^2/8)x^2/2}{1-r^2/4}. \tag{6.159}$$

[19] Invented by Johann Heinrich Lambert 1772 [15, p. 182]. Very commonly used in modern atlases [16, pp 172–173].

Since the determinant $|g| = 1$, the inverse matrix is

$$g^{11} = \frac{1-(1-r^2/8)x^2/2}{1-r^2/4}, \tag{6.160}$$

$$g^{12} = g^{21} = -\frac{xy(1-r^2/8)}{2(1-r^2/4)}, \tag{6.161}$$

$$g^{22} = \frac{1-(1-r^2/8)y^2/2}{1-r^2/4}. \tag{6.162}$$

By (6.146), the eigenvalues of g_{ij} are $(1-r^2/4)^{-1}$ and $1-r^2/4$, and thus the condition number is

$$\varkappa = (1-r^2/4)^{-2}. \tag{6.163}$$

The eccentricity is by (1.7)

$$\varepsilon = \sqrt{1-(1-r^2/4)^2} = \frac{r}{2}\sqrt{2-r^2/4}. \tag{6.164}$$

Example 6.19 (Perspective azimuthal projection)**.** Consider the perspective (geometric) projection from a point P onto a plane tangent to the sphere; we assume that P is on the axis through the centre and the tangent point. We may assume that the plane is tangent to the sphere at the North Pole and $P = (0, 0, -a)$ where $a \neq -1$. Note that the case $a < -1$ means that the point P is above the sphere, and the projection shows how the sphere looks from the point P in space.

Simple geometry shows that this yields an azimuthal projection as in Example 6.16 with

$$r(\varphi) = (a+1)\frac{\sin\varphi}{a+\cos\varphi}. \tag{6.165}$$

Note the three special cases: $a = 0$ (gnomonic projection, Example 6.5), $a = 1$ (stereographic projection, Example 6.3), and the limiting case $a = \infty$ (orthographic projection, Example 6.6).

It is geometrically obvious that if $-1 < a < 1$, so the projection centre P lies inside the sphere, then we can only project the region $\cos\varphi + a > 0$, i.e., $\varphi < \arccos a$. Similarly, in order to avoid double points, if $a > 1$ (P is under the sphere) we have to assume $\varphi < \pi - \arccos(1/a)$ and if $a < -1$ (P is above the sphere) we assume $\varphi < \arccos(1/|a|)$. (If $a = 1$, we can, as said in Example 6.1, map everything except the South Pole.)

The relation (6.165) yields

$$r^2 = (a+1)^2\frac{\sin^2\varphi}{(a+\cos\varphi)^2} = (a+1)^2\frac{1-\cos^2\varphi}{(a+\cos\varphi)^2} \tag{6.166}$$

and thus $r^2(a+\cos\varphi)^2 = (a+1)^2(1-\cos^2\varphi)$ and

$$\bigl((r^2+(a+1)^2\bigr)\cos^2\varphi + 2ar^2\cos\varphi + r^2a^2 - (a+1)^2 = 0 \tag{6.167}$$

with the solution (where we have to take the + sign, since $\varphi = 0$ for $r = 0$)

$$\begin{aligned}\cos\varphi &= \frac{-ar^2 + |a+1|\sqrt{(a+1)^2 + (1-a^2)r^2}}{r^2 + (a+1)^2} \\ &= \frac{\sqrt{1 + \frac{1-a^2}{(a+1)^2}r^2} - \frac{a}{(a+1)^2}r^2}{1 + r^2/(a+1)^2} \end{aligned} \tag{6.168}$$

The formulas (6.141)–(6.146) in Example 6.16 hold, where by (6.165) and simple calculations

$$\begin{aligned}\frac{\sin\varphi}{r} &= \frac{a + \cos\varphi}{a+1} \\ &= \frac{1}{a+1} \cdot \frac{\sqrt{1 + \frac{1-a^2}{(a+1)^2}r^2} + a}{1 + r^2/(a+1)^2} \end{aligned} \tag{6.169}$$

and

$$\begin{aligned}\varphi'(r) = \frac{1}{r'(\varphi)} &= \frac{(a+\cos\varphi)^2}{(a+1)(a\cos\varphi + 1)} \\ &= \frac{a+\cos\varphi}{a\cos\varphi + 1} \cdot \frac{\sin\varphi}{r} \\ &= \frac{\sqrt{1 + \frac{1-a^2}{(a+1)^2}r^2} + a}{(a+1)\Bigl(1 + \frac{r^2}{(a+1)^2}\Bigr)\sqrt{1 + \frac{1-a^2}{(a+1)^2}r^2}}. \end{aligned} \tag{6.170}$$

In particular, the condition number is given by

$$\varkappa = \begin{cases} \Bigl(\frac{a+\cos\varphi}{a\cos\varphi+1}\Bigr)^2 = \frac{1}{1 - \frac{a^2-1}{(a+1)^2}r^2}, & |a| \geqslant 1, \\ \Bigl(\frac{a\cos\varphi+1}{a+\cos\varphi}\Bigr)^2 = 1 + \frac{1-a^2}{(a+1)^2}r^2, & |a| \leqslant 1. \end{cases} \tag{6.171}$$

We see that, as we already know, the projection is conformal ($\varkappa = 1$) if $a = 1$ (Example 6.3), but not otherwise, not even locally except at $r = 0$. Cf. the special cases (6.55) ($a = 0$) and (6.68) ($a = \infty$).

Some specific choices that have been suggested for a, in many cases because of various optimality properties, are $1 + \sqrt{2}/2 \approx 1.707$ (La Hire, 1701), 1.594, $\sqrt{3} \approx 1.732$ and 2.105 (all Parent, 1702), -3 (Seutter, 1740), 1.25 (Buache, 1760), 1.69 (Lowry, 1825), $1/(\pi/2 - 1) \approx 1.752$ (Fischer c. 1850), 1.5 (James, 1857), 1.368 (James and Clarke, 1862), $(1 + \sqrt{5})/2 \approx 1.618$ (Gretschel 1873, 1894), 1.4 (Clarke, 1879), $1 + \sqrt{2}$ (Hammer, 1887), 2.537 (Nowicki, 1962, for a map of the Moon), ≈ -6.61 (the earth as seen from a geostationary satellite); some of these have also found some use. [16, pp. 65–67, 125–126, 128–130 and 170]

Example 6.20 (Breusing geometric projection)**.** Breusing (1880s) proposed [16, p. 130] a compromise between the conformal and equal-area azimuthal projections, by taking the geometric mean of the corresponding formulas

$r = 2\tan(\varphi/2)$ and $r = 2\sin(\varphi/2)$, see Table 3. (These projections both give rather large distortions away from the pole, but partly in different ways. Of course, the compromise is neither conformal nor area-preserving.) This is called the *Breusing geometric projection*, and is thus given by Example 6.16 with

$$r = 2\sqrt{\tan(\varphi/2)\sin(\varphi/2)} = 2\frac{\sin(\varphi/2)}{\cos^{1/2}(\varphi/2)}. \tag{6.172}$$

Squaring (6.172) we find

$$\frac{r^2}{4}\cos(\varphi/2) = \sin^2(\varphi/2) = 1 - \cos^2(\varphi/2) \tag{6.173}$$

and thus

$$\cos\frac{\varphi}{2} = -\frac{r^2}{8} + \sqrt{1 + \frac{r^4}{64}}. \tag{6.174}$$

An elementary calculation yields

$$\sin^2\frac{\varphi}{2} = \frac{r^2}{4}\sqrt{1 + \frac{r^4}{64}} - \frac{r^4}{32}. \tag{6.175}$$

Example 6.21 (Breusing harmonic projection)**.** By taking the harmonic mean instead of the geometric mean in Example 6.20, we obtain the *Breusing harmonic projection*.[20] [16, p. 130] It is thus given by Example 6.16 with

$$\begin{aligned} \frac{1}{r} &= \frac{1}{2}\Bigl(\frac{\cos(\varphi/2)}{2\sin(\varphi/2)} + \frac{1}{2\sin(\varphi/2)}\Bigr) = \frac{1+\cos(\varphi/2)}{4\sin(\varphi/2)} \\ &= \frac{2\cos^2(\varphi/4)}{8\sin(\varphi/4)\cos(\varphi/4)} = \frac{\cos(\varphi/4)}{4\sin(\varphi/4)} \end{aligned} \tag{6.176}$$

and thus

$$r = 4\tan(\varphi/4) \tag{6.177}$$

and

$$\varphi = 4\arctan(r/4). \tag{6.178}$$

The metric is by (6.141)–(6.143) and simple calculations given by, with $r := \sqrt{x^2+y^2} < 4$,

$$g_{11} = \frac{x^2}{r^2(1+r^2/16)^2} + \frac{y^2(1-r^2/16)^2}{r^2(1+r^2/16)^4} = \frac{1+(x^2-y^2)/8+r^4/256}{(1+r^2/16)^4} \tag{6.179}$$

$$g_{12} = g_{21} = \frac{xy}{r^2(1+r^2/16)^2} - \frac{xy(1-r^2/16)^2}{r^2(1+r^2/16)^4} = \frac{xy}{4(1+r^2/16)^4} \tag{6.180}$$

$$g_{22} = \frac{y^2}{r^2(1+r^2/16)^2} + \frac{x^2(1-r^2/16)^2}{r^2(1+r^2/16)^4} = \frac{1+(y^2-x^2)/8+r^4/256}{(1+r^2/16)^4} \tag{6.181}$$

[20] Proposed not by Breusing but by Young (1920) (under this name). [16, p. 130]

By (6.146), the eigenvalues of g_{ij} are

$$\frac{1}{(1+r^2/16)^2} \quad \text{and} \quad \frac{(1-r^2/16)^2}{(1+r^2/16)^4}. \tag{6.182}$$

Thus, or by (6.144), the determinant is

$$|g| = \frac{(1-r^2/16)^2}{(1+r^2/16)^6} \tag{6.183}$$

and thus the invariant measure μ in (1.6) is, see (6.145),

$$\mathrm{d}\mu = \frac{1-r^2/16}{(1+r^2/16)^3}\,\mathrm{d}x\,\mathrm{d}y. \tag{6.184}$$

The condition number is by (6.182)

$$\varkappa = \frac{(1+r^2/16)^2}{(1-r^2/16)^2} \tag{6.185}$$

and the eccentricity is by (1.7)

$$\varepsilon = \sqrt{1-\frac{(1-r^2/16)^2}{(1+r^2/16)^2}} = \frac{r}{2(1+r^2/16)}. \tag{6.186}$$

Example 6.22 (Airy projection)**.** Airy (1861) (corrected by James and Clarke (1862)) proposed an azimuthal projection minimizing a certain error (called "balance of errors" by him); this was an instance of a least squares method. The projection is now called the *Airy projection*.[21] For a map of the spherial cap with radius β (= maximum polar angle), this yields the azimuthal projection in Example 6.16 with

$$r(\varphi) = 2\cot\frac{\varphi}{2}\ln\cos^{-1}\frac{\varphi}{2} + 2\tan\frac{\varphi}{2}\cot^2\frac{\beta}{2}\ln\cos^{-1}\frac{\beta}{2}. \tag{6.187}$$

See [16, pp. 126–128].

Example 6.23 (Conic projections)**.** In a *conic projection*, as in an azimuthal projection (Example 6.16), the meridians are (parts of) straight lines that intersect in a single point, and parallels are (parts of) concentric circles. However, the intersection point of the meridians does not necessarily correspond to the North Pole, and the meridians do not intersect at true angles; instead, the angles between the meridians are reduced by some constant factor, in order to reduce distortion and mid-latitudes. Thus, each parallel is a circular arc, but not a full circle. (There is necessarily a cut at some

[21]George Biddell Airy (1801-1892) had been Lucasian professor of mathematics at Cambridge, and was now astronomer royal in Greenwich. The projection attracted considerable scholarly attention, but very little practical use although it was used for a map of the United Kingdom (1903-36). [16, pp. 126, 128].

meridian, for example at $\theta = \pm\pi$.) The meridians are equispaced on each parallel, and are orthogonal to parallels.[22]

Let again $\varphi = \pi/2 - \phi$ be the polar angle on the sphere and θ the longitude, see Examples 6.7–6.8, and let the chosen factor for the angles at the North Pole be c (the *cone constant*).[23] Then (up to a possible rotation or reflection and translation, which we ignore) the polar coordinates of a conic projection are $(r(\varphi), c\theta)$ for some (smooth) function $r(\varphi)$, and the coordinates are thus

$$\bigl(r(\varphi)\cos(c\theta),\ r(\varphi)\sin(c\theta)\bigr). \tag{6.188}$$

Note that we do not require $r(0) = 0$. (Also, we do not require that $r(\varphi)$, and thus the projection, is defined for all φ.) Note also that the azimuthal projections are the special case $r(0) = 0$ and $c = 1$. (As for the azimuthal projections, we consider here only projections with $r'(\varphi) > 0$; for maps of (parts of) the southern hemisphere, these can be modified in an obvious way.)

Let $\varphi(r)$ be the inverse function to $r(\varphi)$; then the inverse map to (6.188) is, with $r := \sqrt{x^2+y^2}$, cf. (6.73),

$$(x,y) \mapsto \Bigl(\sin\varphi(r)\cos\Bigl(c^{-1}\arctan\frac{y}{x}\Bigr),\ \sin\varphi(r)\sin\Bigl(c^{-1}\arctan\frac{y}{x}\Bigr),\ \cos\varphi(r)\Bigr). \tag{6.189}$$

A calculation shows that the metric is given by, still with $r := \sqrt{x^2+y^2}$,

$$g_{11} = \frac{x^2}{r^2}\varphi'(r)^2 + c^{-2}\frac{y^2}{r^4}\sin^2\varphi(r), \tag{6.190}$$

$$g_{12} = g_{21} = \frac{xy}{r^2}\varphi'(r)^2 - c^{-2}\frac{xy}{r^4}\sin^2\varphi(r), \tag{6.191}$$

$$g_{22} = \frac{y^2}{r^2}\varphi'(r)^2 + c^{-2}\frac{x^2}{r^4}\sin^2\varphi(r), \tag{6.192}$$

cf. (6.141)–(6.143). Furthermore, (6.190)–(6.192) yield

$$|g| = c^{-2}\frac{\varphi'(r)^2\sin^2\varphi(r)}{r^2} \tag{6.193}$$

and thus the invariant measure μ in (1.6) is

$$\mathrm{d}\mu = c^{-1}\frac{\varphi'(r)\sin\varphi(r)}{r}\,\mathrm{d}x\,\mathrm{d}y. \tag{6.194}$$

By (1.8), the two eigenvalues of g_{ij} are

$$\varphi'(r)^2 \quad\text{and}\quad c^{-2}\frac{\sin^2\varphi(r)}{r^2}. \tag{6.195}$$

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[22] The name *conic projection* comes from the fact that the projection may be regarded as a longitude-preserving map from the sphere to a tangent cone with apex above the North Pole, followed by cutting the cone at a meridian and flattening it. See Example 6.25 for a simple example.

[23] For a conic projection constructed using a cone as in Footnote 22, elementary geometry shows that if the cone has opening half-angle ϕ_1, and thus is tangent to the sphere at latitude ϕ_1, then $c = \sin\phi_1$.

	$r(\varphi)$	$\varphi(r)$
equidistant (Example 6.24)	$\varphi - \varphi_0$	$r + \varphi_0$
perspective (Example 6.25)	$\tan(\varphi - \varphi_1) + \tan\varphi_1$	$\varphi_1 + \arctan(r - \tan\varphi_1)$
conformal (Example 6.26)	$A\tan^c \frac{\varphi}{2}$	$2\arctan\bigl((r/A)^{1/c}\bigr)$
equal-area (Example 6.27)	$\sqrt{\frac{2}{c}(A - \cos\varphi)}$	$\arccos\Bigl(A - \frac{c}{2}r^2\Bigr)$

Table 4. Some conic projections. The cone factor $c > 0$ is arbitrary, except for the perspective conic projection where $c = \cos\varphi_1$. The constant $A > 0$ is arbitrary; it can be taken as 1 for the conformal conic projection without loss of generality.

This has a simple explanation: the inverse scale in the radial direction is $\varphi'(r)$ and in the orthogonal (tangential) direction it is $\sin\varphi(r)/(cr)$, and these two orthogonal direction are orthogonal on the map too.

The projection is (locally) conformal at a point where the two eigenvalues in (6.195) are equal, i.e., when

$$\varphi'(r) = \frac{\sin\varphi(r)}{cr}. \tag{6.196}$$

This depends only on r, and thus on the latitude but not on the longitude. Parallels where (6.196) holds are called *standard parallels*.

Example 6.24 (Simple or equidistant conic projection)**.** By taking $r = \varphi - \varphi_0 = \phi_0 - \phi$ in Example 6.23, for some fixed φ_0 and $\phi_0 = \frac{\pi}{2} - \varphi_0$, the distances along meridians are preserved; thus, the parallels are equispaced. This is called the *simple conic projection* or *equidistant conic projection.*[24]

In practice, $\phi_0 > \frac{\pi}{2}$ and thus $\varphi_0 < 0$; then the North Pole is mapped to a circular arc, so the projection is obviously singular there. (The projection is also singular at the South Pole.) In this case, the sphere minus a meridian is mapped onto a sector of angular width $2\pi c$ of the annulus $\{\varphi_0 < r < \varphi_0 + \pi\}$. (We assume $c \leqslant 1$, which is the case used in practice; in principle we can also take $c > 1$, but then the projection has to be restricted to longitudes in an interval of length $2\pi/c$ to prevent overlaps.) Also if $\phi_0 = \frac{\pi}{2}$ so $\varphi_0 = 0$, the projection is singular at the North Pole, except in the case $c = 1$ when we have the azimuthal equidistant projection (Example 6.17). In principle, we can also take $\phi_0 < \frac{\pi}{2}$ so $\varphi_0 > 0$; then the projection is defined only for latitude $\phi < \phi_0$.

[24] A form of the equidistant conic projection was introduced by Claudius Ptolemy (c. 90–170). It was further developed by Matteo Contarini in 1506 and Johannes Ruysch in 1507–08 and by Joseph Nicolas De l'Isle and others in the 18th century. [16, pp. 10–12, 29–32, 68–74]. "The projection has remained a staple for atlas coverage of small temperate -zone countries to the present day." [16, p. 68]; see also [16, pp. 176–178].

The inverse function to $r(\varphi)$ is $\varphi(r) = r + \varphi_0$. The metric is thus, by (6.190)–(6.192), again with $r := \sqrt{x^2 + y^2}$,

$$g_{11} = \frac{x^2}{r^2} + c^{-2}\frac{y^2 \sin^2(r + \varphi_0)}{r^4}, \tag{6.197}$$

$$g_{12} = g_{21} = \frac{xy(r^2 - c^{-2}\sin^2(r + \varphi_0))}{r^4}, \tag{6.198}$$

$$g_{22} = \frac{y^2}{r^2} + c^{-2}\frac{x^2 \sin^2(r + \varphi_0)}{r^4}, \tag{6.199}$$

and the invariant measure μ is by (6.194)

$$\mathrm{d}\mu = c^{-1}\frac{\sin(r + \varphi_0)}{r}\,\mathrm{d}x\,\mathrm{d}y. \tag{6.200}$$

By (6.195), the eigenvalues of g_{ij} are 1 and $c^{-2}\sin^2(r + \varphi_0)/r^2$, and the condition (6.196) for the projection to be locally conformal is

$$cr = \sin(r + \varphi_0) \tag{6.201}$$

or, equivalently,

$$c(\varphi - \varphi_0) = \sin\varphi. \tag{6.202}$$

In other words, latitude ϕ is a standard parallel if and only if

$$c(\phi_0 - \phi) = \cos\phi. \tag{6.203}$$

(This is geometrically obvious, by considering the scale on the parallel.) We can interpret (6.202) (or (6.203)) as saying that the standard parallels are given by the intersection points of a sine curve and a straight line; since $\sin\varphi$ is concave for $\varphi \in [0, \pi]$, there are at most two solutions, i.e., at most two standard parallels. Conversely, given any two distinct latitudes ϕ_1 and ϕ_2 with $\phi_1 + \phi_2 > 0$, there is a unique straight line through the corresponding points on the sine curve, and thus a φ_0 and a c yielding a unique equidistant conic projection with the given two latitudes as standard parallels. (We assume $\phi_1 + \phi_2 > 0$ in order to get $c > 0$. In the case $\phi_1 + \phi_2 < 0$, we should use a conic projection centered on the South Pole. In the case $\phi_1 + \phi_2 = 0$, i.e., $\phi_2 = -\phi_1$, we obtain as a limiting case the equirectangular projection with standard parallels $\pm\phi_1$, see Example 6.9.)

Explicitly, we obtain from (6.203) the solution

$$\phi_0 = \frac{\phi_1\cos\phi_2 - \phi_2\cos\phi_1}{\cos\phi_2 - \cos\phi_1}, \tag{6.204}$$

$$c = \frac{\cos\phi_2 - \cos\phi_1}{\phi_1 - \phi_2}. \tag{6.205}$$

Equivalently, (6.204)–(6.205) can be written

$$\varphi_0 = \frac{\varphi_1\sin\varphi_2 - \varphi_2\sin\varphi_1}{\sin\varphi_2 - \sin\varphi_1}, \tag{6.206}$$

$$c = \frac{\sin\varphi_2 - \sin\varphi_1}{\varphi_2 - \varphi_1}. \tag{6.207}$$

Note that $0 < c < 1$ (assuming $\phi_1+\phi_2 > 0$) by (6.205), using the mean value theorem. Moreover, $x \mapsto \frac{\sin x}{x}$ is decreasing on $[0,\pi]$, as is easily verified by derivation (or by noting that $\frac{\sin x}{x} = \int_0^1 \cos tx \, dt$), and thus, assuming as we may $\varphi_1 < \varphi_2$, by (6.206)–(6.207),

$$\varphi_0 = \frac{\varphi_1\varphi_2}{c} \cdot \frac{\sin\varphi_2/\varphi_2 - \sin\varphi_1/\varphi_1}{\varphi_2 - \varphi_1} < 0. \tag{6.208}$$

Hence, an equidistant conic projection with two standard parallels has $\varphi_0 < 0$, and equivalently $\phi_0 > \frac{\pi}{2}$. (This partially justifies the claim above that in practice, $\varphi_0 < 0$ and $c < 1$.)

Note the special case (really a limiting case) $\phi_2 = \frac{\pi}{2}$, i.e. $\varphi_2 = 0$: we use (6.204)–(6.207) in this case too, for any $\phi_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and obtain simply

$$\varphi_0 = 0, \qquad \phi_0 = \tfrac{\pi}{2}, \tag{6.209}$$

$$c = \frac{\sin\varphi_1}{\varphi_1} = \frac{\cos\phi_1}{\frac{\pi}{2} - \phi_1} < 1. \tag{6.210}$$

We call this the *equidistant conic projection with standard parallels* ϕ_1 *and* $\frac{\pi}{2}$.[25] Indeed, it has ϕ_1 as a standard parallel by (6.203). However, this projection is really not conformal at latitude $\frac{\pi}{2}$ (the North Pole); by the construction of the projection, angles at the pole are multiplied by $c < 1$; moreover, by (6.195) (with $\varphi(r) = r$), the projection is not asymptotically conformal as $\phi \nearrow \frac{\pi}{2}$.

Another interesting limiting case is obtained by letting $\phi_2 \to \phi_1$ ($\varphi_2 \to \varphi_1$) in (6.204)–(6.207), for some $\phi_1 \in (0, \frac{\pi}{2})$. (This corresponds to a double root of (6.203) at ϕ_1, and thus to a line tangent to the cosine curve in (6.203).) Using l'Hôpital's rule, we obtain

$$\phi_0 = \phi_1 + \cot\phi_1 > \tfrac{\pi}{2}, \tag{6.211}$$

$$c = \sin\phi_1 = \cos\varphi_1 < 1, \tag{6.212}$$

$$\varphi_0 = \varphi_1 - \tan\varphi_1 < 0. \tag{6.213}$$

This is called the *equidistant conic projection with one standard parallel* ϕ_1. Indeed, it has ϕ_1 as a standard parallel by (6.203), and this is the only standard parallel since we have a double root in (6.203). Note that for the same reason, the distortion as measured by $\kappa-1$ or ε in (1.7) is $O((\phi-\phi_1)^2)$, i.e., quadratically small near the standard parallel.

For the equidistant conic projection with one standard parallel ϕ_1, the radius of the standard parallel is $\phi_0-\phi_1 = \cot\phi_1$, which by simple trigonometry equals the length of a line from a point on the standard parallel to a point above the North Pole that is tangent to the sphere. Hence, this projection is obtained by the construction in Footnote 22 with a cone that is tangent at the standard parallel; the standard parallel is mapped to itself

[25]This version was suggested by the chemist Dmitri Mendeleev (1834–1907) (better known as the discoverer of the periodic table of elements) and used for a map of Russia with $\phi_1 = 55°$ [16, p. 177]:

and the other parallels to equidistant horizontal circles on the cone. (Note that the formulas in Footnote 23 and (6.212) for the cone constant agree.)

Note that if we take the limiting case $\phi_1 = \frac{\pi}{2}$ ($\varphi_1 = 0$) in (6.211)–(6.213), we obtain $\varphi_0 = 0$ and $c = 1$, which as said above yields the azimuthal equidistant projection (Example 6.17).

In the opposite limiting case $\phi_1 = 0$ ($\varphi_1 = \frac{\pi}{2}$) we obtain (by first translating by $(\cot\phi_1, 0)$ and then letting $\phi_1 \searrow 0$) the coordinates $(-\phi, \theta)$, which after a rotation is the latitude–longitude coordinate system, as a map projection called plate carrée or plane chart (a special case of the equirectangular projection), see Examples 6.8–6.9. Indeed, this projection has standard parallel $0°$ (the equator).

Consider now the practical problem of choosing ϕ_0 and c for a suitable equidistant conic projection for a map of a given region; suppose that the region has latitudes in the interval $[\phi_{\mathsf{c}} - \delta, \phi_{\mathsf{c}} + \delta]$, where thus ϕ_{c} is the middle latitude of the region and $\delta > 0$ is half its North–South width. Several choices have been suggested (and sometimes used), for example the following; see further [16, pp. 70–74 and 176–177]:

(i) A simple rule is to use the equidistant conic projection with standard parallels $\phi_{\mathsf{c}} \pm \frac{1}{2}\delta$.[26]

(ii) Patrick Murdoch (c. 1700–1774) wanted the total area of the region between latitudes $\phi_{\mathsf{c}} \pm \delta$ to be correct. (The projection is not area-preserving, so local areas are still incorrect; see Example 6.27.) It is easily seen that this holds if and only if (with $\varphi_{\mathsf{c}} := \frac{\pi}{2} - \phi_{\mathsf{c}}$)

$$2\delta c r(\varphi_{\mathsf{c}}) = \sin(\phi_{\mathsf{c}} + \delta) - \sin(\phi_{\mathsf{c}} - \delta) = 2\cos\phi_{\mathsf{c}} \sin\delta, \tag{6.214}$$

i.e.,

$$c(\phi_0 - \phi_{\mathsf{c}}) = \cos\phi_{\mathsf{c}} \frac{\sin\delta}{\delta}. \tag{6.215}$$

Murdoch made two suggestions of this type, based on geometric considerations; the first takes $c = \sin\phi_{\mathsf{c}}$ and is thus by (6.215) given by

$$\phi_0 = \frac{\sin\delta}{\delta} \cot\phi_{\mathsf{c}} + \phi_{\mathsf{c}}, \tag{6.216}$$

$$c = \sin\phi_{\mathsf{c}}; \tag{6.217}$$

the second takes instead

$$\phi_0 = \delta \cot\delta \cot\phi_{\mathsf{c}} + \phi_{\mathsf{c}}, \tag{6.218}$$

$$c = \frac{\sin\delta \tan\delta}{\delta^2} \sin\phi_{\mathsf{c}}; \tag{6.219}$$

(iii) The distances on the map along the parallel at latitude ϕ are $c(\phi_0 - \phi)$ for each longitude unit, while the corresponding distance on the sphere is $\cos\phi$. The error is thus

$$c(\phi_0 - \phi) - \cos\phi. \tag{6.220}$$

[26] Used by De l'Isle (1688–1768), at least for a slightly different projection [16, p. 68].

