A NOTE ON POLYA URNS: THE WINNER MAY LEAD ALL THE TIME

SVANTE JANSON

We consider a Pólya urn where, as in Pólya's original urn [2; 3], a ball always is replaced together with additional balls of the same colour only; however, we allow the urn to be unfair in the sense that the number of additional balls may depend on the colour. To be precise, the urn contains balls of q colours; we draw a ball (uniformly at random) and if the drawn ball has colour i, we replace it together with m_i additional balls of colour i, for some given (fixed) integers $m_i > 0$. We assume that the urn start with some set of balls, containing at least one ball of each colour.

It is well-known, and easy to see by the argument below, that if there is a colour i such that $m_i > m_j$ for all colours $j \neq i$, then the proportion of balls of colour i tends to 1 a.s. Of course, even if colour 1, say, wins eventually, it is possible that colour 2 is lucky initially, and that at some time n, most balls are of colour 2. The purpose of this note is to show that this happens with probability strictly less than 1, provided it does not occur already in the initial position. Moreover, this extends to the case when some or all of the numbers m_i are equal, say with $m_1 = m_2 \geqslant m_3 \geqslant \dots$; in this case, the proportion of balls of colour 1 converges to some random limit in $(0,1)$, and the probability that colour 1 eventually dominates lies strictly between 0 and 1; we show that there is also in this case a positive probability that colour 1 dominates at all times, if it does so initially.

To be precise, we have the following theorem, where we for simplicity consider two colours only. (The proof is simple and this has presumably been observed before, but we do not know a reference and therefore give a complete proof.)

Theorem 1. *Consider a Pólya urn with balls of two colours, black and white, with the replacement rule that if a ball of colour* $i \in \{\mathsf{b}, \mathsf{w}\}$ *is drawn, it is replaced together with* m_i *additional balls of the same colour, for some given positive integers* $m_{\mathsf{b}}, m_{\mathsf{w}}$*. Let* B_n *and* W_n *be the numbers of black and white balls after* n *draws, with initial conditions* $B_0 = b_0$ *and* $W_0 = w_0$ *for some given* b_0, w_0*. Suppose that* $m_{\mathsf{b}} \geqslant m_{\mathsf{w}}$ *and that* $b_0 > w_0 \geqslant 0$*. Then there is a positive probability that* $B_n > W_n$ *for all* $n \geqslant 0$*.*

Proof. We use the standard method of embedding the urn process into a continuous time Markov branching process, see e.g. [1, Section V.9]; we give each ball an exponential clock, that rings after a random time with the distribution Exp(1). When the clock rings at a ball of colour i, the ball is

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replaced by $m_i + 1$ new balls of the same colour i, each with a new clock. (All clocks are independent.) Then the original urn process is the same (i.e., has the same distribution) as this continuous time process observed at the times some clock rings. More formally, let $\overline{B}_t$ and $\overline{W}_t$ be the number of black and white balls at time $t \geqslant 0$, and let τ_n be the time the n-th clock rings. Then, the process $(\overline{B}_{\tau_n}, \overline{W}_{\tau_n})_{n\geqslant 0}$ has the same distribution as the Pólya urn process, and we may assume $B_n = \overline{B}_{\tau_n}$, $W_n = \overline{W}_{\tau_n}$.

In the continuous time process, the black and white balls act independently, so $\overline{B}_t$ and $\overline{W}_t$ are two independent continuous-time branching processes. Hence, see [1, Theorems III.7.1 and III.7.2], there exist independent positive random variables $Y_{\mathsf{b}}, Y_{\mathsf{w}}$ such that a.s.

$$e^{-m_{\mathsf{b}}t}\overline{B}_t \to Y_{\mathsf{b}} > 0, \tag{1}$$

$$e^{-m_{\mathsf{w}}t}\overline{W}_t \to Y_{\mathsf{w}} > 0. \tag{2}$$

Consequently, as $t \to \infty$, a.s.,

$$e^{(m_{\mathsf{b}}-m_{\mathsf{w}})t}\frac{\overline{W}_t}{\overline{B}_t} \to \frac{Y_{\mathsf{w}}}{Y_{\mathsf{b}}} < \infty, \tag{3}$$

and thus

$$\frac{\overline{W}_t}{\overline{B}_t} \to Z := \begin{cases} 0, & m_{\mathsf{b}} > m_{\mathsf{w}}, \\ Y_{\mathsf{w}}/Y_{\mathsf{b}}, & m_{\mathsf{b}} = m_{\mathsf{w}}. \end{cases} \tag{4}$$

