THRESHOLDS QUANTIFYING PROPORTIONALITY CRITERIA FOR ELECTION METHODS

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Abstract. We define several different thresholds for election methods by considering different scenarios, corresponding to different proportionality criteria that have been proposed by various authors. In particular, we reformulate the criteria known as DPC, PSC, JR, PJR, EJR in our setting. We consider multi-winner election methods of different types, using ballots with unordered lists of candidates or ordered lists, and for comparison also methods using only party lists. The thresholds are calculated for many different election methods. The considered methods include classical ones such as BV, SNTV and STV (with some results going back to e.g. Droop and Dodgson in the 19th century); we also study in detail several perhaps lesser known methods by Phragmén and Thiele. There are also many cases left as open problems.

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1. Introduction

We consider election methods where a number of persons are elected from some set of candidates. For example, this is the case in a multi-member constituency in a parliamentary election or a local election, but also in many other situations such as electing a board or a committee in an organization. We will here use the language of a parliamentary election; some other authors instead talk about e.g. *committee voting rules*. We assume throughout that the number of *seats*, i.e., the number of elected representatives, is given in advance; we denote the number of seats by S.

Many different election methods have been suggested for this type of elections, and many of them are, or have been, used somewhere. For some important examples and discussions, from both mathematical and political aspects, see e.g. [17; 37; 38]; for the election methods actually used at present see [23] for parliamentary elections around the world, and [38] for the European Parliament; see also my surveys [24] (in Swedish) and [26, Appendix E].

One important aspect of election methods is whether they are *proportional* or not, i.e., whether different groups of voters (parties) get numbers of seats that are (more or less) proportional to their numbers of votes. Whether an election method is proportional or not is not precisely defined, even when there are formal parties. Obviously, there are necessarily deviations from exact proportionality since the number of seats is an integer for each party. Moreover, in practice, whether the outcome of an election is (approximatively) proportional depends not only on the method, but also on the number of seats in the constituency, and on other factors such as the number of parties and their sizes.

One way that has been used by many authors to study proportionality of election methods theoretically is to formulate some criterion, which is supposed to be a desirable requirement for a method to be regarded as proportional, and then investigate whether a particular method satisfies this criterion or not. A typical example is the *Droop Proportionality Criterion* (DPC) by Woodall [48] (see also Dummett [14, p. 283]); this is formulated for STV type elections where each ballot contains a ranked list of candidates, i.e., election methods with *ordered ballots* in the notation introduced in Section 2.

Droop Proportionality Criterion. *If V votes are cast in an election to fill S seats, and, for some whole numbers ℓ and m satisfying $0 < \ell \leqslant m$, more than $\ell \cdot V/(S+1)$ voters put the same m candidates (not necessarily in the same order) as the top m candidates in their preference listings, then at least ℓ of those m candidates should be elected. (In the event of a tie, this should be interpreted as saying that every outcome that is chosen with non-zero probability should include at least ℓ of these m candidates.)*

A related criterion is *Proportionality for Solid Coalitions* (PSC) [46], which differs from the DPC above only in that "more than $\ell V/(S+1)$ voters" is replaced by "at least $\ell V/S$ voters".[1] See further Section 8.

Further examples of similar proportionality criteria are JR, PJR and EJR discussed in Section 6.

Several authors study such criteria and whether specific elections methods satisfy them or not; for example, Woodall [48] points out that any version of STV (Single Transferable Vote) satisfies the DPC above, at least provided the Droop quota $V/(S+1)$ is used without rounding. See Section 8.[2]

The purpose of the present paper is to promote an alternative and more quantitative, but closely related, way of studying such proportionality properties. Instead of, as in DPC and PSC above, fixing a number of voters and asking whether a set of voters of this size always can get at least ℓ candidates elected (under certain assumptions on how these voters vote), we turn the question around and ask how large a set of voters has to be in order to be guaranteed to get at least ℓ candidates elected (again under certain assumptions on how they vote). Corresponding to the DPC and PSC criteria above, we thus define, for a given election method, the threshold $\pi_{\mathsf{PSC}}(\ell, S)$ as the minimum (or rather infimum) fraction of votes that guarantees at least ℓ elected out of S under the conditions above, see Section 8 for details. The DPC criterion then can be formulated as $\pi_{\mathsf{PSC}}(\ell, S) \leqslant \ell/(S+1)$ (for all $S \geqslant 1$ and $\ell \leqslant S$), and the PSC criterion as $\pi_{\mathsf{PSC}}(\ell, S) < \ell/S$ (ignoring a possible minor problem in the case of ties, see further Remark 2.8).