Euler (1707–1783) suggested 1777 the projection that makes this error equal for the extreme parallels $\phi_c \pm \delta$, and equal with opposite sign for the central parallel ϕ_c. This leads to the equations

$$c(\phi_0 - \phi_c \mp \delta) - \cos\phi_c \cos\delta \pm \sin\phi_c \sin\delta = -c(\phi_0 - \phi_c) + \cos\phi_c, \quad (6.221)$$

which yield

$$c\delta = \sin\phi_c \sin\delta \quad (6.222)$$

$$2c(\phi_0 - \phi_c) = \cos\phi_c(\cos\delta + 1) = 2\cos\phi_c \cos^2\frac{\delta}{2}, \quad (6.223)$$

and thus

$$\phi_0 = \phi_c + \frac{\delta\cos^2(\delta/2)}{\sin\delta}\cot\phi_c = \tfrac{\delta}{2}\cot\tfrac{\delta}{2}\cot\phi_c + \phi_c, \quad (6.224)$$

$$c = \frac{\sin\delta}{\delta}\sin\phi_c. \quad (6.225)$$

(iv) Let us instead of the absolute error (6.220) consider the relative error

$$\frac{c(\phi_0 - \phi)}{\cos\phi} - 1 \quad (6.226)$$

and, in analogy with Euler's condition in (iii), require this error to be equal for the extreme parallels $\phi_c \pm \delta$, and equal with opposite sign for the central parallel ϕ_c.[27] This leads to the equations, with $r_c := \phi_0 - \phi_c$,

$$c(r_c \mp \delta) = (1+\eta)(\cos\phi_c \cos\delta \pm \sin\phi_c \sin\delta) \quad (6.227)$$

$$cr_c = (1-\eta)\cos\phi_c, \quad (6.228)$$

which yield

$$r_c/\delta = \cot\phi_c \cot\delta, \quad (6.229)$$

$$\frac{cr_c}{\cos\phi_c}\Bigl(\frac{1}{\cos\delta} + 1\Bigr) = (1+\eta) + (1-\eta) = 2, \quad (6.230)$$

and thus

$$\phi_0 = \delta\cot\delta\cot\phi_c + \phi_c, \quad (6.231)$$

$$c = \frac{2\tan\frac{\delta}{2}}{\delta}\sin\phi_c. \quad (6.232)$$

Example 6.25 (Perspective conic projection)**.** Let $\phi_1 \in (0, \frac{\pi}{2})$, and consider the simple perspective projection from the origin onto the cone C that is tangent to the sphere at the parallel ϕ_1, followed by cutting the cone along a meridian and flattening it, see Footnote 22.[28] The cone C has apex above the North Pole and opening angle $2\phi_1$, and elementary geometry shows that

[27] We do not know any reference for this version, but it has presumably been considered before.

[28] This projection was used 1794 by Christopher Colles (1739-1816), but has apparently not been used further. [16, p. 74]

the resulting projection is a conic projection as in Example 6.23 with, cf. Footnote 23,

$$r = \cot \phi_1 - \tan(\phi - \phi_1), \tag{6.233}$$

$$c = \sin \phi_1. \tag{6.234}$$

Again using the complementary angles $\varphi := \frac{\pi}{2} - \phi$ and $\varphi_1 := \frac{\pi}{2} - \phi_1$, we can write (6.233) as

$$r(\varphi) = \tan(\varphi - \varphi_1) + \tan \varphi_1, \tag{6.235}$$

which has the inverse

$$\varphi(r) = \varphi_1 + \arctan(r - \tan \varphi_1). \tag{6.236}$$

Hence,

$$\varphi'(r) = \frac{1}{1 + (r - \tan \varphi_1)^2}. \tag{6.237}$$

The metric is thus given by (6.190)–(6.192) with $\varphi(r)$ and $\varphi'(r)$ given by (6.236) and (6.237), and thus

$$\sin^2 \varphi(r) = \frac{\tan^2 \varphi(r)}{1 + \tan^2 \varphi(r)} = \frac{r^2}{r^2 + (1 - \tan \varphi_1 (r - \tan \varphi_1))^2}. \tag{6.238}$$

Note that (6.196) holds for $\varphi = \varphi_1$, so the projection is conformal at the parallel ϕ_1, which also follows from the geometric construction.

Example 6.26 (Lambert's conformal conic projection)**.** A conic projection is conformal if (6.196) holds for all r. This equation can be written

$$\frac{1}{c} \,\mathrm{d}(\ln r) = \frac{\mathrm{d}r}{cr} = \frac{\mathrm{d}\varphi}{\sin \varphi} = \mathrm{d}\Bigl(\ln \tan \frac{\varphi}{2}\Bigr), \tag{6.239}$$

with the solution

$$r(\varphi) = A \tan^c \frac{\varphi}{2} = \frac{A}{\tan^c\bigl(\frac{\phi}{2} + \frac{\pi}{4}\bigr)}, \tag{6.240}$$

where the cone factor $c > 0$ is arbitrary, and $A > 0$ is an arbitrary scale factor (that can be taken as 1 without essential loss of generality, although we shall not do so). This is called *Lambert's conformal conic projection*[29], see [16, pp. 78–80].

The case $c = 1$ yields the stereographic projection Example 6.1 (with $A = \rho = 1$; or Example 6.3 with $A = 2\rho = 2$), as is seen by (6.23), cf. Table 3. Note that, if we regard the projections as mappings from the sphere to the complex plane, and $\xi \mapsto \Phi_S(\xi)$ is the stereographic projection, then Lambert's conformal conic projection is simply $a\Phi_S(\xi)^c$, for another scale factor a. The fact that the projection is conformal thus also follows

[29]Invented by Johann Heinrich Lambert in 1772 [16, p. 76]. It was used already in 1777 for a celestial map, but hardly used until it was revived by France for battle maps during the First World War. "It is now standard for much official mapping throughout the world, however, sharing importance in this repect only with Lambert's transverse Mercator." [16, p. 78]. See also [16, pp. 174–175].

from the standard facts that the stereographic projection is conformal and that $z \mapsto z^c$ is analytic with non-vanishing derivative (in the domain we consider), and thus conformal.

The limiting case $c = 0$ (more precisely, the limit $c \searrow 0$ with $A = 1/c$ after a translation by $(-A, 0)$) yields Mercator's projection (Example 6.11), see (6.96) (with a change of sign). Since $(z^c - 1)/c \to \log z$, this is consistent with the fact that Mercator's projection equals $\log \Phi_S$ (with a change of sign), see Example 6.11,

Note also that a conformal conic projection about the South Pole, which is obtained by the substitution $\phi \mapsto -\phi$, i.e. $\varphi \mapsto \pi - \varphi$, (and $\theta \mapsto -\theta$), is given by (6.240) with $c < 0$, since $\tan((\pi - \varphi)/2) = \cot(\varphi/2)$.

The conformal conic projection given by (6.240) has, since it is conformal, a unique scale at each point, which by (6.195) is the inverse of

$$\frac{\sin\varphi}{cr} = \frac{\sin\varphi}{cA\tan^c(\varphi/2)} = \frac{2}{cA}\cdot\frac{\tan^{1-c}(\varphi/2)}{1+\tan^2(\varphi/2)} = \frac{\cos\phi\tan^c(\frac{\phi}{2}+\frac{\pi}{4})}{cA}. \tag{6.241}$$

A parallel at which the scale is unity is called a *standard parallel*. (Note that this differs from the general definition in Example 6.23; the latter is not useful here since the map is conformal everywhere.)

Given any two parallels ϕ_1 and ϕ_2 with $\phi_1, \phi_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\phi_1 + \phi_2 > 0$, there is a unique choice of c and A such that these two parallels are standard parallels. By (6.241), c and A determined by the equations

$$\frac{cA\tan^c(\varphi_1/2)}{\sin\varphi_1} = \frac{cA\tan^c(\varphi_2/2)}{\sin\varphi_2} = 1 \tag{6.242}$$

which have the solution

$$c = \frac{\ln\sin\varphi_1 - \ln\sin\varphi_2}{\ln\tan(\varphi_1/2) - \ln\tan(\varphi_2/2)} = \frac{\ln\cos\phi_1 - \ln\cos\phi_2}{\ln\tan(\frac{\phi_2}{2}+\frac{\pi}{4}) - \ln\tan(\frac{\phi_1}{2}+\frac{\pi}{4})} \tag{6.243}$$

$$A = \frac{\sin\varphi_1}{c\tan^c(\varphi_1/2)} = \frac{\sin\varphi_2}{c\tan^c(\varphi_2/2)} = \frac{1}{c}\cos\phi_1\tan^c\Bigl(\frac{\phi_1}{2}+\frac{\pi}{4}\Bigr). \tag{6.244}$$

It is easily verified that $0 < c < 1$. As in Example 6.24, we here have to assume $\phi_1 + \phi_2 > 0$ in order to have $c > 0$. In the case $\phi_1 + \phi_2 < 0$, there is instead a conic projection about the South Pole with standard parallels ϕ_1 and ϕ_2. (Note that then (6.240) and (6.243) still hold, with $c < 0$, and that the cone factor about the South Pole is $|c|$.) If $\phi_1 + \phi_2 = 0$, (6.243) yields $c = 0$, and as a limiting case we obtain Mercator's projection (Example 6.11) with a suitable scale factor $\cos\phi_1 = \sin\varphi_1$, which indeed is a conformal projection with standard parallels $\pm\phi_1$.

In the limiting case $\phi_2 = \frac{\pi}{2}$, (6.243) yields $c = 1$, i.e., the stereographic projection Example 6.1 (with a suitable scale factor making ϕ_1 a standard parallel).

In the limiting case $\phi_2 \to \phi_1$, we obtain by l'Hôpital's rule from (6.196)

$$c = \cos\varphi_1 = \sin\phi_1. \tag{6.245}$$

This (with A given by (6.244)) is the *conformal conic projection with one standard parallel* ϕ_1. (We here require $\phi_1 > 0$; for $\phi_1 = 0$ we again obtain Mercator's projection as a limiting case.)

The inverse function to (6.240) is

$$\varphi(r) = 2\arctan\bigl((r/A)^{1/c}\bigr) \tag{6.246}$$

with derivative

$$\varphi'(r) = 2c^{-1}A^{-1/c}\frac{r^{1/c-1}}{1+(r/A)^{2/c}} \tag{6.247}$$

(note that (6.196) holds). Since the metric is conformal, it is by (6.195) given by, for simplicity now taking $A = 1$,

$$g_{ij} = \varphi'(r)^2\delta_{ij} = 4c^{-2}\frac{r^{2/c-2}}{(1+r^{2/c})^2}. \tag{6.248}$$

Example 6.27 (Albers's equal-area conic projection)**.** A conic projection is, by (6.194), area-preserving if and only if

$$c^{-1}\frac{\varphi'(r)\sin\varphi(r)}{r} = 1 \tag{6.249}$$

or, equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}r}\cos\varphi(r) = -cr \tag{6.250}$$

with the solution

$$\cos\varphi(r) = A - \frac{c}{2}r^2 \tag{6.251}$$

for some A. This gives

$$r = \sqrt{\tfrac{2}{c}(A-\cos\varphi)} = \sqrt{\tfrac{2}{c}(A-\sin\phi)} \tag{6.252}$$

and

$$\varphi = \arccos\Bigl(A - \frac{c}{2}r^2\Bigr), \tag{6.253}$$

$$\phi = \tfrac{\pi}{2} - \varphi = \arcsin\Bigl(A - \frac{c}{2}r^2\Bigr). \tag{6.254}$$

In practice, $A > 1$; then $\sqrt{\tfrac{2}{c}(A-1)} \leqslant r \leqslant \sqrt{\tfrac{2}{c}(A+1)}$, with the North Pole corresponding to an arc at $r = \sqrt{\tfrac{2}{c}(A-1)}$ and the South Pole to an arc at $r = \sqrt{\tfrac{2}{c}(A+1)}$. (If $A < 1$, only latitudes south of $\arcsin A$ can be mapped.) This projection is called *Albers's equal-area conic projection*.[30]

In the special case $A = 1$, (6.251) can be written

$$\sin^2\frac{\varphi}{2} = \frac{1-\cos\varphi}{2} = \frac{c}{4}r^2, \tag{6.255}$$

[30] Invented by Heinrich Christian Albers (1773–1833) in 1805. It was used for a map of Europe in 1817, and has during the 20th century become one of the most common map projections. [16, pp. 116–117, 90, 175–176]

and thus

$$r = \frac{2}{\sqrt{c}} \sin\frac{\varphi}{2} = \frac{2}{\sqrt{c}} \sin\Bigl(\frac{\pi}{4} - \frac{\phi}{2}\Bigr) = \frac{2}{\sqrt{c}} \cos\Bigl(\frac{\pi}{4} + \frac{\phi}{2}\Bigr), \tag{6.256}$$

$$\varphi = \frac{\pi}{2} - \phi = 2\arcsin\Bigl(\frac{\sqrt{c}}{2} r\Bigr), \tag{6.257}$$

This case is called *Lambert's equal-area conic projection.*[31] Note that $A = 1$ if and only if $\varphi(0) = 0$, i.e., if and only if the North Pole is mapped to a single point (the origin). The Lambert equal-area conic projection maps the sphere (minus a meridian) onto a sector of angle $2\pi c$ and radius $\sqrt{4/c}$.

In any of the cases above, the metric is given by (6.190)–(6.192), with φ given by (6.253) and

$$\varphi'(r) = \frac{cr}{\sin\varphi} = \frac{cr}{\sqrt{1 - (A - \frac{c}{2} r^2)^2}}. \tag{6.258}$$

Using (6.249), it is easily seen that (6.196) holds if and only if

$$\frac{\sin\varphi}{cr} = 1. \tag{6.259}$$

Hence φ is the colatitude of a standard parallel if and only if $\sin\varphi = cr$, which by (6.251) is equivalent to

$$\cos\varphi = A - \frac{1}{2c}\sin^2\varphi = A - \frac{1}{2c} + \frac{1}{2c}\cos^2\varphi. \tag{6.260}$$

This is, for given A and c, a quadratic equation in $\cos\varphi = \sin\phi$, so it has at most two roots. Thus the projection has at most two standard parallels.

Given two parallels with latitudes ϕ_1 and ϕ_2 such that $\phi_1 + \phi_2 > 0$, there is a unique choice of A and c such that the projection has standard parallels ϕ_1 and ϕ_2. In fact, denoting as usually the corresponding colatitudes by φ_1 and φ_2, we obtain from (6.260)

$$\cos\varphi_1 - \cos\varphi_2 = \frac{1}{2c}\bigl(\cos^2\varphi_1 - \cos^2\varphi_2\bigr) \tag{6.261}$$

and thus

$$c = \frac{\cos\varphi_1 + \cos\varphi_2}{2} = \frac{\sin\phi_1 + \sin\phi_2}{2} \tag{6.262}$$

and, by (6.260) again,

$$A = \frac{1}{2c}\bigl(2c\cos\varphi_1 + \sin^2\varphi_1\bigr) = \frac{\cos\varphi_1\cos\varphi_2 + 1}{\cos\varphi_1 + \cos\varphi_2} = \frac{\sin\phi_1\sin\phi_2 + 1}{\sin\phi_1 + \sin\phi_2}. \tag{6.263}$$

Conversely, for this choice of c and A, ϕ_1 and ϕ_2 are standard parallels.

The assumption $\phi_1 + \phi_2 > 0$ makes $0 < c < 1$. Furthermore, (6.263) implies $A > 1$; hence we are in the Albers case. This projection is called the *Albers's equal-area conic projection with standard parallels ϕ_1 and ϕ_2.* As earlier we see that in the case $\phi_1 + \phi_2 < 0$, there is instead a conic projection about the South Pole with these two parallels standard. In the case $\phi_1 + \phi_2 = 0$, we obtain as a limiting case (after translation by $\sqrt{2A/c}$)

[31] Invented by Johann Heinrich Lambert in 1772. It is very rarely used. [16, pp. 90–91]

the modified cylindrical equal-area projection with standard parallels $\pm\phi_1$, see Example 6.14. (More precisely, we obtain as the limit the projection in Example 6.14 with $a = 1/\cos\phi_1$, $b = \cos\phi_1$.)

In the limiting case $\phi_2 \to \phi_1 \in (0, \frac{\pi}{2})$ we obtain *Albers's equal-area conic projection with one standard parallel* ϕ_1, given by

$$c = \cos\varphi_1 = \sin\phi_1, \tag{6.264}$$

$$A = \frac{1}{2c}\bigl(2c\cos\varphi_1 + \sin^2\varphi_1\bigr) = \frac{1+\cos^2\varphi_1}{2\cos\varphi_1} = \frac{1+\sin^2\phi_1}{2\sin\phi_1}. \tag{6.265}$$

We have here implicitly assumed $\phi_1, \phi_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Consider now the limiting case $\phi_2 = \frac{\pi}{2}$, i.e., $\varphi_2 = 0$. Note that $\varphi = 0$ satisfies (6.260) if and only if $A = 1$, i.e., the Lambert case. Hence Lambert's equal-area conic projection can (formally, as a limiting case) be regarded as Albers's equal-area conic projection with a standard parallel at $\frac{\pi}{2}$. (As for other conic projections, the projection is really not conformal at the North Pole.) Hence, Lambert's equal-area conic projection has at most one standard parallel (apart from $\frac{\pi}{2}$); it is easily seen from (6.260), or from (6.262), that it is given by

$$\sin\phi_1 = \cos\varphi_1 = 2c - 1. \tag{6.266}$$

Conversely, for any $\phi_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, there is exactly one such projection, obtained by choosing $A = 1$ and

$$c = \frac{1+\sin\phi_1}{2} = \frac{1+\cos\varphi_1}{2} = \cos^2\frac{\varphi_1}{2}. \tag{6.267}$$

This is *Lambert's equal-area conic projection with standard parallel* ϕ_1. Note that (6.267) yields $0 < c < 1$.

In the limiting case $\phi_1 = \frac{\pi}{2}$ ($\varphi_1 = 0$), both Albers's equal-area conic projection with one standard parallel ϕ_1 (6.264) and Lambert's equal-area conic projection (6.267) yield $c = 1$, and we obtain Lambert's equal-area azimuthal projection Example 6.18, see (6.255) and (6.155).

Example 6.28 (Bonne)**.** The *Bonne projection*[32] has, as the equidistant conic projection (Example 6.24), parallels that are arcs of equispaced concentric circles; thus, the parallel with latitude ϕ has radius[33]

$$r = r(\phi) = \phi_0 - \phi \tag{6.268}$$

for some ϕ_0 (with $\phi_0 \geqslant \frac{\pi}{2}$, at least if we want to map the whole sphere). However, unlike the conic projections, the meridians are not straight lines

[32]Named after Rigobert Bonne (1727–1795) who used it in 1752; however, it was used by others before him. It has continued to be used until the 20th century. [16, pp. 60–61, 173, 180–181]

[33]This is the version suitable for the northern hemisphere. We leave the opposite case to the reader.

(except the central meridian); instead, the scale is correct along each parallel. Hence, the polar coordinates of a point (ϕ,θ) are

$$\Bigl(r(\phi), \frac{\cos\phi}{r(\phi)}\,\theta\Bigr) = \Bigl(\phi_0-\phi, \frac{\cos\phi}{\phi_0-\phi}\,\theta\Bigr), \tag{6.269}$$

and the Euclidean coordinates are thus

$$x = (\phi_0-\phi)\cos\Bigl(\frac{\cos\phi}{\phi_0-\phi}\,\theta\Bigr), \tag{6.270}$$

$$y = (\phi_0-\phi)\sin\Bigl(\frac{\cos\phi}{\phi_0-\phi}\,\theta\Bigr). \tag{6.271}$$

Conversely, let the polar coordinates of (x,y) be (r,t); thus

$$r = \sqrt{x^2+y^2}, \qquad t = \arctan(y/x). \tag{6.272}$$

Then $r=\phi_0-\phi$ by (6.268), and furthermore

$$t = \frac{\cos\phi}{r}\,\theta = \frac{\cos\phi}{\phi_0-\phi}\,\theta. \tag{6.273}$$

Hence

$$\phi = \phi_0 - r = \phi_0 - \sqrt{x^2+y^2}, \tag{6.274}$$

$$\theta = \frac{r}{\cos\phi}\,t = \frac{r}{\cos(\phi_0-r)}\arctan\frac{y}{x}. \tag{6.275}$$

The metric is given by, using (6.80) and (6.274)–(6.275),

$$\begin{aligned}
|\mathrm{d}s|^2 &= |\mathrm{d}\phi|^2 + \cos^2\phi\,|\mathrm{d}\theta|^2 \\
&= |\mathrm{d}r|^2 + \cos^2\phi\,\Bigl|\frac{r}{\cos\phi}\,\mathrm{d}t + t\Bigl(\frac{\mathrm{d}r}{\cos\phi} - \frac{r\sin\phi\,\mathrm{d}r}{\cos^2\phi}\Bigr)\Bigr|^2 \\
&= |\mathrm{d}r|^2 + \bigl|r\,\mathrm{d}t + t\,(1-r\tan\phi)\,\mathrm{d}r\bigr|^2 \\
&= \bigl(1+t^2\,(1-r\tan\phi)^2\bigr)|\mathrm{d}r|^2 + 2rt\,(1-r\tan\phi)\,\mathrm{d}r\,\mathrm{d}t + r^2|\mathrm{d}t|^2
\end{aligned} \tag{6.276}$$

We can express this in $\mathrm{d}x$ and $\mathrm{d}y$, using

$$\mathrm{d}r = \frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{r}, \qquad \mathrm{d}t = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{r^2}. \tag{6.277}$$

However, we omit this and use (6.276) directly, noting that the metric is conformal at a point if and only if $|\mathrm{d}s|^2$ is a multiple of, using (5.2),

$$|\mathrm{d}x|^2 + |\mathrm{d}y|^2 = |\mathrm{d}r|^2 + r^2|\mathrm{d}t|^2. \tag{6.278}$$

Comparing (6.276) and (6.278), we see that the metric is conformal at a point if and only if $t(1-r\tan\phi)=0$, i.e., if $t=0$ or if $r=\cot\phi$. (And then the scale is 1, i.e. correct, at that point.) In other words, the metric is conformal (and the scale is correct) at the central meridian, and at each parallel where $r\tan\phi=1$, i.e., using (6.268) again,

$$\phi_0 - \phi = \cot\phi. \tag{6.279}$$

A parallel where the metric is conformal, i.e., a parallel satisfying (6.279), is called a *standard parallel*. Since $\phi + \cot\phi$ is decreasing from ∞ to $\frac{\pi}{2}$ on $(0, \frac{\pi}{2})$, there is at most one standard parallel for each ϕ_0, and exactly one standard parallel for each $\phi_0 > \frac{\pi}{2}$.

Conversely, given any ϕ_1 with $0 < \phi_1 < \frac{\pi}{2}$, there is a unique Bonne projection with standard parallel ϕ_1; by (6.279), it is given by

$$\phi_0 = \phi_1 + \cot\phi_1. \tag{6.280}$$

(Note that this $\phi_0 > \frac{\pi}{2}$.) In other words, the parallel with latitude ϕ has radius

$$r = r(\phi) = \cot\phi_1 + \phi_1 - \phi, \tag{6.281}$$

exactly as the equidistant conic projection with one standard parallel ϕ_1, see (6.211).

We see from (6.276) that the metric tensor in the (r, t) coordinates is

$$\begin{pmatrix} 1 + t^2 (1 - r\tan\phi)^2 & rt\,(1 - r\tan\phi) \\ rt\,(1 - r\tan\phi) & r^2 \end{pmatrix}. \tag{6.282}$$

This matrix has determinant r^2, and thus the invariant measure (1.6) is

$$\mathrm{d}\mu = r\,\mathrm{d}r\,\mathrm{d}t = \mathrm{d}x\,\mathrm{d}y. \tag{6.283}$$

In other words, the Bonne projection is area-preserving.

In the limiting case $\phi_1 \searrow 0$, which by (6.280) is the same as $\phi_0 \to \infty$, it is easy to see that the limit (after suitable translation and rotation, as for corresponding limit for the equidistant conic projection, see Example 6.24) is the sinusoidal projection in Example 6.31 below. The sinusoidal projection can thus be regarded as the Bonne projection with the equator as standard parallel.

In the limiting case $\phi_1 = \frac{\pi}{2}$, (6.280) yields $\phi_0 = \frac{\pi}{2}$, and thus (6.268) becomes

$$r = \frac{\pi}{2} - \phi = \varphi, \tag{6.284}$$

the polar angle, and the polar coordinates of a point $(\phi, \theta) = (\frac{\pi}{2} - \varphi, \theta)$ are thus

$$\Big(\varphi, \frac{\sin\varphi}{\varphi}\,\theta\Big). \tag{6.285}$$

This yields a heart-shaped map of the sphere (minus a meridian), see Example 6.29.

Example 6.29 (Stabius, Werner)**.** Stabius (c. 1500) invented three cordiform (heart-shaped) projections that were further publicized by Werner (1514), see [16, pp. 33–37].[34] They are all of the same type and similar to the later Bonne projection in Example 6.28, more precisely to the case $\phi_0 = \frac{\pi}{2}$ in (6.285), but the scale along the paralells may differ from the

[34] The first was never used; the second was used on some maps from 1530 to at least 1609; the third had also some use 1536–1560. [16, p. 37].

scale on the central meridian. Thus, a point with latitude and longitude $(\phi, \theta) = (\frac{\pi}{2} - \varphi, \theta)$ has polar coordinates

$$\Bigl(\varphi,\, c\,\frac{\sin\varphi}{\varphi}\,\theta\Bigr) = \Bigl(\frac{\pi}{2} - \phi,\, c\,\frac{\cos\phi}{\frac{\pi}{2} - \phi}\,\theta\Bigr), \tag{6.286}$$

where $c > 0$ is a constant.

The case $c = 1$ thus gives (6.285), i.e. the special case $\phi_0 = \frac{\pi}{2}$ of the Bonne projection discussed above in Example 6.28. This is the case when the scale along the central meridian is the same as the scale along the parallels. This was the second of Stabius' and Werner's projections and it is now called the *Werner projection*.

The first Stabius–Werner projection had $c = \frac{\pi}{2}$. Thus the equator ($\varphi = \frac{\pi}{2}$) is mapped to a full circle. Since $\sin\varphi/\varphi$ is decreasing on $[0, \frac{\pi}{2}]$, the factor $c\sin\varphi/\varphi > 1$ for any $\varphi < \frac{\pi}{2}$, and thus the map will overlap itself unless restricted to $|\theta| < \frac{2}{\pi}\frac{\varphi}{\sin\varphi}\pi = 2\frac{\varphi}{\sin\varphi}$ (or θ in another interval of the same length); if we want to allow all $\varphi \geqslant 0$, we thus have to assume $|\theta| < 2$ (or another interval of 4 radians $\approx 229°11'$).

The third Stabius–Werner projection had $c = \frac{\pi}{3} \approx 1.05$. (On the equator $\varphi = \frac{\pi}{2}$, (6.286) becomes $(\varphi, \frac{2}{3}\theta)$; thus two points spaced $\frac{\pi}{2}$ apart on the equator are mapped to two points spaced $\frac{\pi}{3}$ on the equator, and thus the Euclidean distance between the two points on the map (i.e., the length of the chord between them) equals the equator radius, i.e., the distance between a point on the equator and the pole. This was the reason for the choice $c = \frac{\pi}{3}$.) The result does in practise not differ much from the case $c = 1$, but the parallel $\varphi = \frac{\pi}{6}$ (i.e., $\phi = \frac{\pi}{3} = 60°$) will be a full circle, and parallels above $60°$ latitude will overlap themselves.

Letting, as in Example 6.28, (r, t) be the polar coordinates of (x, y), the only difference from the Bonne projection in Example 6.28 with $\phi_0 = \frac{\pi}{2}$ is that t is multiplied by the constant c. The metric is thus, by (6.276), given by

$$\begin{aligned} |\mathrm{d}s|^2 &= |\mathrm{d}r|^2 + c^{-2}\bigl|r\,\mathrm{d}t + t\,(1 - r\tan\phi)\;\mathrm{d}r\bigr|^2 \\ &= \bigl(1 + c^{-2}t^2\,(1 - r\tan\phi)^2\bigr)|\mathrm{d}r|^2 + 2c^{-2}rt\,(1 - r\tan\phi)\;\mathrm{d}r\,\mathrm{d}t + c^{-2}r^2|\mathrm{d}t|^2, \end{aligned} \tag{6.287}$$

which can be expressed in $\mathrm{d}x$ and $\mathrm{d}y$ by (6.277).

Since the Bonne projection is area-preserving, it follows immediately that also any Stabius–Werner projection is area-preserving (up to a constant factor, viz. c^{-1}).

Example 6.30 (Trapezoidal projection)**.** The *trapezoidal projection* is the coordinates

$$\bigl(\phi, (a - b\phi)\theta\bigr), \tag{6.288}$$

where ϕ and θ are latitude and longitude, and $a > 0$ and $b > 0$ are constants.[35][36] (We consider only domains where $a - b\phi > 0$.) Note that the limiting case $b = 0$ yields the equirectangular projection in Example 6.9.