As a consequence, as $n \to \infty$, a.s.,

$$\frac{W_n}{B_n} = \frac{\overline{W}_{\tau_n}}{\overline{B}_{\tau_n}} \to Z. \tag{5}$$

Furthermore, both Y_{b} and Y_{w} have support on the entire positive half-axis. Consequently, $\mathbb{P}(Y_{\mathsf{w}} < Y_{\mathsf{b}}) > 0$ and thus (4) implies $\mathbb{P}(Z < 1) > 0$ for any m_{b} and m_{w}. Hence, (5) implies

$$\mathbb{P}\bigl(\limsup_{n\to\infty} W_n/B_n < 1\bigr) \geqslant \mathbb{P}(Z < 1) > 0. \tag{6}$$

It follows that there exists an integer N such that

$$\mathbb{P}\bigl(W_n/B_n < 1 \text{ for all } n \geqslant N\bigr) > 0. \tag{7}$$

There is only a finite number of possible outcomes of (B_N, W_N), and consequently (7) implies that there are integers (b_N, w_N) such that

$$\mathbb{P}\bigl((B_N, W_N) = (b_N, w_N) \text{ and } W_n/B_n < 1 \text{ for all } n \geqslant N\bigr) > 0 \tag{8}$$

and consequently,

$$\mathbb{P}\bigl((B_N, W_N) = (b_N, w_N)\bigr) > 0, \tag{9}$$

and

$$\mathbb{P}\bigl(W_n/B_n < 1 \text{ for all } n \geqslant N \mid (B_N, W_N) = (b_N, w_N)\bigr) > 0. \tag{10}$$

By (9), it is possible to reach (b_N, w_N) by some sequence of N draws, starting at (b_0, w_0); thus $b_N = b_0 + k_{\mathsf{b}}m_{\mathsf{b}}$ and $w_N = w_0 + k_{\mathsf{w}}m_{\mathsf{w}}$, where $k_{\mathsf{b}} + k_{\mathsf{w}} = N$.

Consider now the event that we draw a black ball in the first k_{b} draws, and then a white ball in the following k_{w} draws; this evidently has positive probability. Furthermore, then $(B_N, W_N) = (b_N, w_N)$. Moreover, in this case, for $0 \leqslant n \leqslant N$, $B_n - W_n$ first increases and then decreases, and since

$B_0 - W_0 = b_0 - w_0 > 0$ and $B_N - W_N = b_N - w_N > 0$, we have $B_n - W_n > 0$ for every $n \leqslant N$.

Since the urn process is a Markov process, it is by (10) possible to continue, with positive probability, so that $B_n > W_n$ also for all $n \geqslant N$. Hence, with positive probability, $B_n > W_n$ for all $n \geqslant 0$. □

Remark 2. The theorem, and its proof, readily extends to any (finite) number of colours. Several versions are possible. For example, if we assume that $m_1 \geqslant m_j$ for every colour $j \neq 1$, and a majority of the balls at time 0 are of colour 1, then there is a positive probability that a majority has colour 1 at every time n. Alternatively, still assuming that $m_1 \geqslant m_j$ for every colour $j \neq 1$, if a plurality (relative majority) of the balls at time 0 are of colour 1, i.e., the number of balls of colour 1 is larger than the number of any other given colour, then there is a positive probability that a plurality has colour 1 at every time n. We leave the details, and further variations, to the reader.

References

[1] Krishna B. Athreya & Peter E. Ney, *Branching Processes.* Springer-Verlag, Berlin, 1972.

[2] F. Eggenberger & G. Pólya, Über die Statistik verketteter Vorgänge. *Zeitschrift Angew. Math. Mech.* **3** (1923), 279–289.

[3] G. Pólya, Sur quelques points de la théorie des probabilités. *Ann. Inst. Poincaré* 1, 117–161, 1931.

Department of Mathematics, Uppsala University, PO Box 480, SE-751 06 Uppsala, Sweden

Email address: `svante.janson@math.uu.se`

URL: `http://www.math.uu.se/svante-janson`