An advantage of this quantitative version of proportionality criteria is thus that it treats both DPC and PSC at once. Moreover, it is not necessary to consider only the thresholds $\ell/(S+1)$ and ℓ/S. In fact, Aziz and Lee [2] have defined a property q-PSC, where the conclusion above is supposed to hold for every set of at least $q\ell$ voters; with $q = \kappa V$ this is thus the same as $\pi_{\mathsf{PSC}}(\ell, S) < \ell\kappa$. Thus, to consider this parametric family of proportionality criteria is equivalent to considering the quantity π_{PSC}. As another example,

[1]The formulation of the PSC in [46] is somewhat different, but is easily seen to be equivalent to our version; cf. "Theorem" 3.9.

[2]Recall that there are many versions of STV, see Appendix A.3.1; recall also that in many versions, the Droop quota is rounded to an integer, which may lead to minor violations of the DPC as stated above.

note that the property *majority* in [48] in our notation can be written as $\pi_{\mathsf{PSC}}(1, S) \leqslant 1/2$.

Another advantage of the quantitative version is that it enables us to see not only whether a criterion hold or not, but also how badly it fails if it does. Conversely, if a criterion holds, perhaps it turns out that it holds with some marginal. A threshold such as π_{PSC} therefore gives more information, and may give more insight in the behaviour of an election method. Furthermore, even if the goal is to study a specific criterion, the arithmetic of the thresholds may simplify or clarify proofs. (See e.g. Remark 7.33 and Example 7.40 for examples of this.)

We have here used DPC and PSC as examples only. Much of our work below deals with other similar properties, and we will define a number of thresholds $\pi(\ell, S)$ for different scenarios.

Our point of view is far from new. In fact, thresholds of the type $\pi(\ell, S)$ figure prominently in the 19th century discussion on parliamentary reform in (for example) Britain, e.g. in Droop [13] and Dodgson [12]; they were at about the same time computed for D'Hondt's method by D'Hondt [11] and Hagenbach-Bischoff [21, 22]; for party list systems, they were defined by Rae, Hanby and Loosemore [40], [39] under the name *threshold of exclusion*, see also Lijphart and Gibberd [30]. Nevertheless, the idea is still useful and seems to deserve further attention; in the present paper we extend the study of such thresholds to further scenarios and further election methods.

We consider many different election methods. These include several classical ones such as D'Hondt's method, Sainte-Laguë's method, Block Vote, Limited Vote, SNTV and STV; we also study, in a rather large part of the paper, several election methods by Phragmén and Thiele. (Nevertheless, many election methods remain to be studied.) We partly survey known results, but most of the results seem to be new.

Remark 1.1. In the present paper, we aim at computing $\pi(\ell, S)$ in many cases, but we do not try to give any criteria for what is good and what is not. Nevertheless, note that historically, the two main thresholds that have been proposed for various scenarios are ℓ/S and $\ell/(S+1)$; in other words that the required number of votes $\mathcal{W}$ to guarantee ℓ seats is ℓ times the *Hare quota* V/S or the *Droop quota* $V/(S+1)$, which both have been put forth as a natural number of votes deserving a seat. The Hare quota is simpler and perhaps more intuitive; see e.g. "Theorem" 3.9 below; the Droop quota was proposed by Droop [13] based on the argument that if a candidate has more than $1/(S+1)$ of the votes, then there cannot be S others with at least as many votes, so the candidate ought to be among the S elected;[3] see also [12] and [46]. In our context, $\ell/(S+1)$ is essentially the best (i.e., smallest) threshold that can be achieved, see Remark 3.6; hence $\pi(\ell, S) = \ell/(S+1)$ can be regarded as the optimal value. We will see many cases where this

[3]This argument holds for e.g. SNTV (Appendix A.2.3) and Cumulative Vote (Appendix A.2.5; the case really discussed by Droop [13]).

value is achieved, and many where it is not. (See Table 2 in Appendix B for some numerical values.)