The trapezoidal projection is similar to conic projections in that the meridians are straight lines intersecting in a common point (off the map) and that the meridians are equispaced on each parallel. However, unlike conic projections, the parallels are parallel lines; furthermore, they are equispaced. The meridians have different angles to the parallels. (Only the central meridian is perpendicular to them.)

The length scale on the parallels is correct for the latitudes satisfying

$$\cos\phi = a - b\phi, \tag{6.289}$$

which can be chosen as any two given latitudes; these parallels are called *standard parallels.* (However, the scale is not correct on the meridians, except the central one.) The trapezoidal projection with standard parallels ϕ_1 and ϕ_2 is given by

$$a = \frac{\phi_1\cos\phi_2 - \phi_2\cos\phi_1}{\phi_1 - \phi_2}, \tag{6.290}$$

$$b = \frac{\cos\phi_2 - \cos\phi_1}{\phi_1 - \phi_2}. \tag{6.291}$$

(Cf. (6.204)–(6.205); note that (6.289) and (6.203) are the same equation up to notational changes.) Here we as usual assume $\phi_1 + \phi_2 > 0$ in order to have $b > 0$; if $\phi_1 + \phi_2 < 0$ (6.291) yields $b < 0$ which is still valid, see Footnote 35, and if $\phi_2 = -\phi_1$, we obtain $b = 0$ and the equirectangular projection Example 6.9. As in Example 6.24, we may also consider the limiting case $\phi_2 = \phi_1$, the *trapezoidal projection with one standard parallel* ϕ_1 (where $0 < \phi_1 < \frac{\pi}{2}$ if we want $b > 0$). In this case we have

$$a = \cos\phi_1 + \phi_1\sin\phi_1, \tag{6.292}$$

$$b = \sin\phi_1. \tag{6.293}$$

Denoting the coordinates by (x, y), we have $\phi = x$ and $\theta = y/(a - bx)$, and thus, using (6.80),

$$|\mathrm{d}s|^2 = |\mathrm{d}\phi|^2 + \cos^2\phi\,|\mathrm{d}\theta|^2 = |\mathrm{d}x|^2 + \cos^2 x\left|\frac{by}{(a-bx)^2}\,\mathrm{d}x + \frac{1}{a-bx}\,\mathrm{d}y\right|^2, \tag{6.294}$$

---

[35] This is for the northern hemisphere; for the southern hemisphere one takes $b < 0$. The projection is not useful for maps covering parts of both hemispheres; in such cases one usually composes two such maps, one for each hemisphere, in which case meridians (except the central one) will be broken lines with a corner at the equator.

[36] The trapezoidal projection was used in a rudimentary form for star maps by Conrad of Dyffenbach 1426, and perhaps already by Hipparchus about 150 B.C.; "it was treated by later map hisorians as an invention of Donnus Nicolaus Germanus (ca. 1420–ca. 1490) who claimed it as such in 1482 and used it more precisely in several manuscripts and printed editions of Ptolemy's *Geography*, beginning in 1466." [16, pp. 8–9].

which gives the metric tensor

$$(g_{ij}) = \begin{pmatrix} 1+\frac{b^2y^2\cos^2 x}{(a-bx)^4} & \frac{by\cos^2 x}{(a-bx)^3} \\ \frac{by\cos^2 x}{(a-bx)^3} & \frac{\cos^2 x}{(a-bx)^2} \end{pmatrix}. \tag{6.295}$$

The determinant is

$$|g| = \frac{\cos^2 x}{(a-bx)^2} \tag{6.296}$$

and thus the invariant measure (1.6) is

$$\mathrm{d}\mu = \frac{\cos x}{(a-bx)}\,\mathrm{d}x\,\mathrm{d}y; \tag{6.297}$$

the area scale is thus $(a-bx)/\cos x$.

Example 6.31 (Sinusoidal projection)**.** In the *sinusoidal projection*, parallels are equispaced parallel lines, and the scale is correct on each parallel, as well as on the central meridian. Hence the coordinates are

$$\bigl(\phi, \theta\cos\phi\bigr) \tag{6.298}$$

which maps the sphere (minus a meridian) onto the domain bounded by the sinusoidal curves $y = \pm\pi\cos x$, which has given the projection its name.[37]

It follows from (6.298) that meridians (except the central one) also are sine curves, more precisely scaled versions of $\cos\phi$; hence meridians are curved, and do not intersect parallels at right angles. The map is thus not conformal.

The inverse map to (6.298) is

$$(\phi,\theta) = \Bigl(x, \frac{y}{\cos x}\Bigr). \tag{6.299}$$

Hence it follows from (6.80) that

$$\begin{aligned} |\mathrm{d}s|^2 &= |\mathrm{d}\phi|^2 + \cos^2\phi\cdot|\mathrm{d}\theta|^2 = |\mathrm{d}x|^2 + \cos^2 x\Bigl(\frac{\mathrm{d}y}{\cos x} + \frac{y\sin x\,\mathrm{d}x}{\cos^2 x}\Bigr)^2 \\ &= |\mathrm{d}x|^2 + (\mathrm{d}y + y\tan x\,\mathrm{d}x)^2. \end{aligned} \tag{6.300}$$

Hence the metric tensor is

$$g_{11} = 1 + y^2\tan^2 x, \qquad g_{12} = g_{21} = y\tan x, \qquad g_{22} = 1, \tag{6.301}$$

or in matrix form

$$(g_{ij}) = \begin{pmatrix} 1+y^2\tan^2 x & y\tan x \\ y\tan x & 1 \end{pmatrix}. \tag{6.302}$$

The determinant $|g| = 1$, and thus the invariant measure (1.6) is $\mathrm{d}\mu = \mathrm{d}x\,\mathrm{d}y$, i.e., the sinusoidal projection is area preserving. (This can also be seen directly from the construction.)

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[37] It seems to have been used first by Jean Cossin of Dieppe in 1570, and was later used by various map makers during the 17th and 18th centuries, including a star atlas in 1729 by Flamsteed; it has remained in use [16, pp. 50, 60, 166–167, 180–181].

Note that the metric (6.302) is of the type (4.75) with $f(x,y) = y \tan x$. The Christoffel symbols are thus given by (4.79)–(4.82). Further, as a check, (4.84) yields the curvature

$$K = f_{,xy} - f f_{,yy} - f_{,y}^2 = \frac{\mathrm{d}\tan x}{\mathrm{d}x} - 0 - \tan^2 x = 1. \tag{6.303}$$

Example 6.32 (Globular projection, al-Biruni, Nicolosi)**.** The name *globular projection* refers to a group of map projections that map a hemisphere onto a circular disc (which we always take as the unit disc).

There are several versions. (See Examples 6.33–6.35 for some of them; see further e.g. [16, p. 40].) The parallels can be parallel straight lines, with distance from the equator either proportional to the latitude (i.e., with equal spacing along the central meridian), or with distance proportional to the sine of the latitude (i.e., with equal spacing along the perimeter); they can also be circular arcs with equal spacing both on the central meridian and along the perimeter. The central meridian is a vertical line, perpendicular to the parallels; the other meridians are either semiellipses or circular arcs through the poles, in both cases with equal spacing on the equator.

Note that these projections all differ from the equatorial aspect of the ortographic projection in Example 6.6, i.e. orthogonal projection to a plane tangent to the sphere at a point on the equator. The latter has parallels that are straight lines, equally spaced on the perimeter (as Example 6.35), and the meridians are semiellipses, but they are not equispaced on the equator.

The oldest known globular projection, and also the most widely used (and sometimes just called *the globular projection*), has the parallels as circular arcs with equal spacing both on the central meridian and on the perimeter, and the meridians as circular arcs equidistant on the equator.[38]

The meridian with longitude $\theta \in (-\pi/2, \pi/2)$ (and $\theta \neq 0$) is a circular arc through $(1,0)$, $(-1,0)$ and $(0, \frac{2}{\pi}\theta)$; if this arc has radius r_θ and centre $(0, -y_\theta)$, we find for $\theta > 0$,

$$r_\theta^2 = 1 + y_\theta^2 = (y_\theta + \tfrac{2}{\pi}\theta)^2 \tag{6.304}$$

which yields

$$\tfrac{4}{\pi}\theta y_\theta + \tfrac{4}{\pi^2}\theta^2 = 1 \tag{6.305}$$

and

$$y_\theta = \frac{1 - \frac{4}{\pi^2}\theta^2}{\frac{4}{\pi}\theta}, \tag{6.306}$$

$$r_\theta = y_\theta + \tfrac{2}{\pi}\theta = \frac{1 + \frac{4}{\pi^2}\theta^2}{\frac{4}{\pi}\theta}. \tag{6.307}$$

[38]This projection was developed and used for star maps by Abū al-Rayḥān Muḥammad ibn Aḥmad al-Bīrūnī (about 973–after 1050) about 1000. It was reinvented in 1660 by Giovanni Battista Nicolosi (1610–1670). It became the standard projection for maps of the eastern and western hemispheres in the 19th century, but has hardly been used after that. [16, pp. 14, 40–42, 178]

Similarly, the parallel with latitude $\phi \in (-\pi/2, \pi/2)$ (and $\phi \neq 0$) is a circular arc through $(\sin\phi, \pm\cos\phi)$ and $(\frac{2}{\pi}\phi, 0)$; if this arc has radius r_ϕ and centre $(x_\phi, 0)$, we find for $\phi > 0$,

$$r_\phi^2 = (x_\phi - \sin\phi)^2 + \cos\phi^2 = (x_\phi - \tfrac{2}{\pi}\phi)^2 \tag{6.308}$$

which yields

$$-(2\sin\phi)x_\phi + 1 = -\tfrac{4}{\pi}\phi x_\phi + \tfrac{4}{\pi^2}\phi^2 \tag{6.309}$$

and

$$x_\phi = \frac{1 - \frac{4}{\pi^2}\phi^2}{2(\sin\phi - \frac{2}{\pi}\phi)}, \tag{6.310}$$

$$r_\phi = x_\phi - \tfrac{2}{\pi}\phi = \frac{1 - \frac{4}{\pi}\phi\sin\phi + \frac{4}{\pi^2}\phi^2}{2(\sin\phi - \frac{2}{\pi}\phi)}. \tag{6.311}$$

The coordinates (x, y) of a point with latitude ϕ and longitude θ is thus given by solving the system of equations

$$x^2 + (y + y_\theta)^2 = r_\theta^2, \tag{6.312}$$

$$(x - x_\phi)^2 + y^2 = r_\phi^2, \tag{6.313}$$

with the parameters given by (6.306)–(6.307) and (6.310)–(6.311). (At least for $\phi, \theta > 0$. In fact, it is easily seen that (6.312)–(6.313) hold for all $\phi, \theta \neq 0$, although the sign may be incorrect in (6.307) or (6.311).) Subtracting (6.313) from (6.312) yields

$$2x_\phi x + 2y_\theta y = r_\theta^2 - r_\phi^2 + x_\phi^2 - y_\theta^2, \tag{6.314}$$

and substitution in e.g. (6.312) and solving a quadratic yields

$$x = \frac{x_\phi(x_\phi^2 + y_\theta^2 + r_\theta^2 - r_\phi^2) - y_\theta\sqrt{4r_\theta^2 r_\phi^2 - (x_\phi^2 + y_\theta^2 - r_\theta^2 - r_\phi^2)^2}}{2(x_\phi^2 + y_\theta^2)} \tag{6.315}$$

$$y = \frac{-y_\theta(x_\phi^2 + y_\theta^2 - r_\theta^2 + r_\phi^2) + x_\phi\sqrt{4r_\theta^2 r_\phi^2 - (x_\phi^2 + y_\theta^2 - r_\theta^2 - r_\phi^2)^2}}{2(x_\phi^2 + y_\theta^2)}. \tag{6.316}$$

For $\phi = 0$ we have $x = 0$ and $y = \frac{2}{\pi}\theta$; for $\theta = 0$ we have $x = \frac{2}{\pi}\phi$ and $y = 0$.

The algebraic formulas are thus rather complicated, while the geometric construction above is very simple.

Example 6.33 (Globular projection, Apian I)**.** Another globular projection[39] has the parallels as straight lines, equidistant on the central meridian; the meridians are circular arcs with equal spacing on the equator.

The meridian with longitude $\theta \in (-\pi/2, \pi/2)$ (and $\theta \neq 0$) is as in Example 6.32 a circular arc with radius r_θ and centre $(0, -y_\theta)$ given by, for $\theta > 0$,

[39]By Peter Apian (or Bienewitz) in 1524, and used for world maps later in the 16th century. [16, p. 14].

(6.306)–(6.307). A point with latitude ϕ and longitude θ has coordinates (x, y) with $x = \frac{2}{\pi}\phi$ and

$$x^2 + (y + y_\theta)^2 = r_\theta^2, \tag{6.317}$$

and thus

$$\begin{aligned} y &= \sqrt{r_\theta^2 - x^2} - y_\theta = \sqrt{r_\theta^2 - x^2} - \sqrt{r_\theta^2 - 1} \\ &= \frac{1 - x^2}{\sqrt{r_\theta^2 - x^2} + \sqrt{r_\theta^2 - 1}} = \frac{1 - x^2}{\sqrt{r_\theta^2 - x^2} + y_\theta}. \end{aligned} \tag{6.318}$$

Consequently, and in this form valid for all $\theta \in (-\pi/2, \pi/2)$ by symmetry (the case $\theta = 0$ is trivial),

$$x = \frac{2}{\pi}\phi, \tag{6.319}$$

$$y = \frac{\frac{4}{\pi}(1 - x^2)\theta}{\sqrt{\left(1 + \frac{4}{\pi^2}\theta^2\right)^2 - \frac{16}{\pi^2}\theta^2 x^2} + 1 - \frac{4}{\pi^2}\theta^2} \tag{6.320}$$

$$= \frac{\frac{4}{\pi}(1 - \frac{4}{\pi^2}\phi^2)\theta}{\sqrt{\left(1 + \frac{4}{\pi^2}\theta^2\right)^2 - \frac{64}{\pi^4}\theta^2 \phi^2} + 1 - \frac{4}{\pi^2}\theta^2}. \tag{6.321}$$

Conversely, (6.304) and (6.317) yield $x^2 + y^2 + 2yy_\theta = 1$ and thus, for $y > 0$,

$$y_\theta = \frac{1 - x^2 - y^2}{2y} \tag{6.322}$$

and, using (6.304) again,

$$\begin{aligned} \frac{2}{\pi}\theta &= \sqrt{1 + y_\theta^2} - y_\theta = \frac{1}{\sqrt{1 + y_\theta^2} + y_\theta} \\ &= \frac{2y}{\sqrt{(1 - x^2 - y^2)^2 + 4y^2} + 1 - x^2 - y^2}. \end{aligned} \tag{6.323}$$

This holds by symmetry also for $y < 0$, and trivially for $y = 0$. Hence, for all (x, y) in the unit disc,

$$\phi = \frac{\pi}{2}x, \tag{6.324}$$

$$\theta = \frac{\pi y}{\sqrt{(1 - x^2 - y^2)^2 + 4y^2} + 1 - x^2 - y^2} \tag{6.325}$$

$$= \frac{\sqrt{\pi(1 - x^2 - y^2)^2 + 4y^2} - (1 - x^2 - y^2)}{4y}. \tag{6.326}$$

Example 6.34 (Globular projection, Apian II)**.** Another globular projection[40] has the parallels as straight lines, equidistant on the central meridian

[40] Also by Peter Apian (or Bienewitz) in 1524. [16, p. 14].

as in Example 6.33, but the meridians are semiellipses with equal spacing on the equator. This means that the scale is constant along each parallel.

We immediately obtain, cf. (6.319)–(6.321),

$$x = \frac{2}{\pi}\phi, \tag{6.327}$$

$$y = \frac{2}{\pi}\sqrt{1-x^2}\,\theta, \tag{6.328}$$

$$= \frac{2}{\pi}\theta\sqrt{1-\frac{4}{\pi^2}\phi^2}, \tag{6.329}$$

and, conversely,

$$\phi = \frac{\pi}{2}x, \tag{6.330}$$

$$\theta = \frac{\pi y}{2\sqrt{1-x^2}}. \tag{6.331}$$

Example 6.35 (Globular projection, Bacon)**.** Another globular projection[41] has the parallels as straight lines, equidistant on the bounding circle, and the meridians as arcs of circles with equal spacing on the equator.

The meridians are as in Example 6.33, but we now have $x = \sin\phi$ instead of (6.319). Consequently, cf. (6.319)–(6.321) and (6.324)–(6.326),

$$x = \sin\phi, \tag{6.332}$$

$$y = \frac{\frac{4}{\pi}(1-x^2)\theta}{\sqrt{\left(1+\frac{4}{\pi^2}\theta^2\right)^2 - \frac{16}{\pi^2}\theta^2x^2} + 1 - \frac{4}{\pi^2}\theta^2} \tag{6.333}$$

$$= \frac{\frac{4}{\pi}\cos^2\phi\theta}{\sqrt{\left(1+\frac{4}{\pi^2}\theta^2\right)^2 - \frac{16}{\pi^2}\theta^2\sin\phi^2} + 1 - \frac{4}{\pi^2}\theta^2} \tag{6.334}$$

and

$$\phi = \arcsin x, \tag{6.335}$$

$$\theta = \frac{\pi y}{\sqrt{(1-x^2-y^2)^2+4y^2} + 1 - x^2 - y^2} \tag{6.336}$$

$$= \frac{\sqrt{\pi(1-x^2-y^2)^2+4y^2} - (1-x^2-y^2)}{4y}. \tag{6.337}$$

Example 6.36 (Mollweide)**.** *Mollweide's projection*[42] has as the globular projection in Example 6.34 meridians that are semiellipses, equally spaced on the equator, and parallels that are horizontal lines; however, the parallels are not equispaced, instead their spacing is chosen such that the map is area-preserving. Moreover, the projection is extended to the whole sphere minus

[41] Presumably invented and used by Robert Bacon about 1265, and used by Franciscus Monachus about 1527. [16, p. 14].

[42] Invented by Karl Brandan Mollweide (1774–1825) in 1805, and used for some maps in the 19th and 20th centuries. [16, pp. 112–113 and 167–168]

a meridian, mapping the boundary meridian ($\theta = \pm\pi$) to an ellipse with axes in ratio $2:1$, and thus the sphere minus the meridian to the interior of this ellipse. (The hemisphere $|\theta| \leqslant \frac{\pi}{2}$ is mapped to a circle as in globular projections.)

Since the unit sphere has area 4π, an area-preserving map onto an ellipse with semiaxes a and b has to satisfy $\pi ab = 4\pi$, i.e., $ab = 4$. Since we also want $a = 2b$, the projection maps to the ellipse with semiaxes $2\sqrt{2}$ and $\sqrt{2}$.[43] The projection is thus given by

$$x = x(\phi) \tag{6.338}$$

$$y = \frac{2\sqrt{2-x^2}}{\pi}\,\theta \tag{6.339}$$

for some (smooth and increasing) function $x(\phi) : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-\sqrt{2}, \sqrt{2}]$ that remains to be found. The inverse mapping is given by

$$\phi = \phi(x), \tag{6.340}$$

$$\theta = \frac{\pi y}{2\sqrt{2-x^2}}, \tag{6.341}$$

where $\phi(x) : [-\sqrt{2}, \sqrt{2}] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the inverse function of $x(\phi)$. The metric is thus given by

$$\begin{aligned} |\mathrm{d}s|^2 &= |\mathrm{d}\phi|^2 + \cos^2\phi\,|\mathrm{d}\theta|^2 \\ &= \phi'(x)^2|\mathrm{d}x|^2 + \frac{\pi^2}{4}\cos^2\phi\left|\frac{\mathrm{d}y}{\sqrt{2-x^2}} + \frac{y\,\mathrm{d}x}{(2-x^2)^{3/2}}\right|^2 \end{aligned} \tag{6.342}$$

and thus the metric tensor is

$$(g_{ij}) = \begin{pmatrix} \phi'(x)^2 + \frac{\pi^2 y^2\cos^2\phi}{4(2-x^2)^3} & \frac{\pi^2 y\cos^2\phi}{4(2-x^2)^2} \\ \frac{\pi^2 y\cos^2\phi}{4(2-x^2)^2} & \frac{\pi^2\cos^2\phi}{4(2-x^2)} \end{pmatrix} \tag{6.343}$$

with determinant

$$|g| = \phi'(x)^2\frac{\pi^2\cos^2\phi}{4(2-x^2)}. \tag{6.344}$$

The condition that the projection is area-preserving, i.e. $|g| = 1$, is thus equivalent to

$$\phi'(x)\cos\phi = \frac{2}{\pi}\sqrt{2-x^2}, \tag{6.345}$$

or

$$\mathrm{d}(\sin\phi) = \frac{2}{\pi}\sqrt{2-x^2}\,\mathrm{d}x. \tag{6.346}$$

[43] We shall use this normalization of Mollweide's projection. Since we generally ignore linear scaling, we might as well divide the coordinates by $\sqrt{2}$ and obtain an equivalent version that maps to an ellipse with semiaxes 2 and 1, and with the hemisphere $|\theta| \leqslant \frac{\pi}{2}$ mapped to the unit circle; this projection multiplies all areas by the constant factor $\frac{1}{2}$.

To solve this we introduce $\psi := \arcsin(x/\sqrt{2})$, so $x = \sqrt{2}\sin\psi$ and (6.346) can be written

$$\mathrm{d}(\sin\phi) = \tfrac{2}{\pi}\sqrt{2}\cos\psi\,\mathrm{d}\big(\sqrt{2}\sin\psi\big) = \tfrac{4}{\pi}\cos^2\psi\,\mathrm{d}\psi = \tfrac{2}{\pi}(1+\cos 2\psi)\,\mathrm{d}\psi \tag{6.347}$$

with the solution (choosing $x = 0$ for $\phi = 0$)

$$\sin\phi = \frac{2\psi + \sin 2\psi}{\pi}. \tag{6.348}$$

Summarizing, Mollweide's projection is given by

$$x = \sqrt{2}\,\sin\psi, \tag{6.349}$$

$$y = \frac{2\sqrt{2}}{\pi}\theta\cos\psi, \tag{6.350}$$

where $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is given by

$$2\psi + \sin 2\psi = \pi\sin\phi. \tag{6.351}$$

(Note that the left-hand side is a strictly increasing function of $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ that maps $[-\frac{\pi}{2}, \frac{\pi}{2}]$ onto $-[\pi, \pi]$, so (6.351) defines ψ uniquely for every $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.)

Conversely, (6.349) gives

$$\psi = \arcsin(x/\sqrt{2}), \tag{6.352}$$

$$\sin\psi = x/\sqrt{2}, \tag{6.353}$$

$$\cos\psi = \sqrt{1 - x^2/2}, \tag{6.354}$$

$$\sin 2\psi = x\sqrt{2 - x^2}. \tag{6.355}$$

Hence, using also (6.350)–(6.351),

$$\phi = \arcsin\Big(\frac{2}{\pi}\arcsin\frac{x}{\sqrt{2}} + \frac{1}{\pi}x\sqrt{2 - x^2}\Big), \tag{6.356}$$

$$\theta = \frac{\pi y}{2\sqrt{2 - x^2}}. \tag{6.357}$$

The metric tensor is thus (6.343) with $\phi'(x)$ given by (6.345) and, by (6.356),

$$\cos^2\phi = 1 - \Big(\frac{2}{\pi}\arcsin\frac{x}{\sqrt{2}} + \frac{1}{\pi}x\sqrt{2 - x^2}\Big)^2. \tag{6.358}$$

Example 6.37 (Conformal circle-preserving projections)**.** Consider the class of (orientation-preserving) conformal projections (of some domain in $\mathbb{S}^2$) such that all (parts of) circles on the sphere are represented by (parts of) circles or lines. Note that the stereographic projection Φ_S in Example 6.1 has this property. It follows from Example C.3 that any such projection equals $g \circ \Phi_S$ for some Möbius transformation Φ_S, and conversely.

This includes the general stereographic projections in Example 6.4. In fact, it can be seen that any such projection $g \circ \Phi_S$ is given by some stereographic projection as in Example 6.4 followed by a translation and a multiplication by a positive constant. Hence, the class of conformal projections that preserve circles is essentially given by the general stereographic projections.

Example 6.38 (Lagrange)**.** Lambert (1772) and Lagrange [11] studied 1779 conformal projections such that all parallels and meridians are (parts of) circles or lines.[44] (This includes the projections in Example 6.37, but the present class is larger class since only some special circles are required to be mapped to circles or lines.)

We note that Mercator's projection Example 6.11 is of this type, with parallels and meridians mapped to vertical and horizontal lines (in this order, if we use the non-standard orientation in Example 6.11). By (6.97) and the comments after it, we may write Mercator's projection as the map $\log \Phi_S$ into $\mathbb{C}$, where Φ_S is the stereographic projection in Example 6.1. It follows from Example C.4 that the class of all (orientation-preserving) conformal projections that map parallels and meridians to circles or lines (even when defined only in some small domain in $\mathbb{S}^2$) is given by the three types

(i) $h(\log \Phi_S)$
(ii) $h(e^{c \log \Phi_S}) = h(\Phi_S^c)$
(iii) $h(e^{-\mathrm{i}c \log \Phi_S}) = h(e^{\theta + \mathrm{i}\zeta}) = h\big(e^{\theta} e^{-\mathrm{i} \log \tan(\varphi/2)}\big)$

where h is a Möbius transformation and $c > 0$. (Furthermore, we see from Example C.4 that it suffices to require that either parallels or meridians are projeted to circles or lines.)

Type (i) is thus the set of Möbius transformations of Mercator's projection.

Type (ii) includes Lambert's conformal conic projection Example 6.26, obtained with h the identity (or, more generally, $h(z) = az$). A general projection of this type is thus a Möbius transformation of Lambert's conformal conic projection.

A special case, constructed by Lambert (1772) but called the *Lagrange projection* is obtained with $c = 1/2$ and $h(z) = (z-1)/(z+1)$, i.e.,

$$\xi \mapsto \frac{\Phi_S(\xi)^{1/2} - 1}{\Phi_S(\xi)^{1/2} + 1} = \frac{\tan^{1/2}(\frac{\varphi}{2}) e^{\mathrm{i}\theta/2} - 1}{\tan^{1/2}(\frac{\varphi}{2}) e^{\mathrm{i}\theta/2} + 1}. \tag{6.359}$$

This is defined on $\mathbb{S}^2$ minus the meridian $\theta = \pi$; Φ_S maps this domain to $\mathbb{C} \setminus (-\infty, 0]$, so $\Phi_S^{1/2}$ maps onto the right half-plane $\operatorname{Re} z > 0$, which is mapped by h onto the unit disc.[45] The Lagrange projection thus maps the whole sphere, except a meridian, onto a disc, with all meridians and parallels circular arcs (lines for the equator and central meridian). (This is somewhat similar to the globular projections in e.g. Example 6.32, but the Lagrange

[44] Such projections were constructed by Lambert (1772) and studied further by Lagrange [11] (1779), see [16, pp. 80–82].

[45] The North Pole is mapped to -1, the South Pole to 1 and the equator to a vertical line between $-\mathrm{i}$ and i. The traditional orientation is obtained by multiplying (6.359) by $-\mathrm{i}$.

projection maps the whole sphere, except for a meridian, and is furthermore conformal.)

If we denote the stereographic coordinates by $w \in \mathbb{C}$ and the Lagrange projection by z, so (6.359) becomes $z = (w^{1/2} - 1)/(w^{1/2} + 1)$, then the inverse is

$$w = \Bigl(\frac{1+z}{1-z}\Bigr)^2. \tag{6.360}$$

It follows by (6.6) that the metric is given by

$$\begin{aligned} |\mathrm{d}s| &= \frac{2}{1+|w|^2}|\mathrm{d}w| = \frac{2}{1+|w|^2}\Bigl|\frac{\mathrm{d}w}{\mathrm{d}z}\Bigr|\,|\mathrm{d}z| \\ &= \frac{2|1-z|^4}{|1-z|^4+|1+z|^4}\left|\frac{4(1+z)}{(1-z)^3}\right|\,|\mathrm{d}z| \\ &= \frac{8|1-z^2|}{|1-z|^4+|1+z|^4}\,|\mathrm{d}z| =: H(z)|\mathrm{d}z| \end{aligned} \tag{6.361}$$

or equivalently

$$g_{ij} = H(z)^2\delta_{ij} \tag{6.362}$$

with $H(z)$ defined in (6.361).

Type (iii) seems less interesting. For one thing, it will self-overlap unless φ is restricted.