2. Notations and general definitions

We consider only election methods where each voter votes by leaving a ballot in a single round. The outcome is then determined by these ballots and some mathematical algorithm. Furthermore, we assume that either each ballot contains the name of a party, or that each ballot contains a list of one or several candidates. (In fact, the first case can be regarded as a special case of the second, since each party may be regarded as a predefined list of candidates.) Moreover, we consider both methods where the order of the names on the ballot matter and those where it does not, and it is usually important to distinguish between these two cases. Hence, we consider the following three different types of election methods; we will see that these three types often have to be considered separately.[4]

party ballots: The candidates are organized in parties, with separate lists of candidates. Each ballot contains the name of a party, and seats are distributed among the parties. In this case we are only interested in the distribution of seats among the parties. (We thus ignore how seats are distributed within each party.)

unordered ballots: Each ballot contains a set of one or several candidates; their order on the ballot does not matter.

ordered ballots: Each ballot contains a list of one or several candidates, in order of preference.[5]

2.1. **Some notation.** Let $\mathcal{V}$ denote the set of all voters, and $V := |\mathcal{V}|$ the number of votes. Formally, an election may be regarded as a family $(\sigma_\nu)_{\nu\in\mathcal{V}}$ of ballots, where each ballot belongs to the set $\mathcal{B}$ of possible ballots.[6] Let $\mathcal{C}$ be the set of parties in the party ballot case and otherwise the set of candidates. Then, in the party ballot case, $\mathcal{B} = \mathcal{C}$ (the set of parties); with unordered ballots, $\mathcal{B}$ is the set of subsets of $\mathcal{C}$; with ordered ballots, $\mathcal{B}$ is the set of sequences of distinct elements of $\mathcal{C}$. (We assume tacitly that $\mathcal{C}$ is finite, but sufficiently large when needed.)

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[4]A more general type of ballot, which includes both unordered ballots and ordered ballots as special cases, is a ballot with a *weak ordering* of the candidates. Election methods for such ballots are sometimes discussed in the literature, and have even been used, see for example [26, §18.1-2] and the references there. We will not consider such ballots in the present paper, but see [2] for versions of PSC (and implicitly of our π_{PSC}) for such ballots and methods.

[5]An extreme version, more popular in theoretic work than in practical use, is that each voter ranks *all* candidates in a linear order.

[6]We consider only election methods that are *anonymous*, so the order of the ballots does not matter. Hence, an election can equivalently be described by a sequence $(x_\sigma)_{\sigma\in\mathcal{B}}$ of integers $\geqslant 0$, where x_σ is the number of votes for σ, i.e., the number of ballots σ.

Remark 2.1. For convenience, we may identify the voters with their ballots, and write e.g. $\sigma \in \mathcal{W} \subset \mathcal{V}$, thus treating $\mathcal{V}$ as a multiset of ballots. Similarly, we often abuse notation and write $\sum_{\sigma\in\mathcal{V}}$ instead of $\sum_{\nu\in\mathcal{V}}$ (with summands depending on σ or σ_ν, respectively).

We consider only election methods where the number of seats, S, is fixed in advance. We are mainly interested in the case $S \geqslant 2$, but the case $S = 1$ (single-member constituency) is included for completeness and comparisons.

The *outcome* of the election is the set $\mathcal{E}$ of elected candidates. Thus, for elections with unordered or ordered ballots, $\mathcal{E}$ is a subset of $\mathcal{C}$ with $|\mathcal{E}| = S$. (In the case of elections with party ballots, $\mathcal{E}$ can be regarded as a multi-set; we will not use the notation $\mathcal{E}$ in this case.) Note that there might be ties and therefore several possible outcomes for a given set of ballots $(\sigma_\nu)_{\nu\in\mathcal{V}}$. Formally, we regard the different possibilities as different elections; thus an election can formally be defined as a pair $\bigl((\sigma_\nu)_{\nu\in\mathcal{V}}, \mathcal{E}\bigr)$ of a sequence of ballots and a possible outcome.

Some general mathematical notation: $x \wedge y := \min\{x, y\}$; $x \vee y := \max\{x, y\}$; $[n] := \{1, \dots, n\}$; $|\mathcal{S}|$ is the number of elements in a set $\mathcal{S}$; $H_n := \sum_{i=1}^{n} 1/i$, the n-th harmonic number.

Remark 2.2. It is possible to let different voters have different weights, which in principle could be any positive real numbers. Then, the "number of voters" in a subset of $\mathcal{V}$ has to be interpreted as their total weight, which thus is a (positive) real number, not necessarily an integer; similarly $\sum_{\sigma\in\mathcal{V}}$ (see Remark 2.1) has to be interpreted with weights. We leave the trivial modifications for this extension to the reader, and continue to talk about "numbers" of votes.

Remark 2.3. The election methods considered below are almost all *homogeneous*, meaning that the outcome remains the same if the number of votes for each type of ballot is multiplied by the same number; in other words, only the proportions of votes matter.[7] Hence the methods are well-defined also when the "number of votes" for each type of ballot is a positive rational number. In fact, the homogeneous methods considered here (and all reasonable homogeneous methods) all apply to the case when these numbers are arbitrary positive real numbers as in Remark 2.2.

In some cases, we find it convenient to use this and allow vote numbers to be arbitrary positive real numbers. This is not essential; real numbers can be approximated by rational numbers if necessary, and rational numbers can be multiplied by a common denominator to get an equivalent election with integer numbers of votes.

[7]Note that this fails for quota methods and STV if the quota is rounded (up or down) to an integer, which is often the case in practical uses. This is mathematically inconvenient, and theoretically bad also for other reasons, but has very little importance in large elections. We consider mainly mathematically ideal versions without rounding, and they are homogeneous, as are all other methods considered here. For an inhomogenous case, see Remark 4.7.

2.2. **Proportionality thresholds.** As explained in the introduction, our goal is to define and study thresholds of proportionality $\pi(\ell, S) = \pi^{\mathfrak{M}}(\ell, S)$, where $\mathfrak{M}$ denotes the election method. Informally, this is the smallest proportion of votes that a set $\mathcal{W}$ of voters may have in order to be guaranteed at least ℓ elected candidates out of S. Here, the set $\mathcal{W}$ can be an organized group (party) or not, and we may consider several different situations regarding both the votes from $\mathcal{W}$ and the votes from the other voters $\mathcal{W}^{\mathsf{c}} := \mathcal{V} \setminus \mathcal{W}$. For example, the voters in $\mathcal{W}$ may all vote with identical ballots, or according to some organized tactical scheme, or be unorganized with different ballots that happen to contain some common candidate(s). Hence, we will define several such thresholds π, which we distinguish by subscripts. (As above, the election method is indicated by superscript.) Further versions are possible, and are left to the readers imagination.

In general, a subscript $\mathfrak{S}$ defines some *scenario*, i.e., a class of possible elections with some restrictions on the votes from the set $\mathcal{W}$ and possibly also on the votes from $\mathcal{W}^{\mathsf{c}}$; furthermore, $\mathfrak{S}$ specifies, for a given ℓ and S, what we mean by a *good* outcome. This should mean, in some sense, that $\mathcal{W}$ gets at least ℓ candidates elected; however, as is seen below, this can be made precise in different ways, since in some scenarios the voters in $\mathcal{W}$ might vote for partly different candidates, and some voters in $\mathcal{W}^{\mathsf{c}}$ also might vote for some of these. We say that an outcome that is not good is *bad*. ("Good" and "bad" are always from the perspective of the chosen set $\mathcal{W}$ of voters. $\mathcal{W}$ will always denote such a set of voters, arbitrary but with the scenario specifying some assumption on their votes.)

Given a scenario $\mathfrak{S}$ and an election method $\mathfrak{M}$, we define $\pi(\ell, S) = \pi^{\mathfrak{M}}_{\mathfrak{S}}(\ell, S)$ as the smallest proportion of votes that guarantees a good outcome. More precisely and formally:

Definition 2.4.

$$\pi^{\mathfrak{M}}_{\mathfrak{S}}(\ell, S) := \sup\Bigl\{\frac{|\mathcal{W}|}{|\mathcal{V}|} : \textit{elections satisfying } \mathfrak{S} \textit{ with a bad outcome}\Bigr\}. \tag{2.1}$$

Remark 2.5. One reason for the form (2.1) of the definition is that when there are ties, the same set of ballots might lead to both a good and a bad outcome. (In practice, the result is typically decided by lot.) The formulation (2.1) includes such cases; thus the idea is that $|\mathcal{W}|/|\mathcal{V}| > \pi(\ell, S)$ guarantees a good outcome, even in the event of ties. Similarly, there exist election methods that are randomized algorithms, so the outcome may be random. (E.g. some versions of STV, see Appendix A.3.1.) In such cases, any case where there is a positive probability of a bad outcome is included in (2.1).