Example 6.39 (Retroazimuthal projections)**.** An azimuthal projection Φ centered at at point $P_0 \in \mathbb{S}^2$ (different from the north pole considered in Example 6.16, or the south pole) has the property that the direction from P_0 to any point P on the sphere is the same as the direction from $\Phi(P_0)$ to $\Phi(P)$ in the plane. (The direction on the sphere, measured clockwise from North, is known as the *azimuth*. In the plane we measure directions from a fixed direction, and assume the the map is oriented properly.[46])

A *retroazimuthal projection* is a projection with the opposite property, for a fixed point P_0: the direction on the sphere from any point P in the domain to P_0 is the same as the direction from $\Phi(P)$ to $\Phi(P_0)$ on the map.[47]

If P_0 has latitude and longitude $(\phi_0, 0) = (\frac{\pi}{2} - \varphi_0, 0)$ and P has latitude and longitude $(\phi, \theta) = (\frac{\pi}{2} - \varphi, \theta)$, then the azimuth α from P to P_0 is (e.g. by spherical trigonometry applied to the triangle NPP_0, see [8]) given by

$$\cot\alpha = \frac{\sin\varphi_0\,\cos\varphi\,\cos\theta - \cos\varphi_0\sin\varphi}{\sin\varphi_0\,\sin\theta}. \tag{6.363}$$

[46]Traditionally, the azimuth is measured from the y-axis clockwise, i.e. in the negative direction.

[47]Such map projections have perhaps greater mathematical than practical interest; they have little practical use. "Retroazimuthal maps are lttle more than curiosities." [16, p. 231]

If $\Phi(P) = (x, y)$ and $\Phi(P_0) = (0, y_0)$, the projection is thus retroazimuthal if and only if, for every P,

$$\frac{y - y_0}{x} = \frac{\sin\varphi_0\,\cos\varphi\,\cos\theta - \cos\varphi_0\sin\varphi}{\sin\varphi_0\,\sin\theta} \tag{6.364}$$

or, equivalently,

$$\frac{y - y_0}{x} = \sin\phi\,\cot\theta - \tan\phi_0\frac{\cos\phi}{\sin\theta}. \tag{6.365}$$

We give some examples in Examples 6.40–6.43.

Note that a retroazimuthal projection in general is not conformal, nor are geodesics straight lines. The geodesic from P to P_0 forms the same angle with the meridian through P as the straight line from $\Phi(P)$ to $\Phi(P_0)$ on the map forms with the fixed ("up") direction, but in general the geodesic is not this straight line and the meridian does not point in the fixed direction.

Example 6.40 (Mecca projection)**.** The *Mecca projection*[48] is the retroazimuthal projection (Example 6.39) with $x = \theta$; the meridians are thus equispaced vertical lines. (So in this case, the angle on the map is measured from the meridian.) By (6.365), the projection is given by

$$x = \theta, \tag{6.366}$$

$$y = \sin\phi\,\frac{\theta\cos\theta}{\sin\theta} - \tan\phi_0\cos\phi\,\frac{\theta}{\sin\theta}. \tag{6.367}$$

This projection cannot be defined on the whole sphere. For example, if $\phi_0 = 0$, so P_0 is on the equator, then any P with longitude $\pm\frac{\pi}{2}$ has azimuth $\pm\frac{\pi}{2}$ to P_0, so these two meridians collapse to one point each; in this case, the projection thus has to be restricted to the hemisphere $|\theta| < \frac{\pi}{2}$, where it is a bijection.

Example 6.41 (Equidistant retroazimuthal projection)**.** The *equidistant retroazimuthal projection* or *Hammer retroazimuthal projection*[49] has correct azimuth α and distance r to P_0 from any point P. Thus the coordinates of a point P are $(-r\sin\alpha, -r\cos\alpha)$ (choosing $y_0 = 0$). The distance r is by the spherical cosine theorem for the triangle NPP_0 [8] (or (6.79) and linear algebra) given by

$$\cos r = \sin\phi_0\,\sin\phi + \cos\phi_0\,\cos\phi\,\cos\theta. \tag{6.368}$$

Furthermore, the spherical sine theorem, for the same triangle NPP_0, yields $\sin\alpha/\sin\varphi_0 = -\sin\theta/\sin r$. Thus, using also (6.363),

$$x = -r\sin\alpha = \frac{r}{\sin r}\cos\phi_0\,\sin\theta, \tag{6.369}$$

[48]Designed in 1909 by James Craig (1868–1952), with Mecca as P_0, thus enabling Muslims to easily find the correct direction (*qibla*) that should be faced during prayers. The projection has occasionally been used also with other centres P_0. [16, pp. 227–228]

[49]Designed in 1910 by Hammer. Reinvented 1929 by Hinks and Reeves who chose the centre P_0 as the radio station in Rugby. [16, pp. 228–229]

$$y = -r\cos\alpha = \frac{r}{\sin r}\bigl(\cos\phi_0\ \sin\phi\ \cos\theta - \sin\phi_0\ \cos\phi\bigr), \tag{6.370}$$

with r given by (6.368).

A calculation shows that the Jacobian $\bigl|\frac{\partial(x,y)}{\partial(\phi,\theta)}\bigr|$ vanishes when $\theta = \pm\frac{\pi}{2}$, so the projection is singular on the meridians with longitude differing by $\pm\frac{\pi}{2}$ from P_0. Moreover, beyond these meridians, the orientation is reversed. In fact, the projection in the remote hemisphere is the same as the projection with centre the antipodal point $\overline{P_0}$, combined with a rotation and an inversion $|z| \mapsto \pi - |z|$. In the special case $\phi_0 = 0$, all points with $\theta = \pm\frac{\pi}{2}$ are mapped to $(\pm\frac{\pi}{2}, 0)$, so the two singular meridians collapse to one point each. (This case follows from (6.369)–(6.370), but it is also easily seen geometrically that in this case, the azimuth from P to P_0 is $\pm\frac{\pi}{2}$.)

Example 6.42 (Littrow)**.** The *Littrow projection*[50] maps the point with latitude and longitude (ϕ, θ) to the point (x, y) with

$$x = \frac{\sin\theta}{\cos\phi} = \frac{\sin\theta}{\sin\varphi}, \tag{6.371}$$

$$y = \cos\theta\tan\phi = \frac{\cos\varphi\cos\theta}{\sin\varphi}. \tag{6.372}$$

In particular, $y_0 = \tan\phi_0 = \cot\varphi_0$.

It is immediately seen that this projection satisfies (6.364) for any φ_0. The Littrow projection thus is retroazimuthal with respect to *any* point on the central meridian. (The Littrow projection is the unique projection with this property. In fact, a projection that is retroazimuthal with respect to just two points is uniquely determined, since the azimuths to these two points determine the image of each point P, as the intersection of two lines through the images of these points.)

We next show that the Littrow projection is conformal. It follows that, moreover, it is the unique conformal retroazimuthal projection (even locally). In fact any other orientation-preserving conformal projection equals the composition $g\circ\Psi$ of the Littrow projection Ψ and an analytic function g; by translation we may assume that both projections map P_0 to 0, and since both projections are retroazimuthal, g preserves the argument, i.e., $g(z)/z$ is real, and thus constant. (If we consider orientation-reversing projections, there is another possibility, with g the inversion in a circle centred at $\Psi(P_0)$.)

Denote the Littrow projection by Ψ and let Φ_Z be the stereographic projection from $Z = (-1, 0, 0)$, see Example 6.4, both regarded as maps into the complex plane $\mathbb{C}$. (Note that both Ψ and Φ_Z map $(0,0)$ to 0.)

[50] Invented by Joseph Johann von Littrow (1781–1840) in 1833. [16, pp. 134–135]

Then (6.371)–(6.372) and (6.44)–(6.47) yield

$$\Psi = x + \mathrm{i}y = \frac{(\Phi_Z + \overline{\Phi}_Z)(1+|\Phi_Z|^2)}{|1+\Phi_Z^2|^2} + \frac{(1-|\Phi_Z|^2)(\Phi_Z - \overline{\Phi}_Z)}{|1+\Phi_Z^2|^2}$$
$$= \frac{2(\Phi_Z + \overline{\Phi}_Z|\Phi_Z|^2)}{|1+\Phi_Z^2|^2} = \frac{2\Phi_Z(1+\overline{\Phi}_Z^2)}{(1+\Phi_Z^2)(1+\overline{\Phi}_Z^2)} = \frac{2\Phi_Z}{1+\Phi_Z^2}. \tag{6.373}$$

This expresses Ψ as an analytic function of the conformal projection Φ_Z, and verifies that the Littrow projection Ψ is conformal. (Except at the points where $\Phi_Z = \pm 1$, where the derivative of $z \mapsto 2z/(1+z^2)$ vanishes, i.e., at E and W, see Table 1. At these two points, angles are doubled.)

We can rewrite (6.373) as

$$\frac{2}{\Psi} = \Phi_Z + \frac{1}{\Phi_Z}. \tag{6.374}$$

Note that (6.374) shows that each value of Ψ correponds to 2 values of Φ_Z, except $\Psi = \pm 1$ which correspond only to $\Phi_Z = \Psi = \pm 1$, which occurs at the East and West Poles E and W, see Table 1. Since Φ_Z is a 1–1 map (of $\mathbb{S}$ onto the extended complex plane $\mathbb{C} \cup \{\infty\}$), it follows that Ψ is a 2–1 map, except at W and E (with N and S mapped to ∞). In fact, it follows directly from (6.371)–(6.372) that (ϕ, θ) is mapped to the same point as $(-\phi, \pi - \theta)$, obtained by rotating the sphere π around the y-axis (the axis through W and E).

Hence, the Littrow projection is a bijection on any open half-sphere with E and W on the boundary, for example the half-sphere $|\theta| < \frac{\pi}{2}$, which is mapped to $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$. (On the boundary, E and W are mapped to 1 and -1, respectively, and N and S both to ∞.)

We can further rewrite (6.373) as

$$\Psi = \frac{2\Phi_Z}{1+\Phi_Z^2} = \frac{(1+\Phi_Z)^2 - (1-\Phi_Z)^2}{(1+\Phi_Z)^2 + (1-\Phi_Z)^2} = \frac{\big((1+\Phi_Z)/(1-\Phi_Z)\big)^2 - 1}{\big((1+\Phi_Z)/(1-\Phi_Z)\big)^2 + 1}. \tag{6.375}$$

It is easily verified, e.g. using Table 1, that $(1+\Phi_Z)/(1-\Phi_Z) = \Phi_E$. Hence,

$$\Psi = \frac{\Phi_E^2 - 1}{\Phi_E^2 + 1}. \tag{6.376}$$

This shows that the Littrow projection can be regarded as an equatorial aspect (since we use Φ_E instead of Φ_S) of the general Lagrange projection in Example 6.38 constructed with $c = 2$ and $h(z) = (z-1)/(z+1)$, cf. (6.359).

The Littrow projection has further interesting geometrical properties. The meridians are (branches of) hyperbolas with the foci 1 and -1, and the parallels are (parts of) ellipses with the same foci (the equator is the degenerate ellips $[-1, 1]$); if we consider the projection as defined in the half-sphere $|\theta| < \frac{\pi}{2}$, a parallel in the northern [southern] hemisphere is the part of such an ellipse in the upper [lower] half-plane. To see this, we may recognise (6.371)–(6.372) with θ or ϕ fixed as a parametric representation of a hyperbola or ellipse, respectively. Alternatively, we may note that an elementary calculation using (6.371)–(6.372) yields the distances to ± 1 as

$$|\Psi - 1| = \frac{1}{\cos\phi} - \sin\theta, \tag{6.377}$$

$$|\Psi + 1| = \frac{1}{\cos\phi} + \sin\theta. \tag{6.378}$$

Hence

$$|\Psi - 1| + |\Psi + 1| = 2/\cos\phi \tag{6.379}$$

is constant on parallels, and

$$|\Psi - 1| - |\Psi + 1| = -2\sin\theta \tag{6.380}$$

is constant on meridians.

The relation (6.376) can be inverted to $\Phi_E^2 = (1+\Psi)/(1-\Psi)$, and thus

$$\Phi_E = \sqrt{\frac{1+\Psi}{1-\Psi}} \tag{6.381}$$

(with the principal branch for $|\theta| < \frac{\pi}{2}$). (This, together with the relation $\Phi_Z = (\Phi_E - 1)/(\Phi_E + 1)$, which can be verified from Table 1, and (6.44)–(6.47) can be used to invert the Littrow projection Ψ; however, it is simpler to use (6.379)–(6.380).)

Since Φ_E induces the same metric (6.6) on $\mathbb{C}$ as Φ_S, see Example 6.4, the metric on $\mathbb{C}$ induced by Ψ is

$$\begin{aligned}
|\mathrm{d}s| &= \frac{2}{1+|\Phi_E|^2}|\mathrm{d}\Phi_E| = \frac{2}{1+|\Phi_E|^2}\left|\frac{\mathrm{d}\Phi_E}{\mathrm{d}\Psi}\right||\mathrm{d}\Psi| \\
&= \frac{2}{1+\bigl|(1+\Psi)/(1-\Psi)\bigr|}\cdot\frac{1}{|1+\Psi|^{1/2}|1-\Psi|^{3/2}}|\mathrm{d}\Psi| \\
&= \frac{2}{(|1+\Psi|+|1-\Psi|)|1-\Psi^2|^{1/2}}|\mathrm{d}\Psi|.
\end{aligned} \tag{6.382}$$

In other words, the conformal metric on $\mathbb{C}$ induced by the Littrow projection is

$$g_{ij} = \frac{4}{(|1+z|+|1-z|)^2|1-z^2|}\delta_{ij}. \tag{6.383}$$

Using (6.377)–(6.378), this can be written as

$$g_{ij} = \frac{\cos^4\phi}{1-\cos^2\phi\sin^2\theta}\delta_{ij}. \tag{6.384}$$

The invariant measure (1.6) is

$$\mathrm{d}\mu = \frac{4}{(|1+z|+|1-z|)^2|1-z^2|}\,\mathrm{d}x\,\mathrm{d}y = \frac{\cos^4\phi}{1-\cos^2\phi\sin^2\theta}\,\mathrm{d}x\,\mathrm{d}y. \tag{6.385}$$

The latter form can also easily be obtained directly from (6.371)–(6.372) by computing the Jacobian.

Example 6.43 (Maurer)**.** The *Maurer projection* is retroazimuthal with respect to two points, both supposed to be on the equator. We suppose that the two centres have longitude $\pm\theta_0$, so their standard coordinates on are $(0, \pm\theta_0)$, where we may assume $0 < \theta_0 < \frac{\pi}{2}$. (The case $\theta_0 = \frac{\pi}{2}$ has to be excluded; it is easily seen that no map can be retroazimuthal with respect to two antipodes.) By the retroazimuthal property of one centre with respect to the other, the two centres are mapped to points on the same horizontal line, and we may suppose that they are mapped to $(\pm x_0, 0)$ for some constant $x_0 > 0$. (This x_0 is just an arbitrary scale factor, so we may choose $x_0 = 1$. However, we find another choice more convenient below.)

With these assumptions, (6.365) yields, after a translation and recalling $\phi_0 = 0$,

$$\frac{y}{x \pm x_0} = \sin\phi \, \cot(\theta \pm \theta_0) \tag{6.386}$$

or, equivalently,

$$\frac{x \pm x_0}{y} = \frac{\tan(\theta \pm \theta_0)}{\sin\phi}. \tag{6.387}$$

This yields

$$\begin{aligned}
\frac{2x}{y} &= \frac{\tan(\theta+\theta_0) + \tan(\theta-\theta_0)}{\sin\phi} \\
&= \frac{(\tan\theta + \tan\theta_0)(1+\tan\theta\,\tan\theta_0) + (\tan\theta - \tan\theta_0)(1-\tan\theta\,\tan\theta_0)}{\sin\phi(1-\tan\theta\,\tan\theta_0)(1+\tan\theta\,\tan\theta_0)} \\
&= \frac{2\tan\theta + 2\tan\theta\,\tan^2\theta_0}{\sin\phi(1-\tan^2\theta\,\tan^2\theta_0)} = \frac{2\tan\theta(1+\tan^2\theta_0)}{\sin\phi(1-\tan^2\theta\,\tan^2\theta_0)} \\
&= \frac{2\sin\theta\,\cos\theta}{\sin\phi(\cos^2\theta\,\cos^2\theta_0 - \sin^2\theta\,\sin^2\theta_0)} \\
&= \frac{\sin 2\theta}{\sin\phi(\cos^2\theta - \sin^2\theta_0)}
\end{aligned} \tag{6.388}$$

and, similarly (e.g., by interchanging θ and θ_0),

$$\frac{2x_0}{y} = \frac{\tan(\theta+\theta_0) - \tan(\theta-\theta_0)}{\sin\phi} = \frac{\sin 2\theta_0}{\sin\phi(\cos^2\theta - \sin^2\theta_0)} \tag{6.389}$$

Hence, $x/x_0 = \sin 2\theta / \sin 2\theta_0$. We choose for simplicity $x_0 = \sin 2\theta_0$, and find from (6.388)–(6.389)

$$x = \sin 2\theta, \tag{6.390}$$

$$y = 2\sin\phi(\cos^2\theta - \sin^2\theta_0) = \sin\phi(\cos 2\theta + \cos 2\theta_0). \tag{6.391}$$

Note that $(0, \theta)$ and $(0, \frac{\pi}{2} - \theta)$ map to the same point $(\sin 2\theta, 0)$, so the map is singular at $\theta = \pm\frac{\pi}{4}$ and the equator will self-overlap around $\theta = \pm\frac{\pi}{4}$. Furthermore, $y = 0$ if $\cos\theta = \sin\theta_0$, i.e., $\theta = \pm(\frac{\pi}{2} - \theta_0)$, so the map is singular at these longitudes. Consequently, we have to restrict the map to

$|\theta| < \min(\frac{\pi}{4}, \frac{\pi}{2} - \theta_0)$. In this region, the map is a diffeomorphism. If we want to include the centres in the map, we thus need $|\theta_0| < \frac{\pi}{4}$, and then the map is defined for $|\theta| < \frac{\pi}{4}$.

It follows from (6.390)–(6.391) that the meridians are vertical lines (not equispaced), and that the parallels are (parts of) ellipses. More precisely, if the map is defined for $|\theta| < \frac{\pi}{4}$, then the parallels are semi-ellipses (the upper part in the northern hemisphere, and the lower in the southern hemisphere), where the omitted part of the ellips goes through the centres (corresponding to $\theta = \pm(\frac{\pi}{2} - \theta_0)$). (The poles correspond to parts of circles.) The equator is mapped to the straight line between the centres.

The Jacobian is easily computed from (6.390)–(6.391), and it follows, using (6.84), that the invariant measure (1.6) is

$$d\mu = \frac{dx\,dy}{2\cos 2\theta\,(\cos 2\theta + \cos 2\theta_0)}, \tag{6.392}$$

where we again see that we need to restrict to the region $|\theta| < \min(\frac{\pi}{4}, \frac{\pi}{2} - \theta_0)$.

7. Examples, hyperbolic space

We give several examples of hyperbolic metrics. Some are global and give different models of the hyperbolic space; see in particular Example 7.1 (Poincaré half-space model), Example 7.3 (Poincaré ball model), Example 7.7 (Klein ball model) and Example 7.5 (hyperboloid model) for some important models. Some others give models only of parts of hyperbolic space. Usually we consider for simplicity only the case $K = -1$; any other constant negative curvature may be obtained by scaling the metric by a constant factor; see Section 3.5 and Example 7.4. (All examples with the same dimension and the same curvature K are at least locally isometric.)

As in Section 6, we begin with some examples for arbitrary dimension n but then specialise to $n = 2$ (the hyperbolic plane) in many further examples. (Some of these examples too can be generalized to arbitrary n, but we leave that to the reader.) Of particular importance is the connection to complex analysis, so we routinely identify $\mathbb{R}^2$ and the complex plane $\mathbb{C}$, and use the complex coordinate $z = x + iy$ in several examples.

Example 7.1 (Poincaré metric in a half-space)**.** Let $H^n := \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ with

$$ds = \frac{|dx|}{x_n}, \tag{7.1}$$

i.e.,

$$g_{ij} = x_n^{-2}\delta_{ij}. \tag{7.2}$$

This is a conformal metric (3.10) with $f(x) = x_n^{-1}$ and $\varphi(x) = -\log x_n$. Thus, cf. e.g. [4, Section 8.3],

$$\varphi_{,i} = -x_n^{-1}\delta_{in} \tag{7.3}$$

and (3.14) yields

$$\Gamma_{ij}^k = -x_n^{-1}\big(\delta_{in}\delta_{jk} + \delta_{jn}\delta_{ik} - \delta_{kn}\delta_{ij}\big). \tag{7.4}$$

Furthermore,

$$\varphi_{,ij} = x_n^{-2}\delta_{in}\delta_{jn} = \varphi_{,i}\varphi_{,j}, \tag{7.5}$$

$$\|\nabla\varphi\|^2 = |\varphi_{,n}|^2 = x_n^{-2}, \tag{7.6}$$

$$\Delta\varphi = \varphi_{,nn} = x_n^{-2}. \tag{7.7}$$

Thus, by (3.15),

$$R_{ijkl} = -x_n^{-4}\big(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\big) = -\big(g_{ik}g_{jl} - g_{il}g_{jk}\big) = -\tfrac{1}{2}g_{ij} \odot g_{kl}. \tag{7.8}$$

H^n is thus a manifold with constant curvature, see (2.43). By (3.16)–(3.17) or (2.38)–(2.39),

$$R_{ij} = -(n-1)x_n^{-2}\delta_{ij} = -(n-1)g_{ij}, \tag{7.9}$$

$$R = -n(n-1). \tag{7.10}$$

The sectional curvature is, by (2.35), (2.36), (2.38) or (2.39),

$$K = -1. \tag{7.11}$$

The distance $d(x,y)$ between x and y in H^n has the explicit formulas

$$\cosh d(x,y) = 1 + \frac{|x-y|^2}{2x_ny_n}, \tag{7.12}$$

$$\sinh(d(x,y)/2) = \frac{|x-y|}{2\sqrt{x_ny_n}}, \tag{7.13}$$

$$\cosh(d(x,y)/2) = \frac{|x-\bar{y}|}{2\sqrt{x_ny_n}}, \tag{7.14}$$

where $\overline{(y',y_n)} := (y',-y_n)$.

The invariant measure (1.6) is by (4.3)

$$\mathrm{d}\mu = x_n^{-n}\,\mathrm{d}x_1\cdots\,\mathrm{d}x_n. \tag{7.15}$$

The Laplace–Beltrami operator is, by (3.22),

$$\Delta F = x_n^2\sum_{i=1}^n F_{,ii} - (n-2)x_nF_{,n}. \tag{7.16}$$

The isometries of H^n is the Möbius group of (proper and improper) Möbius maps of $\mathbb{R}^n$ that preserve H^n, which is isomorphic to the Möbius group of $\mathbb{R}^{n-1}$. (The Möbius group is generated by inversions, see [7, I.7 and II.7].) In particular, if $n=2$, the isometries are the analytic Möbius maps

$$z \mapsto \frac{az+b}{cz+d} \tag{7.17}$$

with $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$ (we may assume $ad - bc = 1$), and the conjugate analytic

$$z \mapsto \frac{-a\bar{z} + b}{-c\bar{z} + d} \tag{7.18}$$

with $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$.

The geodesics in H^n are

(i) lines orthogonal to the boundary $\partial H^n = \mathbb{R}^{n-1}$ (i.e., the x_n-axis and any line parallel to it);

(ii) (half) circles (in two-dimensional planes) orthogonal to ∂H^n (i.e., circles with centres in ∂H^n in planes orthogonal to ∂H^n).

Unit speed parametrizations of the two types of geodesics are

$$t \mapsto (x', e^t) \tag{7.19}$$

for any fixed $x' \in \mathbb{R}^{n-1}$ and,

$$t \mapsto \Big(x'_0 + (\tanh t)v, \frac{y_0}{\cosh t}\Big), \tag{7.20}$$

where $x'_0, v \in \mathbb{R}^{n-1}$ and $y_0 = \|v\|$ (Euclidean norm). (This is perhaps easiest to see by considering the case $n = 2$, and using Möbius invariance.)

Example 7.2. The mapping $(x', x_n) \mapsto (x', -\log x_n)$ is a diffeomorphism of the upper half-space H^n in Example 7.1 onto $\mathbb{R}^n$, and it maps the Poincaré metric (7.1) onto the metric

$$|\mathrm{d}s|^2 = e^{2x_n}|\mathrm{d}x'|^2 + |\mathrm{d}x_n|^2. \tag{7.21}$$

This is (after relabelling) a metric of the type (4.33) in Example 4.6, with $\tilde{M} = \mathbb{R}^{n-1}$ with the usual (flat) Euclidean metric and $w(r) = e^r$. (This metric is clearly not conformal.) Since (4.55) holds (with $\tilde{K} = 0$), we see from (4.56) that the space has constant curvature $K = -1$, as we also saw in Example 7.1.

In this example, the vertical lines are geodesics with correct scaling.

Example 7.3 (Poincaré metric in a ball (disc))**.** Let D^n be the unit ball in $\mathbb{R}^n$ with the metric

$$\mathrm{d}s = \frac{2\,|\mathrm{d}x|}{1 - |x|^2}, \tag{7.22}$$

i.e.,

$$g_{ij} = \frac{4}{(1 - |x|^2)^2}\delta_{ij}. \tag{7.23}$$

This is a conformal metric of the form (4.2) with $a = 2$ and $c = -1$; thus (3.10) holds with $f(x) = 2/(1 - |x|^2)$ and $\varphi(x) = \log 2 - \log(1 - |x|^2)$. (Formally, this can be seen as the metric (6.7) of a sphere with imaginary radius $\rho = \mathrm{i}$.) We have, specializing the formulas in Example 4.1,

$$g^{ij} = \tfrac{1}{4}(1 - |x|^2)^2\delta_{ij} \tag{7.24}$$

and

$$\Gamma^k_{ij} = \frac{2}{1-|x|^2}\bigl(x_i\delta_{jk} + x_j\delta_{ik} - x_k\delta_{ij}\bigr); \tag{7.25}$$

the metric has constant sectional curvature

$$K = -1, \tag{7.26}$$

by (4.10) and our choice of a, and thus

$$R_{ijkl} = -\frac{1}{2} g_{ij} \odot g_{kl} = -(g_{ik}g_{jl} - g_{il}g_{jk}) = -\frac{8}{(1-|x|^2)^4}\delta_{ij} \odot \delta_{kl}, \tag{7.27}$$

$$R_{ij} = -(n-1)g_{ij} = -\frac{4(n-1)}{(1-|x|^2)^2}\delta_{ij}, \tag{7.28}$$

$$R = -n(n-1). \tag{7.29}$$

The invariant measure (1.6) is by (4.3),

$$\mathrm{d}\mu = \frac{2^n}{(1-|x|^2)^n}\,\mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{7.30}$$

The distance $d(x,y)$ between x and y in D^n is given by

$$\cosh d(x,y) = 1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}. \tag{7.31}$$

In particular,

$$\cosh d(x,0) = 1 + \frac{2|x|^2}{1-|x|^2} = \frac{1+|x|^2}{1-|x|^2}, \tag{7.32}$$

$$\sinh d(x,0) = \frac{2|x|}{1-|x|^2}, \tag{7.33}$$

$$\tanh d(x,0) = \frac{2|x|}{1+|x|^2}, \tag{7.34}$$

$$\cosh(d(x,0)/2) = \frac{1}{\sqrt{1-|x|^2}}, \tag{7.35}$$

$$\sinh(d(x,0)/2) = \frac{|x|}{\sqrt{1-|x|^2}}, \tag{7.36}$$

$$\tanh(d(x,0)/2) = |x|, \tag{7.37}$$

$$d(x,0) = \log\frac{1+|x|}{1-|x|} = \log(1+|x|) - \log(1-|x|). \tag{7.38}$$

The Laplace–Beltrami operator is, by (3.22),

$$\Delta F = \frac{(1-|x|^2)^2}{4}\sum_{i=1}^n F_{,ii} + (n-2)\frac{1-|x|^2}{2}\sum_{i=1}^n x_i F_{,i}. \tag{7.39}$$

In particular, for a radial function $F(r)$, with $r = |x|$,

$$\Delta F = \frac{(1-r^2)^2}{4}\Big(F''(r) + (n-1)\frac{F'(r)}{r}\Big) + (n-2)\frac{1-r^2}{2} rF'(r)$$
$$= \frac{1-r^2}{4r}\Big(r(1-r^2)F''(r) + \big(n-1+(n-3)r^2\big)F'(r)\Big). \tag{7.40}$$

Examples 7.1 and 7.3 are isometric. Writing $x = (x', x_n)$, the inversion ψ in the sphere with centre $(0, -1)$ and radius $\sqrt{2}$, which in coordinates is given by

$$\psi(x', x_n) = 2\frac{(x', x_n + 1)}{|x|^2 + 2x_n + 1} + (0, -1) = \frac{(2x', 1 - |x|^2)}{|x|^2 + 1 + 2x_n}, \tag{7.41}$$

is an isometry of D^n onto H^n. Since ψ is its own inverse, ψ is also is an isometry of H^n onto D^n.

The inversion ψ inverts the orientation; orientation-preserving isometries $D^n \to H^n$ are given by

$$\psi_1(x', x_n) := \psi(x', -x_n) = \frac{(2x', 1 - |x|^2)}{1 + |x|^2 - 2x_n} \tag{7.42}$$

with inverse

$$\psi_1^{-1}(y', y_n) := \frac{(2y', |y|^2 - 1)}{|y|^2 + 1 + 2y_n} \tag{7.43}$$

and

$$\psi_2(x', x_n) := \psi(-x', x_n) = \frac{(-2x', 1 - |x|^2)}{1 + |x|^2 + 2x_n} \tag{7.44}$$

which is its own inverse.

In the case $n = 2$, identifying $\mathbb{R}^2$ and $\mathbb{C}$,

$$\psi(z) = \frac{2}{\overline{z + \mathrm{i}}} - \mathrm{i} = -\mathrm{i}\frac{\bar z + \mathrm{i}}{\bar z - \mathrm{i}}, \tag{7.45}$$

$$\psi_1(z) = \psi(\bar z) = -\mathrm{i}\frac{z + \mathrm{i}}{z - \mathrm{i}}, \tag{7.46}$$

$$\psi_1^{-1}(z) = \mathrm{i}\frac{z - \mathrm{i}}{z + \mathrm{i}}, \tag{7.47}$$

$$\psi_2(z) = \psi(-\bar z) = -\mathrm{i}\frac{z - \mathrm{i}}{z + \mathrm{i}}. \tag{7.48}$$

The isometries of D^n is the Möbius group of (proper and improper) Möbius maps of $\mathbb{R}^n$ that preserve D^n. (See [7, I.7 and II.6].) In particular, if $n = 2$, the isometries are the analytic Möbius maps

$$z \mapsto \frac{az + b}{\bar b z + \bar a} \tag{7.49}$$

with $a, b \in \mathbb{C}$ and $|a|^2 > |b|^2$ (we may assume $|a|^2 - |b|^2 = 1$), and the conjugate analytic

$$z \mapsto \frac{a\bar z + b}{\bar b \bar z + a} \tag{7.50}$$

with $a, b \in \mathbb{C}$ and $|a|^2 > |b|^2$.

The geodesics in D^n are

(i) lines through the centre 0 (and thus orthogonal to the boundary ∂D^n);
(ii) circles (in two-dimensional planes) orthogonal to ∂D^n.

Example 7.4 (Scaled Poincaré metric in a ball (disc))**.** Consider, more generally, the unit ball $D^n \subset \mathbb{R}^n$ with the metric

$$\mathrm{d}s = \frac{a\,|\mathrm{d}x|}{1-|x|^2}, \tag{7.51}$$

i.e.,

$$g_{ij} = \frac{a^2}{(1-|x|^2)^2}\delta_{ij}, \tag{7.52}$$

with $a > 0$. This is a conformal metric of the form (4.2) with $c = -1$; Example 7.3 is the case $a = 2$.

By Example 7.3 and Section 3.5, or directly from Example 4.1, we obtain, for example,

$$\Gamma_{ij}^k = \frac{2}{1-|x|^2}\big(x_i\delta_{jk} + x_j\delta_{ik} - x_k\delta_{ij}\big) \tag{7.53}$$

(independently of a) and constant sectional curvature

$$K = -4/a^2, \tag{7.54}$$

$$R_{ijkl} = -\frac{4}{a^2}(g_{ik}g_{jl} - g_{il}g_{jk}) = -\frac{2a^2}{(1-|x|^2)^4}\delta_{ij} \odot \delta_{kl}, \tag{7.55}$$

$$R_{ij} = -\frac{4(n-1)}{a^2}g_{ij} = -\frac{4(n-1)}{(1-|x|^2)^2}\delta_{ij}, \tag{7.56}$$

$$R = -\frac{4n(n-1)}{a^2}. \tag{7.57}$$

The distance $d(x, y)$ between x and y in D^n is given by

$$\cosh\frac{d(x,y)}{a} = 1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}. \tag{7.58}$$

Example 7.5 (Hyperboloid)**.** Equip $\mathbb{R}^{n+1}$ with the Lorentzian indefinite inner product

$$\langle (x_1, \dots, x_{n+1}), (y_1, \dots, y_{n+1}) \rangle := \sum_{i=1}^n x_i y_i - x_{n+1}y_{n+1}. \tag{7.59}$$

We write the elements of $\mathbb{R}^{n+1}$ as (x, ξ) with $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$, and thus

$$\langle (x,\xi), (y,\eta) \rangle = \langle x, y \rangle - \xi\eta, \tag{7.60}$$

where $\langle x, y \rangle$ is the ordinary Euclidean inner product in $\mathbb{R}^n$.

The hyperboloid (or pseudosphere)

$$\tilde{H} := \bigl\{(x,\xi) : \langle (x,\xi),(x,\xi)\rangle = -1\bigr\} = \Bigl\{(x_1,\dots,x_{n+1}) : x_{n+1}^2 = \sum_{i=1}^n x_i^2 + 1\Bigr\} \tag{7.61}$$

has two connected components, with $\xi = x_{n+1} > 0$ and < 0, respectively. Let H^+ be one of these, for definiteness

$$H^+ := \bigl\{(x,\xi) : \langle (x,\xi),(x,\xi)\rangle = -1 \text{ and } \xi > 0\bigr\} = \bigl\{\bigl(x, \sqrt{|x|^2+1}\bigr) : x \in \mathbb{R}^n\bigr\}. \tag{7.62}$$

The indefinite inner product (7.59) is positive definite on the tangent space H^+_X at every point $X \in H^+$, and defines thus a Riemannian metric on H^+. The distance between two arbitrary points is given by

$$\cosh d\bigl((x,\xi),(y,\eta)\bigr) = -\langle (x,\xi),(y,\eta)\rangle \tag{7.63}$$

$$= \sqrt{(1+|x|^2)(1+|y|^2)} - \langle x,y\rangle. \tag{7.64}$$

The isometries are the Lorentz group, which is the subgroup of index 2 of maps preserving H^+ in the group $O(n,1)$ of all linear maps of $\mathbb{R}^{n+1}$ onto itself that preserve the indefinite inner product (7.59).

Each geodesic lies in a two-dimensional plane in $\mathbb{R}^{n+1}$, and is the intersection of H^+ and this plane. There is thus a one-to-one correspondence between geodesics and 2-dimensional planes in $\mathbb{R}^{n+1}$ that contain a vector X with $\langle X,X\rangle < 0$.

The geodesics with unit speed are given by

$$\gamma(t) = \cosh t \cdot X + \sinh t \cdot V, \tag{7.65}$$

where $X = \gamma(0) \in H^+$ and $V = \dot\gamma(0) \in \mathbb{R}^{n+1}$ satisfy $\langle X,V\rangle = 0$ and $\langle V,V\rangle = 1$.

The hyperboloid is isometric to the hyperbolic ball D^n in Example 7.3, and thus to the hyperbolic half-space H^n in Example 7.1. An isometry $H^+ \to D^n$ is given by

$$(x,\xi) \mapsto \frac{x}{1+\xi} \tag{7.66}$$

with inverse

$$x \mapsto \frac{(2x, 1+|x|^2)}{1-|x|^2}; \tag{7.67}$$

geometrically these are stereographic projections with centre $(0,-1)$, with D^n seen as the subset $\{(x,0) \in R^{n+1} : |x| < 1\}$.

This isometry and (7.26)–(7.29) show that the metric has constant sectional curvature

$$K = -1, \tag{7.68}$$

and thus, for any coordinates and their metric g_{ij},

$$R_{ijkl} = -\frac12 g_{ij} \odot g_{kl} = -(g_{ik}g_{jl} - g_{il}g_{jk}), \tag{7.69}$$

$$R_{ij} = -(n-1)g_{ij}, \tag{7.70}$$

$$R = -n(n-1). \tag{7.71}$$

Example 7.6 (Hyperboloid, cont.)**.** The obvious coordinate map $(x, \xi) \to x$ identifies the hyperboloid H^+ in Example 7.5 with $\mathbb{R}^n$ equipped with the Riemannian metric

$$g_{ij} = \delta_{ij} - \frac{x_i x_j}{1+|x|^2}. \tag{7.72}$$

(Note that by formally replacing x by ix, (7.72) corresponds to (6.64), which has positive curvature. Furthermore, (7.72) is conformally equivalent to (6.51).)

The inverse matrix is given by

$$g^{ij} = \delta_{ij} + x_i x_j \tag{7.73}$$

and (1.13) and (1.12) yield

$$\Gamma_{kij} = -\frac{x_k \delta_{ij}}{1+|x|^2} + \frac{x_i x_j x_k}{(1+|x|^2)^2} = -\frac{x_k}{1+|x|^2} g_{ij}, \tag{7.74}$$

$$\Gamma^k_{ij} = -x_k \delta_{ij} + \frac{x_i x_j x_k}{1+|x|^2} = -x_k g_{ij}. \tag{7.75}$$

The invariant measure (1.6) is

$$\mathrm{d}\mu = (1+|x|^2)^{-1/2}\, \mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{7.76}$$

By Example 7.5, the metric has constant sectional curvature -1, and (7.68)–(7.71) hold.

The geodesics with unit speed are given by

$$\gamma(t) = \cosh(t) \cdot x + \sinh(t) \cdot v, \tag{7.77}$$

where $x = \gamma(0) \in \mathbb{R}^n$ is arbitrary and $v = \dot\gamma(0)$ is arbitrary with unit length, i.e.,

$$|v|^2 - \frac{\langle x, v\rangle^2}{1+|x|^2} = 1, \tag{7.78}$$

or, equivalently,

$$|\langle x, v\rangle|^2 = (1+|x|^2)(|v|^2 - 1). \tag{7.79}$$

The distance between two arbitrary points is given by (7.64):

$$\cosh d(x,y) = \sqrt{(1+|x|^2)(1+|y|^2)} - \langle x, y\rangle. \tag{7.80}$$

Example 7.7 (Klein ball (disc))**.** The Klein model K^n is the unit ball (disc) $\{x \in \mathbb{R}^n : |x| < 1\}$ equipped with the Riemannian metric

$$|\mathrm{d}s|^2 = \frac{|\mathrm{d}x|^2}{1-|x|^2} + \frac{|\langle \mathrm{d}x, x\rangle|^2}{(1-|x|^2)^2} = \frac{|\mathrm{d}x|^2 - |x \wedge \mathrm{d}x|^2}{(1-|x|^2)^2}, \tag{7.81}$$

i.e.,

$$g_{ij} = \frac{\delta_{ij}}{1-|x|^2} + \frac{x_i x_j}{(1-|x|^2)^2}. \tag{7.82}$$

(Note that by formally replacing x by ix, (7.82) corresponds to (6.51), which has positive curvature. Furthermore, (7.82) is conformally equivalent to (6.64).)

This metric is *not* conformal. The inverse matrix is

$$g^{ij} = (1 - |x|^2)(\delta_{ij} - x_i x_j). \tag{7.83}$$

The invariant measure (1.6) is

$$\mathrm{d}\mu = (1 - |x|^2)^{-(n+1)/2}\,\mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{7.84}$$

Calculations yield

$$\Gamma_{kij} = \frac{x_i\delta_{jk} + x_j\delta_{ik}}{(1 - |x|^2)^2} + \frac{2x_i x_j x_k}{(1 - |x|^2)^3} = \frac{x_i g_{jk} + x_j g_{ik}}{1 - |x|^2}, \tag{7.85}$$

$$\Gamma^k_{ij} = \frac{x_i\delta_{jk} + x_j\delta_{ik}}{1 - |x|^2} \tag{7.86}$$

and further (or by the isometries below)

$$R^i_{jkl} = \delta^i_l g_{jk} - \delta^i_k g_{jl} = -(\delta^i_k g_{jl} - \delta^i_l g_{jk}), \tag{7.87}$$

$$R_{ijkl} = -(g_{ik}g_{jl} - g_{il}g_{jk}), \tag{7.88}$$

$$R_{ij} = -(n-1)g_{ij}, \tag{7.89}$$

$$R = -n(n-1) \tag{7.90}$$

and constant sectional curvature

$$K = -1. \tag{7.91}$$

The connection is of the form (1.37), and thus the geodesics are Euclidean lines. More precisely, the equation (1.36) for a geodesic becomes

$$\ddot{\gamma}^i = -\frac{2\sum_j \gamma^j\dot{\gamma}^j}{1 - |\gamma|^2}\dot{\gamma}^i = \frac{\mathrm{d}\log(1 - |\gamma|^2)}{\mathrm{d}t}\dot{\gamma}^i. \tag{7.92}$$

A geodesic can be parametrized (with unit speed) as

$$\gamma(t) = x + \tanh(t)\cdot v \tag{7.93}$$

where $x = \gamma(0) \in \mathbb{R}^n$, $v = \dot{\gamma}(0) \in \mathbb{R}^n$ with $x \perp v$, $|x| < 1$ and $|x|^2 + |v|^2 = 1$.

The distance between two points $x, y \in K^n$ is given by

$$\cosh d(x,y) = \frac{1 - \langle x, y\rangle}{\sqrt{(1 - |x|^2)(1 - |y|^2)}}. \tag{7.94}$$

In particular,

$$\cosh d(x,0) = \frac{1}{\sqrt{1 - |x|^2}}, \tag{7.95}$$

$$\sinh d(x,0) = \frac{|x|}{\sqrt{1 - |x|^2}}, \tag{7.96}$$

$$\tanh d(x,0) = |x|. \tag{7.97}$$

If the line (= Euclidean line) through x and y has endpoints u and v on ∂D^n, then

$$d(x,y) = \frac{1}{2}\left|\log\frac{|x-u|\,|y-v|}{|x-v|\,|y-u|}\right|. \tag{7.98}$$

Every isometry maps (Euclidean) lines to lines, and is linear on each line.

The Klein ball (disc) K^n is isometric to the models in Examples 7.1–7.6. An isometry of the hyperboloid $H^+ \to K^n$ is given by

$$(x, \xi) \mapsto \frac{x}{\xi} \tag{7.99}$$

with inverse

$$x \mapsto \frac{(x, 1)}{\sqrt{1 - |x|^2}}; \tag{7.100}$$

geometrically these are stereographic projections with centre 0, with K^n seen as the subset $\{(x, 1) \in R^{n+1} : |x| < 1\}$.

Composition with the isometries (7.66)–(7.67) yields the isometry $D^n \to K^n$

$$x \mapsto \frac{2x}{1 + |x|^2} \tag{7.101}$$

with inverse $K^n \to D^n$

$$x \mapsto \frac{x}{1 + \sqrt{1 - |x|^2}}. \tag{7.102}$$

Example 7.8 (Polar coordinates in the hyperbolic plane)**.** Consider Example 4.5 with $w(r) = \sinh r$, $r \in (0, \infty)$. Then $w''/w = 1$, and (4.27) shows that M has constant curvature -1.

In fact, if $M = (0, \infty) \times (\theta_0, \theta_0 + 2\pi)$, for some $\theta_0 \in \mathbb{R}$, then

$$(r, \theta) \mapsto (\tanh(r/2) \cos\theta,\ \tanh(r/2) \sin\theta), \tag{7.103}$$

is an isometry of M onto $D^2 \setminus \ell$, where D^2 is the Poincaré disc in Example 7.3 and ℓ is the ray $\{(\rho \cos\theta_0, \rho \sin\theta_0) : \rho \in [0, 1)\}$. r is the distance to the origin 0, cf. (7.37)–(7.38) or (7.105) below, so this is polar coordinates in the hyperbolic plane.

The corresponding isometry into the Klein disc K^2 (onto $K^2 \setminus \ell$) is

$$(r, \theta) \mapsto (\tanh r\ \cos\theta,\ \tanh r\ \sin\theta), \tag{7.104}$$

see (7.97) and (7.101).

The metric tensor is,

$$g_{11} = 1, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = \sinh^2 r, \tag{7.105}$$

or in matrix form

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2 r \end{pmatrix}. \tag{7.106}$$

The connection coefficients are, by (4.22)–(4.23),

$$\Gamma^1_{22} = -\sinh r \cosh r, \tag{7.107}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \coth r, \tag{7.108}$$

with all other components 0. The invariant measure is

$$\mathrm{d}\mu = \sinh r \, \mathrm{d}r \, \mathrm{d}\theta. \tag{7.109}$$

By Example 7.5, (7.68)–(7.71) hold (with $n = 2$); thus the metric has constant sectional curvature $K = -1$, $R = -2$, $R_{ij} = -g_{ij}$ and, see Section 3.1,

$$R_{1212} = -|g| = -\sinh^2 r. \tag{7.110}$$

Example 7.9 (Hyperbolic azimuthal projections)**.** An *azimuthal projection of the hyperbolic plane* is, in analogy with Example 6.16, obtained by denoting the hyperbolic polar coordinates in Example 7.8 by (ρ, θ) and regarding $(r(\rho), \theta)$ as polar coordinates in the plane, for some smooth function $r(\rho)$. (As in Example 6.16, we consider only $r(\rho)$ with $r(0) = 0$ and $r'(0) > 0$, and assume that $r(\rho)$ can be extended to a smooth odd function.) The (Cartesian) coordinates are thus

$$\bigl(r(\rho)\cos\theta,\ r(\rho)\sin\theta\bigr). \tag{7.111}$$

Let $\rho(r)$ be the inverse function to $r(\rho)$; then the inverse map to (7.111) is

$$(x, y) \mapsto (\rho, \theta) = \Bigl(\rho(r),\ \arctan\frac{y}{x}\Bigr) \qquad \text{with } r := \sqrt{x^2 + y^2}. \tag{7.112}$$

A calculation using (7.112) and (7.105) shows that the metric is given by, still with $r := \sqrt{x^2 + y^2}$,

$$g_{11} = \frac{x^2}{r^2}\rho'(r)^2 + \frac{y^2}{r^4}\sinh^2\rho(r), \tag{7.113}$$

$$g_{12} = g_{21} = \frac{xy}{r^2}\rho'(r)^2 - \frac{xy}{r^4}\sinh^2\rho(r), \tag{7.114}$$

$$g_{22} = \frac{y^2}{r^2}\rho'(r)^2 + \frac{x^2}{r^4}\sinh^2\rho(r). \tag{7.115}$$

Furthermore, (7.113)–(7.115) yield

$$|g| = \frac{\rho'(r)^2\sinh^2\rho(r)}{r^2} \tag{7.116}$$

and thus the invariant measure μ in (1.6) is

$$\mathrm{d}\mu = \frac{\rho'(r)\sinh\rho(r)}{r}\,\mathrm{d}x\,\mathrm{d}y. \tag{7.117}$$

By (1.8), the two eigenvalues of g_{ij} are

$$\rho'(r)^2 \quad \text{and} \quad \frac{\sinh^2\rho(r)}{r^2}. \tag{7.118}$$

Note that the standard embedding of the Poincaré disc into the plane (Example 7.3) is given by the special case $r = \tanh(\rho/2)$, and thus $\rho(r) = 2\operatorname{arctanh} r$, see (7.103). Similarly, the Klein disc is given by the case $r = \tanh(\rho)$, and thus $\rho(r) = \operatorname{arctanh}(r)$, see (7.104). Two further examples are given below; see also Table 5.

Example 7.10 (Hyperbolic azimuthal equidistant projection (native representation))**.** By taking $r = \rho$ in Example 6.16, the distances from the centre point (the origin) are preserved; this projection thus gives correct directions

	$r(\rho)$	$\rho(r)$
Poincaré disc (Example 7.3)	$2\tanh(\rho/2)$	$2\operatorname{arctanh}(r/2)$
Klein disc (Example 7.7)	$\tanh(\rho)$	$\operatorname{arctanh}(r)$
equidistant (Example 7.10)	ρ	r
equal-area (Example 7.11)	$2\sinh(\rho/2)$	$2\operatorname{arcsinh}(r/2)$

Table 5. Some hyperbolic azimuthal projections

and distances from a fixed point, and it may be called the *hyperbolic azimuthal equidistant projection*. It is also called the *native representation* of the hyperbolic plane [10].

The metric is by (7.113)–(7.115) given by, with $r := \sqrt{x^2+y^2}$,

$$g_{11} = \frac{x^2}{r^2} + \frac{y^2\sinh^2 r}{r^4}, \tag{7.119}$$

$$g_{12} = g_{21} = \frac{xy(r^2-\sinh^2 r)}{r^4}, \tag{7.120}$$

$$g_{22} = \frac{y^2}{r^2} + \frac{x^2\sinh^2 r}{r^4}, \tag{7.121}$$

and the invariant measure μ is by (7.117)

$$\mathrm{d}\mu = \frac{\sinh r}{r}\,\mathrm{d}x\,\mathrm{d}y. \tag{7.122}$$

By (7.118), the eigenvalues of g_{ij} are 1 and $\sinh^2 r/r^2$, and thus the condition number is

$$\varkappa = \Bigl(\frac{\sinh r}{r}\Bigr)^2; \tag{7.123}$$

the eccentricity is by (1.7)

$$\varepsilon = \sqrt{1-\Bigl(\frac{r}{\sinh r}\Bigr)^2}. \tag{7.124}$$

Example 7.11 (Hyperbolic azimuthal equal-area projection)**.** The *hyperbolic azimuthal equal-area projection* is obtained by choosing $r(\rho)$ in Example 7.9 such that the invariant measure $\mathrm{d}\mu = \mathrm{d}x\,\mathrm{d}y$; in other words, the map is area-preserving.

By (7.117), the condition for this is

$$1 = \frac{\rho'(r)\sinh\rho(r)}{r} = \frac{\sinh\rho}{r'(\rho)r(\rho)}; \tag{7.125}$$

we thus need

$$(r(\rho)^2)' = 2r'(\rho)r(\rho) = 2\sinh\rho, \tag{7.126}$$

with the solution

$$r(\rho)^2 = 2\cosh\rho - 2 = 4\sinh^2(\rho/2). \tag{7.127}$$

(This also follows from (B.40), since the hyperbolic disc with radius ρ has area $2\pi(\cosh\rho-1)$ and it is mapped onto the Euclidean disc with radius

$r(\rho)$ having area $\pi r(\rho)^2$; these two areas have to be equal.) In other words, we choose

$$r(\rho) = 2\sinh(\rho/2). \tag{7.128}$$

The metric is by (7.113)–(7.115) given by, with $r := \sqrt{x^2+y^2}$,

$$g_{11} = \frac{x^2}{r^2(1+r^2/4)} + \frac{y^2(1+r^2/4)}{r^2} = \frac{1+(1+r^2/8)y^2/2}{1+r^2/4}, \tag{7.129}$$

$$g_{12} = g_{21} = \frac{xy}{r^2(1+r^2/4)} - \frac{xy(1+r^2/4)}{r^2} = -\frac{xy(1+r^2/8)}{2(1+r^2/4)}, \tag{7.130}$$

$$g_{22} = \frac{y^2}{r^2(1+r^2/4)} + \frac{x^2(1+r^2/4)}{r^2} = \frac{1+(1+r^2/8)x^2/2}{1+r^2/4}. \tag{7.131}$$

Since the determinant $|g| = 1$, the inverse matrix is

$$g^{11} = \frac{1+(1+r^2/8)x^2/2}{1+r^2/4}, \tag{7.132}$$

$$g^{12} = g^{21} = \frac{xy(1+r^2/8)}{2(1+r^2/4)}, \tag{7.133}$$

$$g^{22} = \frac{1+(1+r^2/8)y^2/2}{1+r^2/4}. \tag{7.134}$$

By (7.118), the eigenvalues of g_{ij} are $(1+r^2/4)^{-1}$ and $1+r^2/4$, and thus the condition number is

$$\varkappa = (1+r^2/4)^2. \tag{7.135}$$

Example 7.12 (Hyperbolic Mercator projection)**.** Define, in analogy with (6.102) for Mercator's projection, the conformal metric (with coordinates denoted (x,y))

$$g_{ij} = \frac{1}{\cos^2 y}\,\delta_{ij}, \qquad |y| < \pi/2, \tag{7.136}$$

defined in the horizontal strip $\Omega := (-\infty,\infty)\times(-\pi/2,\pi/2)$.

It is easily verified by (3.27) that this metric has constant sectional curvature $K=-1$. The invariant measure (1.6) is

$$\mathrm{d}\mu = (\cos y)^{-2}\,\mathrm{d}x\,\mathrm{d}y. \tag{7.137}$$

By (3.14) (with $\varphi(x,y) = -\log\cos y$),

$$\Gamma^k_{ij} = \bigl(\delta_{i2}\delta_{jk} + \delta_{j2}\delta_{ik} - \delta_{k2}\delta_{ij}\bigr)\tan x, \tag{7.138}$$

i.e., the non-zero coefficients are, see also (4.60) (with the coordinates interchanged),

$$\Gamma^2_{22} = \Gamma^1_{12} = \Gamma^1_{21} = -\Gamma^2_{11} = \tan y. \tag{7.139}$$

The Laplace–Beltrami operator is, by (3.28),

$$\Delta F = \cos^2 y\,\bigl(F_{,xx} + F_{,yy}\bigr). \tag{7.140}$$

The x-axis is a geodesic; a parametrization with unit speed is $(t,0)$. Moreover, every vertical line is a geodesic, as is easily verified by (1.36) and (7.139); more precisely, a calculation shows that

$$\gamma(t)=\Bigl(x_0,2\arctan\frac{e^t-1}{e^t+1}\Bigr)=\Bigl(x_0,2\arctan\Bigl(\tanh\frac{t}{2}\Bigr)\Bigr) \tag{7.141}$$

is a geodesic with unit speed for every $x_0\in\mathbb{R}$.

The horizontal lines are curves with constant distance to the "equator" $y=0$; they are thus hypercycles, see Appendix B.3. This also follows by (4.64), which shows (by a small calculation) that the horizontal line $x\mapsto(x,y)$, with a given y, is a curve with constant curvature $|\sin y|$ (directed towards the equator by (4.65)).

Note that the relation

$$y=2\arctan\bigl(\tanh\frac{t}{2}\bigr) \tag{7.142}$$

can be written in the following equivalent forms:

$$\tan\frac{y}{2}=\tanh\frac{t}{2}, \tag{7.143}$$

$$\sin y=\tanh t, \tag{7.144}$$

$$\tan y=\sinh t, \tag{7.145}$$

$$\cos y=\frac{1}{\cosh t}\quad\text{and}\;\operatorname{sign} y=\operatorname{sign} t. \tag{7.146}$$

Cf. (6.98)–(6.101) which describe the same correspondence; the mapping $t\mapsto y$ here is the inverse of the mapping $\phi\mapsto\zeta$ in (6.95).

Ω with the metric (7.136) is another model of the hyperbolic plane. An isometry $\Omega\to H^2$ (in Example 7.1) is given by

$$(x,y)\mapsto\bigl(e^x\sin y,e^x\cos y\bigr) \tag{7.147}$$

or, in complex notation (changing an immaterial sign)

$$z\mapsto \mathrm{i}e^z=e^{z+\mathrm{i}\pi/2}. \tag{7.148}$$

(This isometry maps the x-axis to the y-axis, and the vertical lines to cirles with centre 0, in accordance with the fact that these curves are geodesics orthogonal to the x-axis or y-axis, respectively. More prcisely, the unit speed geodesic (7.141) is by (7.148) mapped to

$$t\mapsto e^x\Bigl(\tanh t,\frac{1}{\cosh t}\Bigr), \tag{7.149}$$

which indeed is a unit speed geodesic in H^2, see (7.20).

Example 7.13 (Translated hyperbolic Mercator projection)**.** By translating Example 7.12 by $\pi/2$ in the second coordinate, we obtain the equivalent conformal metric

$$g_{ij}=\frac{1}{\sin^2 y}\delta_{ij},\qquad 0<y<\pi, \tag{7.150}$$

defined in the strip $\Omega_1 := (-\infty,\infty)\times(0,\pi)$. An isometry $\Omega_1 \to H^2$ is given by

$$(x,y) \mapsto \bigl(e^x\cos y, e^x \sin y\bigr) \tag{7.151}$$

or, in complex notation

$$z \mapsto e^z. \tag{7.152}$$

Example 7.14 (Hyperbolic cylindrical projection)**.** Denote the hyperbolic Mercator projection in Example 7.12 by (x,ζ), and consider the coordinates (x,y) for a smooth function $y = y(\zeta)$ (with $y'(\zeta) > 0$). A simple calculation using (7.136) shows that the metric tensor is given by, letting the function $\zeta(y)$ be the inverse of $y(\zeta)$,

$$g_{11} = \frac{1}{\cos^2(\zeta(y))}, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = \frac{\zeta'(y)^2}{\cos^2(\zeta(y))}. \tag{7.153}$$

This is a metric of the type in Example 4.7 (interchanging the coordinates).