Remark 2.6. Of course, the outcome might be good also in some cases where $|\mathcal{W}|/|\mathcal{V}| < \pi(\ell, S)$ (depending on, e.g., how other voters vote); to have more that $\pi(\ell, S)$ of the votes is sufficient but not necessary for a good outcome.

Remark 2.7. The case $\ell = 1$ is of particular interest: $\pi_{\mathfrak{S}}(1, S)$ is the smallest proportion that guarantees the election of at least one representative of $\mathcal{W}$. This is called the *threshold of exclusion* [40; 30].

Also the other extreme case $\ell = S$ is of interest: $\pi_{\mathfrak{S}}(S, S)$ is the smallest proportion that guarantees $\mathcal{W}$ to get all seats, excluding all minorities (assuming $\mathfrak{S}$).

As another example, if S is odd, then $\pi_{\mathfrak{S}}\big((S+1)/2, S\big)$ is the proportion of votes that guarantees a majority of the seats.

Remark 2.8. Typically, the supremum in (2.1) is attained because ties might appear when $|\mathcal{W}|/|\mathcal{V}| = \pi_{\mathfrak{S}}(\ell, S)$, cf. Remark 2.5; in such cases, a proportion of votes exactly equal to $\pi_{\mathfrak{S}}(\ell, S)$ is not enough to guarantee a good outcome. We may indicate whether this happens or not by a more refined notation: if p denotes the supremum in (2.1), we may write $\pi_{\mathfrak{S}}(\ell, S) = p+$ or $\pi_{\mathfrak{S}}(\ell, S) = p-$, with $p+$ meaning that the supremum is attained, i.e., that there exists a bad outcome with a proportion of the votes exactly p, and $p-$ meaning that there is no such bad outcome. In other words, if $\pi_{\mathfrak{S}}(\ell, S) = p-$, then a proportion of votes equal to p guarantees a good outcome, while if $\pi_{\mathfrak{S}}(\ell, S) = p+$, then we need strictly more than p in order to be sure.

For example, the formulation of the DPC in Section 1 uses "more than", and is thus equivalent to $\pi_{\mathsf{PSC}}(\ell, S) \leqslant \frac{\ell}{S+1}+$, while PSC uses "at least", and is thus equivalent to $\pi_{\mathsf{PSC}}(\ell, S) \leqslant \frac{\ell}{S}-$.

We will occasionally use the refined notation, but usually not; this is then left to the reader.

Note that almost all values calculated below are of the "+" type, because of the possibility of ties at the threshold, but we note a few, more or less trivial, exceptions with a "−" type:

(i) When $\pi_{\mathfrak{S}}(\ell, S) = 1$: $\pi_{\mathfrak{S}}(\ell, S) = 1+$ means that even if $\mathcal{W}$ comprises all voters, this does not guarantee all seats, which does not happen for reasonable methods and scenarios, while $\pi_{\mathfrak{S}}(\ell, S) = 1-$ is perfectly possible, also for $\ell < S$, see e.g. Theorem 9.3.

(ii) When $\pi_{\mathfrak{S}}(\ell, S)$ is irrational. (Unless we allow real numbers of votes, see Remark 2.2.) This is usually not the case, but can happen e.g. for the divisor method $\mathsf{Div}(\gamma)$ in Theorem 4.3 with irrational γ (never used in practice as far as I know) or for the Estonian method (Footnote 25 in Appendix A.1).