We see from Example 7.12 that every vertical line is a geodesic; these form the set of all hyperbolic lines orthogonal to the "equator" $y = y_0 := y(0)$, which also is a hyperbolic line (a geodesic). Hence, using a term from hyperbolic geometry, the vertical lines form a *pencil of ultraparallel lines*. Moreover, the vertical lines are equispaced, in the sense that their distances (which equal their distances measured along the equator) are preserved by the metric. In analogy with Example 6.10, we call such projections *hyperbolic cylindrical projections*. The horizontal lines are, as in Example 7.12, curves with constant distance to the equator; they are thus hypercycles, see Appendix B.3. More precisely, the line $x \mapsto (x,\zeta)$ with constant ζ has, by Example 7.12 or by (4.65), curvature $|\sin(y(\zeta))|$.

The hyperbolic Mercator projection in Example 7.12 is an example (with $y(\zeta) = \zeta$) and so is the translated version in Example 7.13 (with $y(\zeta) = \zeta + \pi/2$). We give two further examples in Example 7.15 and Example 7.16.

Example 7.15 (Hyperbolic latitude and longitude)**.** Consider the hyperbolic cylindrical projection in Example 7.14 with the choice

$$y = \log\tan\Bigl(\frac{\zeta}{2} + \frac{\pi}{4}\Bigr) \tag{7.154}$$

with the inverse

$$\zeta = 2\arctan\Bigl(\tanh\frac{y}{2}\Bigr). \tag{7.155}$$

This is the same function as in (6.95) and (7.142), and we have, cf. (6.98)–(6.101) and (7.143)–(7.146),

$$\sinh y = \tan\zeta, \tag{7.156}$$

$$\cosh y = \frac{1}{\cos\zeta}, \tag{7.157}$$

$$\tanh y = \sin\zeta, \tag{7.158}$$

$$\tanh\frac{y}{2} = \tan\frac{\zeta}{2}. \tag{7.159}$$

Note also, by differentiating (7.155), or by differentiating e.g. (7.156) and using (7.157),

$$\zeta'(y) = \cos\zeta(y). \tag{7.160}$$

We have chosen y as the (signed) distance from the "equator" $y = 0$, see (7.142) and (7.155). This is also shown by the Riemannian metric, which by (7.153), (7.157) and (7.160) is given by

$$g_{11} = \cosh^2(y), \qquad g_{12} = g_{21} = 0, \qquad g_{22} = 1. \tag{7.161}$$

This is an example of the type in Example 4.5, with $w(r) = \cosh r$.

The vertical lines are geodesics, with the scale correct along them, and along the equator. This is thus the hyperbolic equivalent of longitude and latitude, see Example 6.8.

Example 7.16 (Hyperbolic equal-area cylindrical projection)**.** Consider the hyperbolic cylindrical projection in Example 7.14 with the choice

$$y = \tan\zeta \tag{7.162}$$

and thus

$$\zeta = \arctan y. \tag{7.163}$$

This gives a representation of the hyperbolic plane in $\mathbb{R}^2$ with the metric, by (7.153),

$$g_{11} = 1 + y^2, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = \frac{1}{1+y^2}. \tag{7.164}$$

The determinant $|g| = 1$, so this is an area-preserving projection of the hyperbolic plane. Cf. Example 6.13 and (6.116)–(6.118).

The metric is of the type in Example 4.7 (interchanging the order of the coordinates).

Example 7.17 (Hyperbolic cylindrical projection of polar type)**.** Let (r, θ) be the hyperbolic polar coordinates in Example 7.8 and consider the coordinates

$$(\theta, \zeta) = (\theta, \zeta(r)) \tag{7.165}$$

for a transformation $\zeta = \zeta(r)$, where $\zeta(r)$ is a smooth function with (smooth) inverse $r(\zeta)$. This yields, as in Example 7.14, a projection of (part of) the hyperbolic plane where vertical lines are geodesics (and further equispaced in the sense that every horizontal translation is an isometry where it is defined). However, in this case the vertical lines form (part of) a *pencil of concurrent lines*, viz. the set of lines through the origin O of the polar coordinates (which necessarily is outside the map). The horizontal lines are clearly (parts of) circles with centre O. Note that both these projections and the hyperbolic cylindrical projections in Example 7.14 correspond to the cylindrical projections of the sphere in Example 6.10. (In the spherical case, there is no difference between the pencil of all lines orthogonal to some line and the pencil of all lines through a point.) We may perhaps say that the hyperbolic cylindrical projections in Example 7.14 correspond to spherical cylindrical projections close to the pole, while the hyperbolic cylindrical projections in the present example correspond to spherical cylindrical projections close to the equator. (Note however that $\zeta(r)$ here corresponds to $\zeta(\varphi) = \zeta(\frac{\pi}{2} - \phi)$ in Example 6.10, see Examples 6.7–6.8.)

We may call a projection of this type a *hyperbolic cylindrical projection of polar type*.

The metric tensor is by (7.105)

$$g_{11} = (r'(\zeta))^2, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = \sinh^2(r(\zeta)). \tag{7.166}$$

This is a metric of the type in Example 4.7.

The vertical lines are thus hyperbolic lines through the origin O, while horizontal lines are hyperbolic circles with O as centre. The two families of lines are orthogonal to each other; however, the projection in general is not conformal. In fact, (7.166) shows that the (local) scales along these lines are typically different, and that the projection is locally conformal at a point if and only if

$$r'(\zeta) = \pm \sinh(r(\zeta)), \tag{7.167}$$

or equivalently

$$\zeta'(r) = \frac{1}{r'(\zeta(r))} = \pm \frac{1}{\sinh r}. \tag{7.168}$$

One special case is given by the polar coordinates in Example 7.8, with $\zeta(r) = r$; this case has correct scale along all vertical lines. Two other interesting cases follow in Examples 7.18–7.19.

Example 7.18 (Hyperbolic conformal projection of polar type)**.** In particular, the cylindrical projection (7.165) is conformal if and only if (7.168) holds for all r, which has the solution (up to sign and an additive constant)

$$\zeta(r) = -\log \tanh \frac{r}{2}; \tag{7.169}$$

we choose this sign to make $\zeta > 0$. Consequently, the coordinates

$$(\zeta, \theta) = \left(-\log \tanh \frac{r}{2}, \theta\right), \tag{7.170}$$

yield a conformal coordinate system.

Note that the function (7.169) is its own inverse; we have also

$$r = -\log \tanh \frac{\zeta}{2}; \tag{7.171}$$

Moreover, cf. (6.98)–(6.101),

$$\sinh \zeta = \frac{1}{\sinh r}, \tag{7.172}$$

$$\cosh \zeta = \frac{1}{\tanh r} = \coth r, \tag{7.173}$$

$$\tanh \zeta = \frac{1}{\cosh r}. \tag{7.174}$$

This projection has by (7.166) and (7.172) the conformal metric

$$g_{ij} = \frac{1}{\sinh^2 \zeta}\, \delta_{ij} = \sinh^2 r\, \delta_{ij}, \qquad \zeta > 0. \tag{7.175}$$

This may be called the *hyperbolic Mercator projection of polar type*. (Cf. the metrics for Mercator's projection (6.102) and its hyperbolic variants (7.136) and (7.150).)

It is easily verified directly from (3.27), with $\varphi(\zeta,\theta) = -\log\sinh\zeta$, that (7.175) indeed has the constant sectional curvature $K = -1$.

The invariant measure (1.6) is

$$\mathrm{d}\mu = \sinh^{-2}\zeta\, \mathrm{d}\zeta\, \mathrm{d}\theta. \tag{7.176}$$

By (3.14),

$$\Gamma^k_{ij} = -\big(\delta_{i1}\delta_{jk} + \delta_{j1}\delta_{ik} - \delta_{k1}\delta_{ij}\big) \coth\zeta, \tag{7.177}$$

i.e., the non-zero coefficients are

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = -\Gamma^1_{22} = -\coth\zeta. \tag{7.178}$$

The Laplace–Beltrami operator is, by (3.28),

$$\Delta F = \sinh^2\zeta\big(F_{,\zeta\zeta} + F_{,\theta\theta}\big). \tag{7.179}$$

Example 7.19 (Hyperbolic equal-area projection of polar type)**.** The hyperbolic cylindrical metric (7.166) is area-preserving if and only if $|g| = g_{11}g_{22} = 1$, i.e., if

$$r'(\zeta)\sinh(r(\zeta)) = \pm 1 \tag{7.180}$$

or

$$\zeta'(r) = \frac{1}{r'(\zeta(r))} = \pm\sinh r. \tag{7.181}$$

We choose the solution

$$\zeta(r) = \cosh r, \tag{7.182}$$

with the inverse

$$r = \ln\big(\zeta + \sqrt{\zeta^2 - 1}\big), \tag{7.183}$$

noting that $\zeta \in (1,\infty)$. The Riemannian metric is by (7.166) and (7.180)

$$g_{11} = \frac{1}{\zeta^2 - 1}, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = \zeta^2 - 1. \tag{7.184}$$

This is another example of a metric of the type in Example 4.7. Cf. the (also area-preserving) metric in Example 7.16 (with the coordinates interchanged).

Example 7.20 (Hyperbolic cylindrical projection of a third type)**.** There is also a third type of hyperbolic cylindrical projection, obtained as

$$(x,\zeta) = (x,\zeta(y)) \tag{7.185}$$

from the coordinates (x,y) in the Poincaré half-plane H^2 in Example 7.1, using a smooth function $\zeta(y)$ with (smooth) inverse $y(\zeta)$.

This yields, as in Examples 7.14 and 7.17, a projection of the hyperbolic plane where vertical lines are geodesics (and further equispaced in the sense that every horizontal translation is an isometry where it is defined). In this case the vertical lines form a *pencil of parallel lines*. The horizontal lines are horocycles, see Appendix B.2. (There are three types of pencils of lines in hyperbolic geometry, and the three types of hyperbolic cylindrical projections have one type each for the vertical lines. Similarly, there are three types of curves with constant non-zero curvature in hyperbolic geometry, see Appendix B, and the three types of hyperbolic cylindrical projections have one type each for the horizontal lines.)

The metric tensor is by (7.2)

$$g_{11} = \frac{1}{y(\zeta)^2}, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = \frac{y'(\zeta)^2}{y(\zeta)^2}. \tag{7.186}$$

Again, this is a metric of the type in Example 4.7. It is easily verified directly from (7.186) and (4.65) that a horizontal line is a curve with curvature 1, i.e., a horocycle, as said above.

Again, vertical lines and horizontal lines are are orthogonal to each other, but the projection is in general not conformal. In fact, (7.186) shows that the projection is locally conformal at a point if and only if

$$y'(\zeta) = \pm 1. \tag{7.187}$$

Thus, the Poincaré half-plane Example 7.1 (with $n = 2$), which is the trivial case $\zeta(y) = y$, is essentially the only conformal example.

Another special case is Example 7.2 (for $n = 2$) with $\zeta(y) = -\log y$; this is the case when the scale is correct along the vertical lines.

A third interesting case follows in Example 7.21.

Example 7.21 (Hyperbolic equal-area projection of the third type)**.** The hyperbolic cylindrical metric (7.186) is area-preserving if and only if $|g| = g_{11}g_{22} = 1$, i.e., if

$$y'(\zeta) = \pm y(\zeta)^2 \tag{7.188}$$

or

$$\frac{\mathrm{d}(1/y(\zeta))}{\mathrm{d}\zeta} = \pm 1. \tag{7.189}$$

We choose the solution

$$\zeta(y) = \frac{1}{y} \tag{7.190}$$

with the inverse

$$y = \frac{1}{\zeta}. \tag{7.191}$$

noting that $\zeta \in (0, \infty)$. The Riemannian metric is by (7.186) and (7.188)

$$g_{11} = \zeta^2, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = \frac{1}{\zeta^2}. \tag{7.192}$$

This is another example of a metric of the type in Example 4.7. Cf. the (also area-preserving) metrics in Example 7.16 (with the coordinates interchanged) and Example 7.19.

Example 7.22 (Hyperbolic sinusoidal projection)**.** We can define a hyperbolic analogue of the sinusoidal projection for the sphere (Example 6.31) by choosing a line (geodesic) L as the "equator", an orthogonal line L' as the "central meridian", and letting the "parallels" be the curves of constant distance to L, i.e., hypercycles, see Appendix B.3. Thus, given a point P, let C be the hypercycle through P with constant distance to L, let P' be the intersection of C and L', and take as the coordinates P the distance $d(P, L)$ from P to L, with negative sign in one of the half-spaces defined by L, and the distance along the hypercycle C between P and P', again with negative sign on one side of L'. This clearly maps the hyperbolic space bijectively onto $\mathbb{R}^2$.

To find the Riemannian metric, it is enough to study one example (since all choices of two orthogonal lines in a hyperbolic plane are isometric). We choose the Poincaré half-plane model of hyperbolic space (Example 7.1), with the "equator" as the y-axis, and the "central meridian" as (half) the unit circle. (Note that these are orthogonal geodesics.) We denote the coordinates by (ξ, η), and have by (B.66) (choosing the sign positive for $x > 0$)

$$\sinh \xi = \frac{x}{y}. \tag{7.193}$$

Furthermore, with $r = \sqrt{x^2 + y^2}$, the distance along the hypercycle C is, by (7.1) and because C is a Euclidean straight line through the origin, and using (B.65),

$$\mathrm{d}s = \frac{|\mathrm{d}r|}{y} = \cosh \xi \frac{|\mathrm{d}r|}{r}. \tag{7.194}$$

Hence (recall that by definition, ξ is constant on C), now choosing the sign positive for $r > 1$,

$$\eta = \cosh \xi \ln r = \frac{\sqrt{x^2 + y^2} \ln \sqrt{x^2 + y^2}}{y}. \tag{7.195}$$

The projection is thus (in this case) given by (7.193) and (7.195). The inverse mapping is easily seen to be

$$x = \tanh \xi \cdot e^{\eta / \cosh \xi}, \tag{7.196}$$

$$y = \frac{1}{\cosh \xi} \cdot e^{\eta / \cosh \xi}. \tag{7.197}$$

An elementary calculation using (7.1), (7.193) and (7.196)–(7.197) shows that (we omit the details)

$$\begin{aligned} |\mathrm{d}s|^2 = \frac{|\mathrm{d}x|^2 + |\mathrm{d}y|^2}{y^2} &= \sinh^2 x \cdot |\mathrm{d} \ln x|^2 + |\mathrm{d} \ln y|^2 \\ &= |\mathrm{d}\xi|^2 + |\mathrm{d}\eta - \eta \tanh \xi \, \mathrm{d}\xi|^2. \end{aligned} \tag{7.198}$$

Thus the hyperbolic sinusoid projection has metric tensor

$$g_{11} = 1 + \eta^2 \tanh^2 \xi, \qquad g_{12} = g_{21} = -\eta \tanh \xi, \qquad g_{22} = 1, \tag{7.199}$$

or in matrix form

$$(g_{ij}) = \begin{pmatrix} 1 + \eta^2 \tanh^2 \xi & -\eta \tanh \xi \\ -\eta \tanh \xi & 1 \end{pmatrix}, \tag{7.200}$$

cf. (6.301)–(6.302). The determinant $|g| = 1$, and thus the invariant measure (1.6) is $\mathrm{d}\mu = \mathrm{d}x\,\mathrm{d}y$, i.e., the hyperbolic sinusoidal projection is area preserving.

Note that the metric (7.200) is of the type (4.75) with $f(\xi, \eta) = -\eta \tanh \xi$. The Christoffel symbols are thus given by (4.79)–(4.82). Further, as a check, (4.84) yields the curvature

$$K = f_{,\xi\eta} - f f_{,\eta\eta} - f_{,\eta}^2 = -\frac{\mathrm{d} \tanh \xi}{\mathrm{d}\xi} - 0 - \tanh^2 \xi = -1. \tag{7.201}$$

Example 7.23 (Poincaré metric in a simply connected complex domain)**.** Consider a non-empty proper simply connected open subset $\Omega \subset \mathbb{C}$, or more generally a simply connected Riemann surface that is not isomorphic (biholomorphic) to the Riemann sphere or $\mathbb{C}$; then Ω is biholomorphic to the unit disc D^2, so there exists a holomorphic bijection $\Phi : D^2 \to \Omega$, with inverse $\Psi := \Phi^{-1}$. The map Ψ is not unique, but all possibilities are obtained by composing any fixed choice with Möbius mappings $D^2 \to D^2$. Since the latter preserve the Poincaré metric in D^2, see Example 7.3, there is a unique Riemannian metric in Ω corresponding to the Poincaré metric (7.23) in D^2. This metric is called the *Poincaré metric* in Ω; it is given by, cf. (7.23),

$$|\mathrm{d}s| = \frac{2|\Psi'(z)||\mathrm{d}z|}{1 - |\Psi(z)|^2}, \tag{7.202}$$

i.e.,

$$g_{ij} = \frac{4|\Psi'(z)|^2}{(1 - |\Psi(z)|^2)^2}\, \delta_{ij}. \tag{7.203}$$

The metric is conformal by construction. Moreover, Ω is isometric to D^2, and thus the metric (7.203) has constant (sectional) curvature $K = -1$.

By construction, any analytic bijection between two simply connected domains is an isometry for their Poincaré metrics.

Note that (7.203) can be written

$$g_{ij} = w(z)^2 \delta_{ij}, \tag{7.204}$$

where

$$w(z) = 2|\Psi'(z)| = \frac{2}{|\Phi'(0)|} \tag{7.205}$$

if $\Phi : D^2 \to \Omega$ is any analytic bijection with $\Phi(0) = z$, and $\Psi := \Phi^{-1}$ (so $\Psi(z) = 0$).

Of course, Example 7.3 (with $n = 2$) is the trivial example $\Omega = D^2$, where we may take $\Psi(z) = z$. Moreover, Example 7.1 (with $n = 2$) is the case $\Omega = H^2$, the upper half-plane; we may for example take $\Phi = \psi_1$ in (7.46). Similarly, Example 7.12 is another instance, for the strip Ω; we may choose $\Psi : \Omega \to D^2$ as

$$\Psi(z) = \frac{e^z - 1}{e^z + 1} = \tanh \frac{z}{2}, \tag{7.206}$$

and (7.203) easily yields (7.136). (By translation, Example 7.13 is yet another instance.)

Remark 7.24. More generally, it is easily seen from (3.27) and $\Delta = 4\partial_z\bar{\partial}_z$ that if Ψ is any analytic function on Ω with $|\Psi| < 1$ and $\Psi' \neq 0$, then (7.203) defines a metric with constant curvature -1, since writing (7.203) as $g_{ij} = e^{2\varphi}\delta_{ij}$, we have

$$\begin{aligned}\partial_z\varphi(z) &= \partial_z \tfrac{1}{2} \log \frac{4|\Psi'(z)|^2}{(1-|\Psi(z)|^2)^2} \\ &= \partial_z\Big(\log 2 + \tfrac{1}{2}\log\Psi'(z) + \tfrac{1}{2}\overline{\log\Psi'(z)} - \log\big(1-\Psi(z)\overline{\Psi(z)}\big)\Big) \\ &= \frac{\Psi''(z)}{2\Psi'(z)} + \frac{\Psi'(z)\overline{\Psi(z)}}{1-\Psi(z)\overline{\Psi(z)}}\end{aligned} \tag{7.207}$$

and thus

$$\begin{aligned}\tfrac{1}{4}\Delta\varphi = \overline{\partial_z}\partial_z\varphi(z) &= \frac{\Psi'(z)\overline{\Psi'(z)}}{1-\Psi(z)\overline{\Psi(z)}} + \frac{\Psi'(z)\overline{\Psi(z)}\Psi(z)\overline{\Psi'(z)}}{(1-\Psi(z)\overline{\Psi(z)})^2} \\ &= \frac{\Psi'(z)\overline{\Psi'(z)}}{(1-\Psi(z)\overline{\Psi(z)})^2} = \frac{|\Psi'(z)|^2}{(1-|\Psi(z)|^2)^2} = \tfrac{1}{4}e^{2\varphi(z)}.\end{aligned} \tag{7.208}$$

Hence, (3.27) yields $K = -1$.

Example 7.25 (Poincaré metric in a general complex domain)**.** Consider, more generally, a non-empty proper connected open subset $\Omega \subset \mathbb{C}$, or more generally any connected Riemann surface Ω, and assume that Ω is not isomorphic (biholomorphic) to the Riemann sphere, $\mathbb{C}$, $\mathbb{C} \setminus \{0\} \cong \mathbb{C}/\mathbb{Z}$ or a torus; then Ω has the unit disc $D = D^2$ as its universal covering space, and thus there is a covering map $\Phi : D \to \Omega$. Φ is, in general, not injective, but if $z \in \Omega$ and $\zeta_1, \zeta_2 \in \Phi^{-1}(z)$, then (by the properties of covering spaces) there exists a Möbius map $\Xi : D \to D$ such that $\Xi(\zeta_1) = \zeta_2$ and $\Phi \circ \Xi = \Phi$. Since Ξ preserves the Poincaré metric (7.23) on D, it follows that the local inverses Ψ_1 and Ψ_2 to Φ with $\Psi_j(z) = \zeta_j$, $j = 1, 2$, induce the same metric on some neighbourhood of $z \in \Omega$; similarly, two different covering maps $\Phi_1, \Phi_2 : D \to \Omega$ are related by a Möbius transformation of D, and induce thus the same metric.

Consequently, we obtain a unique conformal Riemannian metric on Ω, which at any point $z \in \Omega$ can be written as (7.203) for any local inverse Ψ of any covering map $\Phi : D \to \Omega$. This metric is called the *Poincaré metric* in Ω and has constant curvature $K = -1$; for a simply connected domain it is the metric in Example 7.23.

Since a composition of two covering maps is a covering map, it follows that any covering map $\Phi : \Omega_1 \to \Omega_2$ between two such domains is a local isometry for the Poincaré metrics; in particular, $\mathrm{d}\Phi$ is an isometry on every tangent space.

Moreover, any analytic function $F : \Omega_1 \to \Omega_2$ between two such domains is a contraction for the Poincaré metrics, both locally ($\mathrm{d}F$ is an operator of norm $\leqslant 1$ on each tangent space) and globally (the distance $d_{\Omega_2}(F(z_1), F(z_2)) \leqslant d_{\Omega_1}(z_1, z_2)$ for any $z_1, z_2 \in \Omega_1$); in both cases with strict inequality unless F is a covering map. (Using covering maps, this is reduced to the case $\Omega_1 = \Omega_2 = D$ and $F(0) = 0$, when the conclusion $|F'(0)| \leqslant 1$ is the Schwarz lemma.) This is (at least for simply connected domains) known as the Schwarz–Pick lemma; a generalization by Ahlfors (the Schwarz–Pick–Ahlfors theorem) extends this to more general Riemann surfaces, see [13] where also further generalizations are given.

We give two examples of Poincaré metrics in multiply connected domains as the following examples. The Poincaré metrics in the complex domains considered in Examples 7.1, 7.3, 7.12, 7.13, 7.26, 7.27, together with some immediate generalizations obtained by linear maps and the inversion $z \mapsto z^{-1}$ are collected in Table 6.

Example 7.26 (Poincaré metric in an annulus)**.** Consider the annulus

$$\Omega := \{z : a < |z| < 1\}, \tag{7.209}$$

with $0 < a < 1$. Let, as in Example 7.13, $\Omega_1 := \{z : 0 < \operatorname{Im} z < \pi\}$, and let $\alpha := \pi/\log(1/a)$. Then $\Phi(z) := e^{\mathrm{i}z/\alpha}$ is a covering map $\Omega_1 \to \Omega$, and by Example 7.25 it is locally an isometry for the Poincaré metrics.

Let

$$\Psi(z) := \Phi^{-1}(z) = -\mathrm{i}\alpha \log z \tag{7.210}$$

be the inverse function (which is multi-valued, and thus only locally an analytic function). Then

$$\operatorname{Im} \Psi(z) = -\alpha \operatorname{Re} \log z = -\alpha \log |z| = \pi \log_a |z|. \tag{7.211}$$

Hence, using (7.150), the Poincaré metric in the annulus (7.209) is

$$g_{ij} = \frac{|\Psi'(z)|^2}{\sin^2 \operatorname{Im} \Psi(z)}\,\delta_{ij} = \frac{\alpha^2}{|z|^2 \sin^2(\alpha \log |z|)}\,\delta_{ij} = \frac{(\pi/|\log a|)^2}{|z|^2 \sin^2(\pi \log_a |z|)}\,\delta_{ij}. \tag{7.212}$$

Example 7.27 (Poincaré metric in punctured disc)**.** Consider the punctured disc

$$\Omega := \{z : 0 < |z| < 1\}. \tag{7.213}$$

(This is the limiting case $a = 0$ of (7.209).) Then $\Phi(z) := e^{\mathrm{i}z}$ is a covering map $H^2 \to \Omega$, and by Example 7.25 it is locally an isometry for the Poincaré metrics.

Let

$$\Psi(z) := \Phi^{-1}(z) = -\mathrm{i} \log z \tag{7.214}$$

domain	$w(z)$
$\operatorname{Im} z > 0$	$\dfrac{1}{\operatorname{Im} z}$
$\lvert z\rvert < r$	$\dfrac{2r}{r^2 - \lvert z\rvert^2}$
$r < \lvert z\rvert \leqslant \infty$	$\dfrac{2r}{\lvert z\rvert^2 - r^2}$
$0 < \operatorname{Im} z < \pi$	$\dfrac{1}{\sin \operatorname{Im} z}$
$a < \operatorname{Im} z < b$	$\dfrac{\pi}{(b-a)\sin \pi \frac{\operatorname{Im} z - a}{b-a}}$
$\lvert \operatorname{Im} z\rvert < \pi/2$	$\dfrac{1}{\cos \operatorname{Im} z}$
$0 < a < \lvert z\rvert < b$	$\dfrac{\pi/\log(b/a)}{\lvert z\rvert \sin\bigl(\pi \log_{b/a}(\lvert z\rvert/a)\bigr)}$
$0 < \lvert z\rvert < b$	$\dfrac{1}{\lvert z\rvert \log(b/\lvert z\rvert)}$
$0 < a < \lvert z\rvert < \infty$	$\dfrac{1}{\lvert z\rvert \log(\lvert z\rvert/a)}$

Table 6. Poincaré metrics in some domains. The metric is $w(z)|\mathrm{d}z|$; the metric tensor is $w(z)^2\delta_{ij}$.

be the inverse function (which is multi-valued, and thus only locally an analytic function). Then

$$\operatorname{Im}\Psi(z) = -\operatorname{Re}\log z = -\log|z|. \tag{7.215}$$

Hence, using (7.2), the Poincaré metric in the punctured disc (7.213) is

$$g_{ij} = \frac{|\Psi'(z)|^2}{(\operatorname{Im}\Psi(z))^2}\,\delta_{ij} = \frac{1}{|z|^2\log^2|z|}\,\delta_{ij}. \tag{7.216}$$

Note that this agrees with the limit of (7.212) as $a \to 0$.

8. **Examples, further**

Example 8.1 (Catenoid)**.** The *catenoid* is the rotation surface

$$\Bigl\{(x,y,z) : \sqrt{x^2+y^2} = \cosh z\Bigr\} \tag{8.1}$$

in the Euclidean space $\mathbb{R}^3$. We give the catenoid the induced Riemannian metric.

The mapping

$$(z,\theta) \mapsto \bigl(\cosh z\,\cos\theta,\ \cosh z\,\sin\theta,\ z\bigr) \tag{8.2}$$

maps the plane $\mathbb{R}^2$ onto the catenoid: this map has period 2π in θ and is a diffeomorphism of the cylinder $\mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$ onto the catenoid. Hence, the restriction to $\theta \in (\theta_0, \theta_0 + 2\pi)$ yields a coordinate system (covering the catenoid except for the line $\theta = \theta_0$). A simple calculation shows that the Riemannian metric in this coordinate system is given by the metric tensor

$$g_{11} = \sinh^2 z + 1 = \cosh^2 z, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = \cosh^2 z, \tag{8.3}$$

or in matrix form

$$(g_{ij}) = \begin{pmatrix} \cosh^2 z & 0 \\ 0 & \cosh^2 z \end{pmatrix}. \tag{8.4}$$

This is a conformal metric of the special type (4.70), with $\varphi(z) = \log \cosh z$.

The connection coefficients are, by (4.71),

$$\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = \tanh z \tag{8.5}$$

with all other components $\Gamma^k_{ij} = 0$.

The curvature tensor is by (4.72)–(4.73) given by

$$R_{1212} = -1, \tag{8.6}$$

which by (3.1)–(3.8) yields all components R_{ijkl}, R_{ij} and R as well as

$$K = \frac{1}{2} R = \frac{R_{1212}}{g_{11}^2} = -\frac{1}{\cosh^4 z}. \tag{8.7}$$

The catenoid is thus a surface with non-constant negative curvature.

Appendix A. Symmetries of the curvature tensor

We explore the consequences of the symmetries (2.5)–(2.6) and the Bianchi identity (2.8). (In this section we consider tensors at a single point only. The section is thus linear algebra, dealing with tensors over a given n-dimensional vector space. We could start with any vector space V of finite dimension and a given non-singular symmetric covariant tensor g_{ij}.)

Let $\mathcal{R}_4$ be the space of all covariant 4-tensors that satisfy (2.5)–(2.6) and (2.8). Thus the curvature tensor $R_{ijkl} \in \mathcal{R}_4$.

As another example, for any symmetric tensors A_{ij} and B_{ij}, the definition (1.22) of the Kulkarni–Nomizu product $A_{ij} \odot B_{kl}$ shows that it satisfies the relations (2.5)–(2.6) and (2.8); i.e., $A_{ij} \odot B_{kl} \in \mathcal{R}_4$.

We begin by finding the dimension of $\mathcal{R}_4$.

It follows from (2.5) that if $R_{ijkl} \neq 0$, then $i \neq j$ and $k \neq l$. In particular, no index occurs more than twice. This leaves three cases.

(i) i, j, k, l are distinct. There are $\binom{n}{4}$ sets of four distinct indices in $\{1, \dots, n\}$. Each such set may be ordered in $4! = 24$ different ways, but as said in Section 2, only 2 of these yield linearly independent components R_{ijkl}.

n	$\dim(\mathcal{R}_4)$	$\dim(\mathcal{R}_{4,0})$	$\dim(\mathcal{R}_{4,2})$	$\dim(\mathcal{R}_{4,4})$	$\dim(\mathcal{R}_2)$	$\dim(\mathcal{R}_0)$
2	1	1	–	0	3	1
3	6	1	5	0	6	1
4	20	1	9	10	10	1
5	50	1	14	35	15	1

Table 7. Dimensions of $\mathcal{R}_4$, $\mathcal{R}_2$, $\mathcal{R}_0$ and some subspaces for small n

(ii) i, j, k, l contains one repetition, say $i = l$. By (2.5)–(2.6), we have to have

$$R_{ijik} = -R_{ijki} = -R_{jiik} = R_{jiki} = R_{ikij} = -R_{ikji} = -R_{kiij} = R_{kiji},$$

while the remaining components $R_{iijk} = R_{iikj} = R_{jkii} = R_{kjii} = 0$. Conversely, if these equalities hold, then also the Bianchi identity (2.8) holds for these indices. Hence there is one linearly independent component for each family $iijk$ with three distinct indices. There are $3\binom{n}{3} = n(n-1)(n-2)/2$ such families.

(iii) i, j, k, l contains two repetitions, say $i = l$ and $j = k$. By (2.5)–(2.6), we have to have

$$R_{ijij} = -R_{ijji} = -R_{jiij} = R_{jiji}$$

while the remaining components $R_{iijj} = R_{jjii} = 0$. Conversely, as in (ii), if these equalities hold, then also the Bianchi identity (2.8) holds for these indices. Hence there is one linearly independent component for each family $iijj$ with two distinct indices. There are $\binom{n}{2} = n(n-1)/2$ such families.

Note also that there is no interaction between components with different multisets $\{i, j, k, l\}$. Hence,

$$\dim(\mathcal{R}_4) = \binom{n}{4} \cdot 2 + 3\binom{n}{3} + \binom{n}{2} = \frac{n^2(n-1)(n+1)}{12} = \frac{n^2(n^2-1)}{12}. \quad \text{(A.1)}$$

See Table 7 for small n.

A.1. **A decomposition.** Let $\mathcal{R}_2$ be the space of symmetric covariant 2-tensors, and let $\mathcal{R}_0 := \mathbb{R}$, the space of scalars (= 0-tensors). Thus

$$\dim(\mathcal{R}_2) = \binom{n+1}{2} = \frac{n(n+1)}{2}, \quad \text{(A.2)}$$

$$\dim(\mathcal{R}_0) = 1. \quad \text{(A.3)}$$

The contractions $\pi_4 : A_{ijkl} \mapsto g^{kl} A_{ikjl}$ and $\pi_2 : A_{ij} \mapsto g^{ij} A_{ij}$ are linear maps $\pi_4 : \mathcal{R}_4 \to \mathcal{R}_2$ and $\pi_2 : \mathcal{R}_2 \to \mathcal{R}_0$. Moreover, we define the maps $\psi_2 : A \mapsto A g_{ij}$ and $\psi_4 : A_{ij} \mapsto A_{ij} \odot g_{kl}$, see (1.22); these are linear maps $\psi_2 : \mathcal{R}_0 \to \mathcal{R}_2$ and $\psi_4 : \mathcal{R}_2 \to \mathcal{R}_4$.

By (1.3), for any $A \in \mathcal{R}_0$,

$$\pi_2\psi_2(A) = \pi_2(Ag_{ij}) = g^{ij}Ag_{ij} = nA, \tag{A.4}$$

i.e., $\pi_2\psi_2 = nI$, where I is the identity map. Hence we have the direct decomposition

$$\mathcal{R}_2 = \mathcal{R}_{2,2} \oplus \mathcal{R}_{2,0}, \tag{A.5}$$

where

$$\mathcal{R}_{2,2} := \ker(\pi_2) = \text{ the set of traceless tensors in } \mathcal{R}_2, \tag{A.6}$$

$$\mathcal{R}_{2,0} := \operatorname{im}(\psi_2) = \{rg_{ij} : r \in \mathbb{R}\}. \tag{A.7}$$

Explicitly, a tensor $A_{ij} \in \mathcal{R}_2$ decomposes as

$$A_{ij} = A_{ij}^{(0)} + A_{ij}^{(2)}, \tag{A.8}$$

where, with $A := \pi_2(A_{ij}) = g^{ij}A_{ij}$,

$$A_{ij}^{(0)} := \frac{1}{n}Ag_{ij} \in \mathcal{R}_{2,0}, \tag{A.9}$$

$$A_{ij}^{(2)} := A_{ij} - A_{ij}^{(0)} = A_{ij} - \frac{1}{n}Ag_{ij} \in \mathcal{R}_{2,2}. \tag{A.10}$$

Moreover, with the inner product on $\mathcal{R}_2$

$$\langle A_{ij}, B_{ij}\rangle := A_{ij}B^{ij} = g^{ik}g^{jl}A_{ij}B_{kl}, \tag{A.11}$$

it is easily seen that $\psi_2^* = \pi_2$, and thus $\mathcal{R}_{2,0} \perp \mathcal{R}_{2,2}$, so the decomposition (A.5) is orthogonal.

We have

$$\dim(\mathcal{R}_{2,2}) = \dim(\mathcal{R}_2) - 1 = \frac{n(n+1)}{2} - 1 = \frac{(n-1)(n+2)}{2}. \tag{A.12}$$

Similarly, we compute for $A_{ij} \in \mathcal{R}_2$, with $A := g^{ij}A_{ij}$,

$$\begin{aligned} \pi_4\psi_4(A_{ij})) &= \pi_4\bigl(A_{ij} \odot g_{kl}\bigr) = g^{kl}\bigl(A_{ik} \odot g_{jl}\bigr) \\ &= g^{kl}\bigl(A_{ij}g_{kl} + A_{kl}g_{ij} - A_{il}g_{kj} - A_{kj}g_{il}\bigr) \\ &= nA_{ij} + Ag_{ij} - A_{ij} - A_{ij} = \bigl((n-2) + \psi_2\pi_2)(A_{ij})\bigr). \end{aligned} \tag{A.13}$$

Hence, using (A.4),

$$\pi_4\psi_4 = \begin{cases} (n-2)I & \text{on } \mathcal{R}_{2,2}, \\ (n-2+n)I = 2(n-1)I & \text{on } \mathcal{R}_{2,0}. \end{cases} \tag{A.14}$$

A.1.1. *The case $n \geqslant 3$.* Suppose now $n \geqslant 3$. Then (A.14) shows that $\pi_4\psi_4$ is an isomorphism of $\mathcal{R}_2 = \mathcal{R}_{2,2} \oplus \mathcal{R}_{2,0}$ onto itself; hence $\pi_4 : \mathcal{R}_4 \to \mathcal{R}_2$ is surjective, $\psi_4 : \mathcal{R}_2 \to \mathcal{R}_4$ is injective and

$$\mathcal{R}_4 = \ker(\pi_4) \oplus \operatorname{im}(\psi_4) = \ker(\pi_4) \oplus \psi_4(\mathcal{R}_2) = \ker(\pi_4) \oplus \psi_4(\mathcal{R}_{2,2}) \oplus \psi_4(\mathcal{R}_{2,0}). \tag{A.15}$$

We write this as

$$\mathcal{R}_4 = \mathcal{R}_{4,0} \oplus \mathcal{R}_{4,2} \oplus \mathcal{R}_{4,4} \tag{A.16}$$

with

$$\mathcal{R}_{4,0} := \psi_4(\mathcal{R}_{2,0}) = \psi_4(\psi_2(\mathcal{R}_0)) = \{r g_{ij} \odot g_{kl} : r \in \mathbb{R}\}, \tag{A.17}$$
$$\mathcal{R}_{4,2} := \psi_4(\mathcal{R}_{2,2}) = \{A_{ij} \odot g_{kl} : A_{ij} \in \mathcal{R}_2 \text{ and } g^{ij} A_{ij} = 0\}, \tag{A.18}$$
$$\mathcal{R}_{4,4} := \ker(\pi_4). \tag{A.19}$$

Explicitly, if $A_{ijkl} \in \mathcal{R}_4$, it has a unique decomposition

$$A_{ijkl} = A^{(0)}_{ijkl} + A^{(2)}_{ijkl} + A^{(4)}_{ijkl} \tag{A.20}$$

with $A^{(m)}_{ijkl} \in \mathcal{R}_{4,m}$, viz., using the contractions $A_{ij} := \pi_4(A_{ijkl}) = g^{kl} A_{ikjl}$ and $A := \pi_2(\pi_4(A_{ijkl})) = g^{ij} A_{ij} = g^{ij} g^{kl} A_{ikjl}$,

$$A^{(0)}_{ijkl} = \frac{1}{2n(n-1)} A\, g_{ij} \odot g_{kl}, \tag{A.21}$$
$$A^{(2)}_{ijkl} = \frac{1}{n-2}\Big(A_{ij} - \frac{1}{n} A g_{ij}\Big) \odot g_{kl}, \tag{A.22}$$
$$\begin{aligned} A^{(4)}_{ijkl} &= A_{ijkl} - A^{(0)}_{ijkl} - A^{(2)}_{ijkl} \\ &= A_{ijkl} - \frac{1}{n-2} A_{ij} \odot g_{kl} + \frac{1}{2(n-1)(n-2)} A\, g_{ij} \odot g_{kl}. \end{aligned} \tag{A.23}$$

In particular, for the Riemann curvature tensor, this gives the decomposition (2.15), where the Weyl tensor $W_{ijkl} := R^{(4)}_{ijkl}$.

With the natural inner product on $\mathcal{R}_4$,

$$\langle A_{ijkl}, B_{ijkl} \rangle := A_{ijkl} B^{ijkl}, \tag{A.24}$$

we have, similarly to $\mathcal{R}_2$ treated above, using the assumption that $A_{ijkl} \in \mathcal{R}_4$, and thus satisfies (2.5)–(2.6),

$$\begin{aligned} \langle \psi_4^*(A_{ijkl}), B_{ij} \rangle &= \langle A_{ijkl}, B_{ij} \odot g_{kl} \rangle \\ &= \langle A_{ijkl}, B_{ik} g_{jl} + B_{jl} g_{ik} - B_{il} g_{jk} - B_{jk} g_{il} \rangle \\ &= 4 \langle A_{ijkl}, B_{ik} g_{jl} \rangle = 4 A_{ijkl} B^{ik} g^{jl} \\ &= 4 \langle \pi_4(A_{ijkl}), B_{ij} \rangle, \end{aligned} \tag{A.25}$$

which shows that $\psi_4^* = 4\pi_4$ and thus $\pi_4^* = \frac{1}{4}\psi_4$. This, together with (A.14), shows that the decomposition (A.16) is orthogonal.

We note also that

$$\begin{aligned} \dim(\mathcal{R}_{4,4}) = \dim(\mathcal{R}_4) - \dim(\mathcal{R}_2) &= \frac{n^2(n+1)(n-1)}{12} - \frac{n(n+1)}{2} \\ &= \frac{n(n+1)(n+2)(n-3)}{12}. \end{aligned} \tag{A.26}$$

A.1.2. *The case $n = 3$.* In the special case $n = 3$, (A.26) shows that $\dim(\mathcal{R}_{4,4}) = 0$, so $A^{(4)}_{ijkl} = 0$ for every tensor $A_{ijkl} \in \mathcal{R}_4$, and (A.20) takes the simpler form

$$\begin{aligned} A_{ijkl} = A^{(0)}_{ijkl} + A^{(2)}_{ijkl} &= \frac{1}{12} A\, g_{ij} \odot g_{kl} + \Big(A_{ij} - \frac{1}{3} A g_{ij}\Big) \odot g_{kl} \\ &= \Big(A_{ij} - \frac{1}{4} A g_{ij}\Big) \odot g_{kl}. \end{aligned} \tag{A.27}$$

A.1.3. *The case $n = 2$.* When $n = 2$, (A.14) shows that $\pi_4\psi_4$ vanishes on $\mathcal{R}_{2,2}$ but is an isomorphism on $\mathcal{R}_{2,0} \cong \mathcal{R}_0$. In fact, (A.1) shows that $\dim(\mathcal{R}_4) = 1 = \dim(\mathcal{R}_{2,0})$; thus ψ_4 restricts to an isomorphism of $\mathcal{R}_{2,0}$ onto $\mathcal{R}_4$ and π_4 maps $\mathcal{R}_4$ onto $\mathcal{R}_{2,0}$ and is an isomorphism $\mathcal{R}_4 \to \mathcal{R}_{2,0}$. Moreover, $\pi_4\psi_4 = 2I$ on $\mathcal{R}_{2,0}$ and $\psi_4\pi_4 = 2I$ on $\mathcal{R}_4$.

Consequently, for every $A_{ijkl} \in \mathcal{R}_4$, with the contractions A_{ij} and A as above, using also (A.9),

$$A_{ijkl} = \tfrac{1}{2} A_{ij} \odot g_{kl} = \tfrac{1}{4} A\, g_{ij} \odot g_{kl}. \tag{A.28}$$

In other words, we still have the decomposition (A.20) even when $n = 2$, with $A^{(0)}_{ijkl}$ given by (A.21) but $A^{(2)}_{ijkl} = A^{(4)}_{ijkl} := 0$. In fact, $\mathcal{R}_{4,4} = \ker(\pi_4) = \{0\}$.

Note that the contraction $A_{ij} = \pi_4(A_{ijkl}) \in R_{2,0}$ for every $A_{ijkl} \in \mathcal{R}_4$, i.e.

$$A_{ij} = \tfrac{1}{2} A\, g_{ij}, \tag{A.29}$$

unlike the case $n \geqslant 3$ where, as we saw above, A_{ij} can be any symmetric tensor.

Appendix B. Curves with constant curvature

Consider a 2-dimensional manifold M. By Section 1.5, a curve γ (with $\dot\gamma \neq 0$) has constant curvature if and only if it can be reparametrized (by arc length) to have constant velocity 1 and an acceleration vector that has constant length and is orthogonal to the velocity vector. For a given non-zero curvature, this gives two choices for the acceleration vector (since we assume $n = 2$), but for a smooth curve, the acceleration is (locally) always on the same side of the velocity vector by continuity (left or right, as defined by the coordinates); hence the acceleration is determined by the velocity vector and the curvature. This gives a (non-linear) second order differential equation for $\gamma(t)$, which has a unique maximal solution $\gamma : (-a, b) \to M$ for each set of initial values, i.e., for each initial point $\gamma(0)$, each unit tangent vector $\dot\gamma(0)$ and each curvature $\kappa \in [0, \infty)$, with an initial direction of the curvature unless it is 0. (The case of curvature 0 yields the geodesics.)

We consider such curves in convenient models of 2-dimensional manifolds with constant curvature.

B.1. **Circles.** A circle with radius r is the set of points of distance $r > 0$ from some point x_0 (the centre).

B.1.1. *Plane, zero curvature.* In the standard Euclidean plane with curvature 0, we take the centre as 0 and can parametrize the circle as

$$\gamma(t) = (r\cos t, r\sin t). \tag{B.1}$$

If we identify $\mathbb{R}^2$ and the complex plane $\mathbb{C}$, we can write this more simply as $\gamma(t) = re^{\mathrm{i}t}$. The radius is r, the velocity vector is

$$\dot\gamma(t) = (-r\sin t, r\cos t) = \mathrm{i}re^{\mathrm{i}t} \tag{B.2}$$

with length r, and the acceleration is

$$\ddot\gamma(t) = (-r\cos t, -r\sin t) = -re^{\mathrm{i}t} \tag{B.3}$$

with length r (orthogonal to $\dot\gamma$ since the velocity is constant). Note that we have chosen a parametrization with constant velocity, but not necessarily velocity 1 (i.e., parametrization by arc length). The curvature is by (1.32)

$$\kappa = \frac{r}{r^2} = \frac{1}{r}. \tag{B.4}$$

The curvature vector is directed towards the centre of the circle.

We see that the curvature $\kappa = 1/r \in (0,\infty)$, and that every positive curvature is obtained by some circle.

The circle is a closed curve which really is given by (B.1) for $t \in [0, 2\pi)$; hence by (B.2) its length is, as has been well-known since antiquity,

$$\ell = 2\pi r. \tag{B.5}$$

Equally well-known is that the area of a circle of radius r is

$$A = \pi r^2, \tag{B.6}$$

which can be verified by integrating the measure (1.6), which now is Lebesgue measure, for $|x| \leqslant r$.

Note that the total (geodesic) curvature of the circle (measured inwards) is

$$\int_\gamma \kappa = \ell\kappa = 2\pi, \tag{B.7}$$

for every circle.

We have here used Euclidean coordinates. As an alternative, we may instead use polar coordinates, see Example 5.2. Then the circle above (except for a point) can be parametrized as

$$\gamma(t) = (r, t), \qquad 0 < t < 2\pi, \tag{B.8}$$

with velocity vector

$$\dot\gamma(t) = (0, 1), \tag{B.9}$$

with length r by (5.2). Furthermore,

$$\ddot\gamma(t) = 0, \tag{B.10}$$

and thus the acceleration is by (1.29), (B.9) and (5.4)–(5.5),

$$D_t^2\gamma(t) = (\Gamma^1_{22}, \Gamma^2_{22}) = (-r, 0), \tag{B.11}$$

with length r by (5.2), which again yields the formula (B.4) for the curvature. (See also the more general (4.29)–(4.32).) We also immediately obtain (B.5) and (B.6) for the length and area, using the invariant measure (5.6) for the latter.

B.1.2. *Sphere, positive curvature.* Consider the sphere $\mathbb{S}^2_\rho$ of radius ρ in (6.1), which has constant curvature $K = \rho^{-2} > 0$ by (6.11). Note that the maximum distance in $\mathbb{S}^2_\rho$ (i.e., its diameter as a metric space) is $\pi\rho$, and thus there are circles only with radii $0 < r < \pi\rho$. (The case $r = \pi\rho$ is degenerate, since it would give a single point only and not a curve.) Note also that a circle of radius r and centre x_0 is the same as a circle of radius $\pi\rho - r$ and centre $-x_0$, the antipode. Hence it suffices to consider the case $0 < r \leqslant \pi\rho/2$, although we allow larger r too below unless we indicate otherwise.

The circles with radius $r = \pi\rho/2$ are the geodesics in $\mathbb{S}^2_\rho$.

All circles of the same radius are congruent by rotations of the sphere, so it suffices to consider one of them. We may also use any convenient coordinate system. As an illustration of the calculations, we consider as in Section B.1 two alternatives.

We first use the stereographic coordinates in Example 6.1. We consider the curve

$$\gamma(t) = (a\cos t, a\sin t), \qquad t \in \mathbb{R}, \tag{B.12}$$

with $a > 0$. This is by (6.22) a circle with centre 0 and radius $r \in (0, \pi\rho)$ given by

$$\tan\Bigl(\frac{r}{2\rho}\Bigr) = \frac{a}{\rho}. \tag{B.13}$$

We have $\dot\gamma(t) = (-a\sin t, a\cos t)$ and thus, using (6.7),

$$\|\dot\gamma(t)\|^2 = \frac{4a^2}{(1+a^2/\rho^2)^2}. \tag{B.14}$$

Furthermore, $\ddot\gamma(t) = (-a\cos t, -a\sin t) = -\gamma(t)$, and it follows from (1.29) and (6.10) that (by rotational symmetry it suffices to consider $t = 0$)

$$D_t^2\gamma^k = \ddot\gamma^k + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j = -\gamma^k + \frac{2a^2}{\rho^2+a^2}\gamma^k = -\frac{\rho^2-a^2}{\rho^2+a^2}\gamma^k, \tag{B.15}$$

and thus

$$\|D_t^2\gamma(t)\| = \frac{|\rho^2-a^2|}{\rho^2+a^2}\|\gamma(t)\| = \frac{|\rho^2-a^2|}{\rho^2+a^2}\frac{2a}{1+a^2/\rho^2}. \tag{B.16}$$

Hence (1.32) yields, noting that the velocity is constant by (B.14) and thus the acceleration $D_t^2\gamma(t) \perp \dot\gamma(t)$ as seen in (B.15), and using (B.13),

$$\kappa = \frac{\|D_t^2\gamma(t)\|}{\|\dot\gamma(t)\|^2} = \frac{|1-a^2/\rho^2|}{2a} = \frac{|1-\tan^2(r/2\rho)|}{2\rho\tan(r/2\rho)} = \frac{1}{\rho\,|\tan(r/\rho)|}. \tag{B.17}$$

If we consider only $r \leqslant \pi\rho/2$, we can write this as

$$\kappa = \frac{\cot(r/\rho)}{\rho}, \qquad 0 < r \leqslant \pi\rho/2. \tag{B.18}$$

Note that for $r = \pi\rho/2$, the curvature vanishes and the circle is a geodesic, as said above.

The absolute value in (B.17) is explained by the fact that we have defined the curvature κ as a non-negative scalar. If we instead consider the curvature vector (1.34), we can say that, for all $r \in (0, \pi\rho)$, the curvature vector is $\cot(r/\rho)/\rho$ directed towards the centre of the circle, i.e., (B.18) holds for all r if we measure the curvature in the direction towards the centre. (See (B.15) and note that this description yields the same vector if we consider the circle as a circle with radius $\pi\rho - r$ about the antipodal centre.)

For $0 < r < \pi\rho/2$, the curvature vector is directed towards the centre of the circle; the curvature $\kappa = \cot(r/\rho)/\rho \in (0, \infty)$, and every positive curvature is obtained by some circle with radius $r < \pi\rho/2$.

The circle is a closed curve which really is given by (B.12) for $t \in [0, 2\pi)$; hence its length is, by (B.14) and (B.13),

$$\ell = 2\pi \frac{2a}{1 + a^2/\rho^2} = 4\pi\rho \frac{\tan(r/2\rho)}{1 + \tan^2(r/2\rho)} = 2\pi\rho \sin\frac{r}{\rho}, \tag{B.19}$$

as is also seen by elementary geometry. The area of a circle of radius r is, by integrating the measure (6.15) for $|x| \leqslant a$ or by elementary geometry,

$$A = 4\pi\rho^2 \sin^2\frac{r}{2\rho} = 2\pi\rho^2\Bigl(1 - \cos\frac{r}{\rho}\Bigr). \tag{B.20}$$

Note that the total (geodesic) curvature of the circle (measured inwards; we here allow $\kappa < 0$) is

$$\int_\gamma \kappa = \ell\kappa = 2\pi\cos(r/\rho) \tag{B.21}$$

while the total curvature of the interior is, using (6.11),

$$\int K \,\mathrm{d}\mu = KA = \frac{1}{\rho^2}A = 2\pi\Bigl(1 - \cos\frac{r}{\rho}\Bigr). \tag{B.22}$$

Hence

$$\int_\gamma \kappa + \int K \,\mathrm{d}\mu = \ell\kappa + KA = 2\pi \tag{B.23}$$

for every circle, an example of the Gauss–Bonnet theorem.

As an alternative (and partly simpler) calculation, we consider polar coordinates on a sphere, see Example 6.7; for simplicity we consider as in Example 6.7 only the unit sphere with $\rho = 1$. Then as in (B.8), the circle above with centre 0 and radius r can be parametrized as

$$\gamma(t) = (r, t), \qquad 0 < t < 2\pi, \tag{B.24}$$

with velocity vector

$$\dot\gamma(t) = (0, 1), \tag{B.25}$$

with length $\sin r$ by (6.74). Furthermore,

$$\ddot\gamma(t) = 0, \tag{B.26}$$

and thus the acceleration is by (1.29), (B.25) and (6.76)–(6.77), in accordance with the more general (4.29),

$$D_t^2\gamma(t) = (\Gamma^1_{22}, \Gamma^2_{22}) = (-\sin r\cos r, 0), \tag{B.27}$$

with length

$$\|D_t^2\gamma(t)\| = |\sin r\cos r| \tag{B.28}$$

by (6.74). Hence the curvature (1.32) is, see also (4.32),

$$\kappa = \frac{\|D_t^2\gamma(t)\|}{\|\dot\gamma(t)\|^2} = \frac{|\sin r\cos r|}{\sin^2 r} = |\cot r| \tag{B.29}$$

in accordance with (B.17)–(B.18) (with $\rho = 1$). Furthermore, we see immediately from (6.74) that the length of the circle is

$$\ell = 2\pi\sin r \tag{B.30}$$

as in (B.19); similarly, the area is, using the invariant measure (6.78),

$$A = \int_{\varphi=0}^{r}\int_{\theta=0}^{2\pi} \sin\varphi \,\mathrm{d}\varphi\,\mathrm{d}\theta = 2\pi\bigl(1-\cos r\bigr) \tag{B.31}$$

as in (B.20).

Finally, consider again a general sphere with radius ρ. If ℓ is a geodesic and $0 < d < \pi\rho/2$, then the set of points of distance d to ℓ form two circles, one in each half-sphere bounded by ℓ, both having radius $\pi\rho/2 - d$ and thus by (B.18) curvature

$$\kappa = \frac{\cot(\pi/2 - d/\rho)}{\rho} = \frac{\tan(d/\rho)}{\rho}. \tag{B.32}$$

(If ℓ is the equator, these circles are parallels, where the latitude is constant.) Conversely, every circle is the set of points of distance $d = \rho\arctan(\rho\kappa)$ to some geodesic ℓ, and further lying in one of the two half-spheres defined by ℓ.

B.1.3. *Hyperbolic plane, negative curvature.* Consider the hyperbolic plane, which can be represented by, for example, the Poincaré half-plane H^2 in Example 7.1 or the Poincaré disc D^2 in Example 7.3. The (sectional) curvature is $K = -1$, see (7.11) and (7.26). (We consider for simplicity only this case. Any other constant negative curvature can be obtained by scaling the metric, see Example 7.4.)

All circles of the same radius are congruent by isometries of the hyperbolic plane, since the isometry group is transitive, so it suffices to consider one of them. We choose to use the model D^2 in Example 7.3 and consider the curve

$$\gamma(t) = (a\cos t, a\sin t), \qquad t\in\mathbb{R}, \tag{B.33}$$

with $0 < a < 1$. This is by (7.37) a circle with centre 0 and radius $r \in (0,\infty)$ given by

$$\tanh\frac{r}{2} = a. \tag{B.34}$$

We have $\dot\gamma(t) = (-a\sin t, a\cos t)$ and thus, using (7.23),

$$\|\dot\gamma(t)\|^2 = \frac{4a^2}{(1-a^2)^2}. \tag{B.35}$$

Furthermore, $\ddot\gamma(t) = (-a\cos t, -a\sin t) = -\gamma(t)$, and it follows from (1.29) and (7.25) that (by rotational symmetry it suffices to consider $t = 0$)

$$D_t^2\gamma^k = \ddot\gamma^k + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j = -\gamma^k - \frac{2a^2}{1-a^2}\gamma^k = -\frac{1+a^2}{1-a^2}\gamma^k, \tag{B.36}$$

and thus

$$\|D_t^2\gamma(t)\| = \frac{1+a^2}{1-a^2}\|\gamma(t)\| = \frac{2a(1+a^2)}{(1-a^2)^2}. \tag{B.37}$$

Hence (1.32) yields, noting that the velocity is constant by (B.35) and thus the acceleration $D_t^2\gamma(t) \perp \dot\gamma(t)$ as seen in (B.36), and using (B.34),

$$\kappa = \frac{\|D_t^2\gamma(t)\|}{\|\dot\gamma(t)\|^2} = \frac{1+a^2}{2a} = \frac{1+\tanh^2(r/2)}{2\tanh(r/2)} = \frac{1}{\tanh r} = \coth r. \tag{B.38}$$

The curvature vector is (as in the Euclidean case, and for small circles on a sphere) directed towards the centre of the circle, see (B.36).

Note that the curvature $\kappa = \coth r \in (1,\infty)$, and that every curvature > 1 is obtained by some circle; however hyperbolic circles cannot have curvature $\leqslant 1$. Hence (unlike the Euclidean and spherical cases), there are also other curves than circles with constant curvature in the hyperbolic plane. We consider these in the next two subsections.

A hyperbolic circle (B.33) is a closed curve and its length is, by (B.35) and (B.34),

$$\ell = 2\pi\frac{2a}{1-a^2} = 4\pi\frac{\tanh(r/2)}{1-\tanh^2(r/2)} = 2\pi\sinh r. \tag{B.39}$$

The area of a circle of radius r is, by integrating the measure (7.30) for $|x| \leqslant a$,

$$A = 4\pi\frac{a^2}{1-a^2} = 4\pi\sinh^2\frac{r}{2} = 2\pi(\cosh r - 1). \tag{B.40}$$

Note that the total (geodesic) curvature of the circle (measured inwards) is

$$\int_\gamma \kappa = \ell\kappa = 2\pi\cosh r \tag{B.41}$$

while the total curvature of the interior is, using (7.26),

$$\int K\,\mathrm{d}\mu = KA = -A = 2\pi(1-\cosh r). \tag{B.42}$$

Hence

$$\int_\gamma \kappa + \int K\,\mathrm{d}\mu = \ell\kappa + KA = 2\pi \tag{B.43}$$

for every hyperbolic circle, another example of the Gauss–Bonnet theorem.

Again, the calculations are simpler if we instead use polar coordinates, see Example 7.8. Then as in (B.8) and (B.24), the circle with centre 0 and radius r can be parametrized as

$$\gamma(t) = (r, t), \qquad 0 < t < 2\pi, \tag{B.44}$$

with velocity vector

$$\dot\gamma(t) = (0, 1), \tag{B.45}$$

with length $\sinh r$ by (7.105). Furthermore,

$$\ddot\gamma(t) = 0, \tag{B.46}$$

and thus the acceleration is by (1.29), (B.45) and (7.107)–(7.108), in accordance with the more general (4.29),

$$D_t^2\gamma(t) = (\Gamma^1_{22}, \Gamma^2_{22}) = (-\sinh r \cosh r, 0), \tag{B.47}$$

with length, by (7.105),

$$\|D_t^2\gamma(t)\| = \sinh r \cosh r \tag{B.48}$$

Hence the curvature (1.32) is, see also (4.32),

$$\kappa = \frac{\|D_t^2\gamma(t)\|}{\|\ddot\gamma(t)\|^2} = \frac{\sinh r \cosh r}{\sinh^2 r} = \coth r \tag{B.49}$$

in accordance with (B.38). Furthermore, we see from (7.105) immediately that the length of the circle is

$$\ell = 2\pi \sinh r \tag{B.50}$$

as in (B.39); similarly, the area is, using the invariant measure (7.109),

$$A = \int_{s=0}^{r}\int_{\theta=0}^{2\pi} \sinh s \,\mathrm{d}s\,\mathrm{d}\theta = 2\pi(\cosh r - 1) \tag{B.51}$$

as in (B.40).

See also the equal-area representation Example 7.11, in particular (7.127).

B.2. **Horocycles in the hyperbolic plane.** We have seen in Section B.1 that in the hyperbolic plane, curves with constant curvature $\leqslant 1$ are not circles. We describe these curves in this and the following subsections; in this subsection we discuss the curves of constant curvature $\kappa = 1$, which are called *horocycles*.

Note that through every point, there is a unique (undirected) horocycle with a given curvature vector $\vec\kappa$ of length 1. In other words, there are two horocycles (with curvature vectors in opposite directions) through each point with a given tangent.

All horocycles are congruent by isometries of the hyperbolic plane, since the isometry group is transitive and the group of isometries fixing a given point is transitive on the tangent vectors there; hence it suffices to consider one horocycle.

We choose the model H^2 in Example 7.1 and consider the curve

$$\gamma(t) = (t, 1), \qquad t \in \mathbb{R}, \tag{B.52}$$

i.e., the Euclidean horizontal line $y = 1$. (We could consider choose $y = y_0$ for any $y_0 > 0$, but we need only one example so we take the simplest.)

We have $\dot\gamma(t) = (1, 0)$ and thus by (7.2)

$$\|\dot\gamma(t)\|^2 = 1. \tag{B.53}$$

Furthermore, $\ddot\gamma(t) = 0$ and it follows from (1.29) and (7.4) that

$$D_t^2\gamma^k = \ddot\gamma^k + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j = \Gamma^k_{11} = \delta_{k2}, \tag{B.54}$$

i.e., $D_t^2\gamma(t) = (0, 1)$, and thus

$$\|D_t^2\gamma(t)\| = 1. \tag{B.55}$$

Hence (1.32) yields, noting that the velocity is constant and thus the acceleration $D_t^2\gamma(t) \perp \dot\gamma(t)$, as also seen directly in (B.54),

$$\kappa = 1, \tag{B.56}$$

verifying that the curve (B.52) indeed is a horocycle. The curvature vector is directed upwards.

The orientation-preserving isometries of the Poincaré half-plane H^2 are the Möbius transformations that preserve H^2, i.e., the mappings $z \mapsto \frac{az+b}{cz+d}$ with a, b, c, d real and $ad - bc > 0$. The horocycle (B.52) is a circle in the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ with exactly one point, viz. ∞, in common with the boundary ∂H^2. It follows that the horocycles in H^2 are precisely the circles in $\mathbb{S}^2$ with one point on ∂H^2 and the rest in H^2, i.e., the Euclidean horizontal lines $y = y_0$ with $y_0 > 0$ (including (B.52)) and the Euclidean circles in the upper half plane $\bar H^2$ that are tangent to the real axis. (The curvature vector is directed upwards in the first case and inwards the circle in the second case. The curvature vector lies on a geodesic that ends at the point where the horocycle meets the boundary ∂H^2.)

Since H^2 and the Poincaré disc D^2 are isometric by a Möbius transformation, it follows also that the horocycles in D^2 are the Euclidean circles that are tangent to the unit circle ∂D (on the inside); the curvature vector is directed inwards this circle. (Again, the curvature vector lies on a geodesic that ends at the point where the horocycle meets the boundary ∂D^2.)

A horocycle has infinite length, and it separates the hyperbolic plane into two parts, both having infinite area. (This is seen by considering the horocycle (B.52).)

B.3. **Hypercycles in the hyperbolic plane.** The curves of constant curvature $\kappa < 1$ in the hyperbolic plane are called *hypercycles*.

Note that through every point, there is a unique (undirected) hypercycle with a given curvature vector $\vec\kappa$ of length < 1. In other words, there are two hypercycles (with curvature vectors in opposite directions) through each point with a given tangent direction and a given curvature $\kappa < 1$.

All hypercycles with the same curvature are congruent by isometries of the hyperbolic plane; hence it suffices to consider one example for each curvature $\kappa < 1$.

We choose the model H^2 in Example 7.1 and consider the curve

$$\gamma(t) = (\alpha e^t, e^t), \qquad t \in \mathbb{R}, \tag{B.57}$$

with $0 < \alpha < \infty$, i.e., a Euclidean line through the origin. (The case $\alpha = 0$ yields a geodesic, and the case $\alpha < 0$ is equivalent to $\alpha > 0$ by reflection in the y-axis.)

We have $\dot\gamma(t) = \gamma(t)$ and thus by (7.2)

$$\|\dot\gamma(t)\|^2 = e^{-2t}(\alpha^2 e^{2t} + e^{2t}) = \alpha^2 + 1. \tag{B.58}$$

Furthermore, $\ddot\gamma(t) = \gamma(t)$ and it follows from (1.29) and (7.4) that

$$\begin{aligned} D_t^2\gamma^k &= \ddot\gamma^k + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j = \gamma^k(t) - e^{-t}\bigl(2e^t\gamma^k(t) - \delta_{k2}(1+\alpha^2)e^{2t}\bigr) \\ &= -\gamma^k(t) + \delta_{k2}(1+\alpha^2)\gamma^2(t), \end{aligned} \tag{B.59}$$

i.e.,

$$D_t^2\gamma = (-\alpha, \alpha^2)e^t, \tag{B.60}$$

and thus, by (7.2),

$$\|D_t^2\gamma(t)\|^2 = \alpha^2 + \alpha^4. \tag{B.61}$$

Hence (1.32) yields, noting that the velocity is constant and thus the acceleration $D_t^2\gamma(t) \perp \dot\gamma(t)$, as also seen directly in (B.61),

$$\kappa = \frac{\alpha\sqrt{1+\alpha^2}}{1+\alpha^2} = \frac{\alpha}{\sqrt{1+\alpha^2}}. \tag{B.62}$$

verifying that the curve (B.57) indeed is a hypercycle. The curvature vector is directed towards the y-axis.

Note that there is a unique $\alpha > 0$ satisfying (B.62) for every given $\kappa \in (0,1)$; hence there is a unique hypercycle (B.57) for every curvature $\kappa < 1$.

The hypercycles are thus the curves in the hyperbolic plane that are congruent to one of the curves (B.57). Since the isometries of H^2 and D^2 are given by Möbius mappings and reflections, it follows that in both H^2 and D^2, the hypercycles are the Euclidean circles and lines that intersect the boundary in two different points, but not orthogonally. (The circles and lines that are orthogonal to the boundary are congruent to (B.57) with $\alpha = 0$, and are thus geodesics.) The angle of intersection with the boundary is invariant under isometries; hence a Euclidean line or circle intersecting the boundary ∂H^2 or ∂D^2 at an angle $\theta \in (0, \pi/2]$ is congruent to (B.57) with $\alpha = \cot\theta$, and is thus by (B.62) a hypercycle with curvature

$$\kappa = \cos\theta, \tag{B.63}$$

unless $\theta = \pi/2$ in which case the curve is a geodesic. (The relation (B.63) holds in the latter case too, with $\kappa = 0$.) Conversely, every hypercycle in H^2 or D^2 is thus a Euclidean line or circle intersecting the boundary in two points, at each point with an angle $\theta \in (0, \pi/2)$ satisfying (B.63), i.e., $\theta = \arccos\kappa$.

Considering again H^2, the y-axis $\{(0,u)\}$ is, as said above, a geodesic. The distance d from a point (x,y) to the y-axis is, by (7.12), given by

$$\begin{aligned}\cosh d &= \inf_{u>0} d\big((x,y),(0,u)\big) = \inf_{u>0}\Big(1+\frac{x^2+(y-u)^2}{2yu}\Big)\\ &= \inf_{u>0}\frac{x^2+y^2+u^2}{2yu}.\end{aligned} \tag{B.64}$$

This infimum is attained for $u=\sqrt{x^2+y^2}$, which gives

$$\cosh d = \frac{\sqrt{x^2+y^2}}{y} \tag{B.65}$$

and thus

$$\sinh d = \frac{|x|}{y}. \tag{B.66}$$

(This can also be seen geometrically; the geodesics orthogonal to the y-axis are the Euclidean circles with centre 0, so the one passing through (x,y) has Euclidean radius $\sqrt{x^2+y^2}$ and intersects the y-axis in $(0,\sqrt{x^2+y^2})$, see (7.20).) Hence, the points on the hypercycle (B.57) all have the same distance d to the y-axis, with $\sinh d=\alpha$. By (B.62), the curvature of the hypercycle is

$$\kappa = \frac{\sinh d}{\sqrt{1+\sinh^2 d}} = \tanh d. \tag{B.67}$$

Applying isometries, we see that for any geodesic ℓ in the hyperbolic plane, and any $d>0$, the points of distance d to ℓ form two hypercycles, one in each (hyperbolic) half-plane bounded by ℓ, both having curvature $\kappa=\tanh d$ given by (B.67). (Cf. (B.18) in the spherical case.) Conversely, every hypercycle is the set of points of distance $d=\operatorname{arctanh}\kappa$ to some geodesic ℓ, and further lying in one of the two half-planes defined by ℓ. Note that the hypercycle and the geodesic have the same endpoints in the boundary (line at infinity) of the hyperbolic plane, which defines the geodesic corresponding to a given hypercycle.

Note also that a hypercycle that makes an angle $\theta\in(0,\pi/2)$ with the boundary (line at infinity) has curvature $\kappa=\cos\theta$ by (B.63), and thus (B.67) shows that the distance d to the corresponding geodesic satisfies

$$\tanh d = \cos\theta \tag{B.68}$$

and hence also

$$\cosh d = \frac{1}{\sin\theta}, \tag{B.69}$$

$$\sinh d = \cot\theta. \tag{B.70}$$

B.4. **Summary for the hyperbolic plane.** For both the Poincaré half-plane H^2 (Example 7.1) and the Poincaré disc D^2 (Example 7.3), we have seen that the circles, horocycles, hypercycles and geodesics together are all Euclidean lines and circles that intersect the half-plane or disc; moreover, they can be distinguished as follows:

(i) Circles do not intersect the boundary (line at infinity); they are Euclidean circles completely included in H^2 or D^2.
(ii) Horocycles intersect the boundary in exactly one point; they are tangent to the boundary and lie inside H^2 or D^2 except for the tangent point.
(iii) Hypercycles intersect the boundary in two distinct points, at an angle $\theta \in (0, \pi/2)$.
(iv) Geodesics intersect the boundary in two distinct points, at an angle $\theta = \pi/2$.

Note that horocycles can be seen as limiting cases of circles (letting the radius $r \to \infty$), and also of hypercycles (letting the angle θ at the line of infinity tend to 0, or equivalently letting the distance d to a geodesic tend to ∞). In both cases, the curvatures given by (B.38) and (B.63) or (B.67) tend to 1, the value for horocycles.

Appendix C. **Curves in the complex plane**

We consider here a (smooth) curve $\gamma(t)$ in the complex plane $\mathbb{C}$, with the usual Euclidean metric. We assume $\dot\gamma(t) \neq 0$. Furthermore, we consider the curvature κ in (1.32) with sign, as explained in Remark 1.2. Thus, see (1.34), $\vec\kappa$ is a positive number times $\mathrm{i}\kappa\dot\gamma$, and it follows from (1.35) that

$$\vec\kappa = \frac{\mathrm{i}\kappa\dot\gamma}{\|\dot\gamma\|} = \frac{\mathrm{i}\kappa\dot\gamma}{|\dot\gamma|}. \tag{C.1}$$

Consequently,

$$\kappa = \frac{\vec\kappa|\dot\gamma|}{\mathrm{i}\dot\gamma} = \frac{\vec\kappa|\dot\gamma|}{\mathrm{i}\dot\gamma} = \frac{P_N\ddot\gamma}{\mathrm{i}\dot\gamma|\dot\gamma|} = \operatorname{Im}\frac{\ddot\gamma}{\dot\gamma|\dot\gamma|}. \tag{C.2}$$

As a consequence, using $|\dot\gamma| = (\dot\gamma\bar{\dot\gamma})^{1/2}$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\kappa(t) &= \operatorname{Im}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\ddot\gamma}{\dot\gamma}|\dot\gamma|^{-1}\Big) = \operatorname{Im}\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\ddot\gamma}{\dot\gamma}\Big)|\dot\gamma|^{-1} - \frac{1}{2}\frac{\ddot\gamma}{\dot\gamma}|\dot\gamma|^{-3}\big(\ddot\gamma\bar{\dot\gamma} + \dot\gamma\bar{\ddot\gamma}\big)\right)\\ &= \frac{1}{|\dot\gamma|}\operatorname{Im}\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\ddot\gamma}{\dot\gamma}\Big) - \frac{1}{2}\left(\frac{\ddot\gamma}{\dot\gamma}\right)^2\right)\end{aligned} \tag{C.3}$$

Define, for a thrice differentiable function $f(x)$ of a real or complex variable, the *Schwarzian derivative* $S(f)$ by

$$S(f)(x) := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{C.4}$$

Then (C.3) can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\kappa(t) = \frac{1}{|\dot\gamma(t)|}\operatorname{Im} S(\gamma)(t). \tag{C.5}$$

A standard calculation yields, for an analytic function g,

$$S(g\circ f)(x) = S(g)(f(x))\cdot f'(x)^2 + S(f)(x). \tag{C.6}$$

Hence, (C.5) implies that if $\tilde\kappa(t)$ is the (signed) curvature of $g\circ\gamma(t)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde\kappa(t) = \frac{1}{|g'(\gamma(t))|\,|\dot\gamma(t)|}\operatorname{Im}\bigl(S(g)(\gamma(t))\cdot\dot\gamma(t)^2\bigr) + \frac{1}{|g'(\gamma(t))|}\frac{\mathrm{d}}{\mathrm{d}t}\kappa(t). \tag{C.7}$$

Remark C.1. If $\gamma(t)$ is parametrized by arc-length, so $|\dot\gamma(t)|=1$, and $\tilde\kappa$ is reparametrized as $\tilde\kappa(s)$ where s is its arc-length, then (C.7) takes the simpler form

$$|g'(\gamma(t))|^2\frac{\mathrm{d}}{\mathrm{d}s}\tilde\kappa(s) = \operatorname{Im}\bigl(S(g)(\gamma(t))\cdot\dot\gamma(t)^2\bigr) + \frac{\mathrm{d}}{\mathrm{d}t}\kappa(t). \tag{C.8}$$

Remark C.2. It is well-known that $S(g)=0$ if and only if g is a Möbius transformation, i.e., a fractional linear map $g(z)=(az+b)/(cz+d)$ (with $a,b,c,d\in\mathbb{C}$ and $ad-bc\neq 0$ so that g is not constant), cf. (6.36). Hence, (C.6) implies that if g is a Möbius transformation, then $S(g\circ f)=S(f)$ for any f. Conversely, it is known that if $S(h)=S(f)$ for two analytic functions, then $h=g\circ f$ for some Möbius transformation g.

If γ is a curve with constant curvature, we see by (C.7) that $g(\gamma)$ also is a curve with constant curvature if and only if

$$S(g)(\gamma(t))\cdot\dot\gamma(t)^2\in\mathbb{R}. \tag{C.9}$$

Example C.3. Let g be an analytic function g, defined in some domain $\Omega\subseteq\mathbb{C}$. It follows from (C.9) that g maps all (parts of) circles and lines to (parts of) circles and lines if and only if $g'\neq 0$ and $S(g)=0$ in Ω, which by Remark C.2 occurs if and only if g is a Möbius transformation.

Example C.4. Let g be an analytic function g, defined in some domain $\Omega\subseteq\mathbb{C}$. It follows from (C.9) that the following are equivalent:

(i) g maps all horizontal lines to (parts of) circles and lines.
(ii) g maps all vertical lines to (parts of) circles and lines.
(iii) $S(g)$ is real.
(iv) $S(g)=a$ for some real constant a.

Since the exponential function $f(z)=e^{bz}$ (with a complex constant $b\neq 0$) by a simple calculation has $S(f)=-\frac12 b^2$, it follows by Remark C.2 that these properties are further equivalent to

(v) $g(z)$ has one of the forms

$$h(z), \tag{C.10}$$

$$h(e^{cz}),\qquad c>0, \tag{C.11}$$

$$h(e^{\mathrm{i}cz}),\qquad c>0, \tag{C.12}$$

for a Möbius transformation h.

Remark C.5. Some alternative formulas for the Schwarzian derivative are

$$S(f)(x) = -2\sqrt{f'}\Big(\frac{1}{\sqrt{f'}}\Big)'' \tag{C.13}$$

$$= 6\frac{\partial^2}{\partial x \partial y}\left(\frac{f(x)-f(y)}{x-y}\right)\Big|_{y=x} \tag{C.14}$$

$$= 6\left(\frac{f'(x)f'(y)}{(f(x)-f(y))^2} - \frac{1}{(x-y)^2}\right)\Big|_{y=x}. \tag{C.15}$$

Appendix D. Geodesics as straight lines

We prove the following result, stated in Section 1.6, characterizing Riemannian metrics such that the geodesics are (parts of) straight lines (possibly traversed with $\dot\gamma$ non-constant).

Theorem D.1. *Geodesics are straight lines in a coordinate system if and only if the connection has the special form*

$$\Gamma^k_{ij} = \delta^k_i a_j + \delta^k_j a_i \tag{1.37}$$

for some functions a_i.

Proof. We have already shown in Section 1.6 that (1.37) implies that $\ddot\gamma$ is parallel to $\dot\gamma$ for a geodesic γ, which thus is a straight line.

Conversely, suppose that every geodesic is a straight line. If γ is a geodesic, then both $\ddot\gamma$ and $\dot\gamma$ are directed along the line formed by γ, and are thus parallel. At any point x, we may choose any non-zero tangent vector $(t_i)_1^n$ and find a geodesic γ with $\gamma(0) = x$ and $\dot\gamma^i(0) = t_i$. By (1.36), then $\ddot\gamma^i(0) = -\Gamma^i_{jk} t_j t_k$. Consequently, for any choice of $(t_i)_1^n$, there exists c such that

$$\Gamma^i_{jk} t_j t_k = c t_i. \tag{D.1}$$

Choosing first $t_\ell = \delta_{i\ell}$, we see that

$$\Gamma^i_{jj} = 0, \qquad i \neq j. \tag{D.2}$$

Similarly, choosing $t_\ell = \delta_{i\ell} + \delta_{j\ell}$, with $i \neq j$, we see, using (D.2), that

$$\Gamma^k_{ij} = 0, \qquad k \neq i, j. \tag{D.3}$$

Next, fixing any two distinct indices i and j and taking $t_i, t_j \neq 0$ but $t_k = 0$ for $k \neq i, j$, we find by (D.1), using (D.2)

$$\Gamma^i_{ii} t_i^2 + 2\Gamma^i_{ij} t_i t_j = c t_i \tag{D.4}$$

$$\Gamma^j_{jj} t_j^2 + 2\Gamma^j_{ij} t_i t_j = c t_j \tag{D.5}$$

(with no implicit summation) and thus

$$\Gamma^i_{ii} t_i + 2\Gamma^i_{ij} t_j = c = \Gamma^j_{jj} t_j + 2\Gamma^j_{ij} t_i. \tag{D.6}$$

Since t_i and t_j are arbitrary non-zero, this yields

$$\Gamma^i_{ii} = 2\Gamma^j_{ij}. \tag{D.7}$$

This is valid for all $i \neq j$, which shows that (1.37) holds with $a_i := \frac{1}{2}\Gamma^i_{ii}$ when $i \neq j$ and $k \in \{i,j\}$; it is furthermore trivially true when $i = j = k$, and the case $k \notin \{i,j\}$ follows by (D.2)–(D.3). □

Appendix E. A matrix lemma

Lemma E.1. *Let $A = (a_{ij})_{i,j=1}^n$ be an $n \times n$-matrix of the special form*

$$a_{ij} = \delta_{ij} + u_i v_j, \qquad i,j = 1,\dots,n, \tag{E.1}$$

for two vectors $u = (u_i)_1^n$ and $v = (v_i)_1^n$. Then the determinant of A is

$$|A| = 1 + \langle u,v\rangle = 1 + \sum_{i=1}^n u_i v_i, \tag{E.2}$$

and if $|A| \neq 0$, then the inverse is

$$A^{-1} = \Bigl(\delta_{ij} - \frac{u_i v_j}{1+\langle u,v\rangle}\Bigr)_{i,j=1}^n = \Bigl(\delta_{ij} - \frac{u_i v_j}{|A|}\Bigr)_{i,j=1}^n. \tag{E.3}$$

Proof. The determinant (E.2) may be calculated in several ways. For example, suppose that $u_1 \neq 0$, and subtract u_i/u_1 times row 1 from row i for each $i = 2,\dots,n$; this gives for each $i \geqslant 2$ a new row with elements

$$\delta_{ij} + u_i v_j - \frac{u_i}{u_1}\bigl(\delta_{1j} + u_1 v_j\bigr) = \delta_{ij} - \frac{u_i}{u_1}\delta_{1j}; \tag{E.4}$$

note that only (at most) two elements are non-zero, the first and the one on the diagonal. Next, subtract $u_1 v_i$ times the new row i from the first row, for each $i \geqslant 2$. This gives a new first row with elements

$$\delta_{1j} + u_1 v_j - \sum_{i=2}^n u_1 v_i\Bigl(\delta_{ij} - \frac{u_i}{u_1}\delta_{1j}\Bigr), \tag{E.5}$$

which vanishes for $j > 1$ and equals $1 + \sum_{i=1}^n u_i v_i$ for $j = 1$. Hence we have reduced the matrix A to a lower triangular matrix (with the same determinant) with one diagonal element $1 + \langle u,v\rangle$ and all other diagonal elements 1; hence (E.2) follows. The case $u_1 = 0$ follows by continuity.

Next, let $\lambda := \langle u,v\rangle$ and $B := A - I = (u_i v_j)$. Then $B^2 = \lambda B$, and thus

$$(I+B)((1+\lambda)I - B) = (1+\lambda)I + (1+\lambda-1)B - B^2 = (1+\lambda)I, \tag{E.6}$$

and (E.3) follows by dividing by $1+\lambda$, provided this is non-zero. □

References

[1] *Explanatory Supplement to the Astronomical Almanac.* P.K. Seidelmann ed., University Science books, Sausalito, CA, 1992.

[2] Shiing-Shen Chern, An elementary proof of the existence of isothermal parameters on a surface, *Proc. Amer. Math. Soc.* **6** (1955), no. 5, 771–782.

[3] É. Cotton, Sur les variétés à trois dimensions. *Annales de la faculté des sciences de Toulouse, Sér. 2*, **1** (1899), no. 4, 385–438.

[4] Manfredo Perdigão do Carmo, *Riemannian Geometry.* Birkhäuser, Boston, 1992.
[5] Philip Hartman and Aurel Wintner. On uniform Dini conditions in the theory of linear partial differential equations of elliptic type. *American Journal of Mathematics* **77** (1955), no. 2, 329–354.
[6] Udo Hertrich-Jeromin, *Introduction to Möbius Differential Geometry.* London Mathematical Society Lecture Note Series 300, Cambridge Univ. Press, 2003.
[7] Birger Iversen, *Hyperbolic Geometry*, Cambridge Univ. Press, 1992.
[8] Svante Janson, Euclidean, spherical and hyperbolic trigonometry. Notes, 2015. http://www.math.uu.se/~svante/papers/sjN16.pdf
[9] A. Korn, Zwei Anwendungen der Methode der sukzessiven Annäherungen, *Schwarz Festschrift*, Berlin (1914), pp. 215–229.
[10] Dmitri Krioukov, Fragkiskos Papadopoulos, Maksim Kitsak, Amin Vahdat, Marián Boguñá, Hyperbolic geometry of complex networks. *Phys. Rev. E (3)* **82** (2010), no. 3, 036106.
[11] Joseph-Louis Lagrange, Sur la construction des Cartes géographiques. *Noveaux Mémoires de l'Académie royale des Sciences et Belles-Lettres de Berlin*, 1779. Reprinted in *Oeuvres de Lagrange*, vol. 4, pp. 637–692, Gauthier-Villars, Paris, 1867–1892.
[12] L. Lichtenstein, Zur Theorie der konformen Abbildung. Konforme Abbildung nicht-analytischer, singularitätenfreier Flächenstücke auf ebene Gebiete. *Bull. Internat. Acad. Sci. Cracovie, Sér. A* (1916), 192–217.
[13] Robert Osserman, From Schwarz to Pick to Ahlfors and beyond. *Notices Amer. Math. Soc.* **46** (1999), no. 8, 868–873.
[14] J. A. Schouten, Über die konforme Abbildung n-dimensionaler Mannigfaltigkeiten mit quadratischer Maßbestimmung auf eine Mannigfaltigkeit mit euklidischer Maßbestimmung. *Math. Z.* **11** (1921), no. 1-2, 58–88.
[15] John P. Snyder, *Map Projections: A Working Manual.* US Geologic Survey Professional Paper 1395, US Government Printing Office, Washington, D.C., 1987. http://pubs.er.usgs.gov/publication/pp1395
[16] John P. Snyder, *Flattening the Earth: Two Thousand Years of Map Projections.* Univ. of Chicago Press, Chicago, IL, 1993.
[17] *UN Flag and Emblem.* http://www.un.org/depts/dhl/maplib/flag.htm
[18] Hermann Weyl, Reine Infinitesimalgeometrie. *Math. Z.* **2** (1918), no. 3-4, 384–411.
[19] Frank W. Warner, *Foundations of Differentiable Manifolds and Lie Groups.* Springer-Verlag, New York, 1983.

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