(iii) Some cases for election methods with special tie-breaking rules. For example, for D'Hondt's method with ties always broken in favour of the largest party, $\pi^{\mathsf{D'H}}_{\mathsf{party}}(\ell, S) = \frac{\ell}{S+1}+$ for $\ell \leqslant (S+1)/2$, but $\pi^{\mathsf{D'H}}_{\mathsf{party}}(\ell, S) = \frac{\ell}{S+1}-$ for $\ell > (S+1)/2$, cf. (4.3). Another example is given in Remark 6.8.

Hence, the property that some given proportion $p \in [0, 1]$ of votes is enough to guarantee ℓ seats, which really means $\pi(\ell, S) \leqslant p-$, is in practice equivalent to $\pi(\ell, S) < p$. (Note that this holds even in the exceptional case (ii), since then $\pi(\ell, S) = |\mathcal{W}|/V$ cannot occur.) For example, in practice, PSC can be written as $\pi_{\mathsf{PSC}}(\ell, S) < \frac{\ell}{S}$, with strict inequality, when $\ell < S$.

Remark 2.9. Formally, the refined notation in Remark 2.8 means that we regard $\pi_{\mathfrak{S}}(\ell, S)$ as a member of the *split interval* (also called *two arrow space*), which contains two elements $x+$ and $x-$ for each $x \in [0, 1]$, and that (2.1) is interpreted in this space with its natural order, and a real x identified with $x+$. The split interval, with its order topology, has interesting topological properties and is a standard example in topology, but we do not need any of this here.

In some cases, $\pi_{\mathfrak{S}}(\ell, S)$ may be determined by some bad outcome with small $V = |\mathcal{V}|$, while a smaller proportion of votes suffices to get a good outcome when V is large. This may happen, for example, because of rounding effects in the election method as in Remark 4.7. It may also happen because some tactical voting scheme requires votes to be split e.g. equally between two candidates, which is impossible when V is odd, see Example 5.8. In such cases it is more interesting to consider only large V, or more precisely, the limit as $V \to \infty$, and define

$$\overline{\pi}_{\mathfrak{S}}(\ell, S) := \limsup_{|\mathcal{V}| \to \infty} \Big\{ \frac{|\mathcal{W}|}{|\mathcal{V}|} : \text{elections satisfying } \mathfrak{S} \text{ with a bad outcome} \Big\}. \tag{2.2}$$

Note that by the definitions (2.1) and (2.2), always

$$\overline{\pi}_{\mathfrak{S}}(\ell, S) \leqslant \pi_{\mathfrak{S}}(\ell, S). \tag{2.3}$$

Remark 2.10. For an homogeneous election method, and a scenario that does not assume that the voters in $\mathcal{W}$ agree to split their votes on different lists according to some strategy, it is easy to see that $\overline{\pi}^{\mathfrak{M}}_{\mathfrak{S}}(\ell, S) = \pi^{\mathfrak{M}}_{\mathfrak{S}}(\ell, S)$. In fact, consider any election with a bad outcome, and replace each ballot by N identical ballots, for some large integer N. This gives a new election with (by our assumptions on $\mathfrak{M}$ and $\mathfrak{S}$) the same elected set $\mathcal{E}$ and thus a bad outcome with the same proportion $|\mathcal{W}|/V$, but V replaced by NV. Letting $N \to \infty$, we obtain $\overline{\pi}^{\mathfrak{M}}_{\mathfrak{S}}(\ell, S) \geqslant |\mathcal{W}|/V$, and thus $\overline{\pi}^{\mathfrak{M}}_{\mathfrak{S}}(\ell, S) \geqslant \pi^{\mathfrak{M}}_{\mathfrak{S}}(\ell, S)$, showing that the general inequality (2.3) is an equality in this case. This applies to all election methods considered in the present paper except versions of quota methods and STV with rounding of the quota, and to all scenarios defined below except tactic.

In most cases studied below, $\overline{\pi} = \pi$ is obvious from Remark 2.10, or follows from the given proofs; we will usually not comment on this or mention $\overline{\pi}$ at all in such cases, leaving this to the reader. We will thus discuss $\overline{\pi}$ mainly in cases where it differs from π. (In such cases we instead often leave π to the reader.)

In the next section we give a few general properties of the definition. In the following sections we put life in the general definitions above by specifying some scenarios $\mathfrak{S}$, and calculating $\pi^{\mathfrak{M}}_{\mathfrak{S}}(\ell, S)$ for some methods $\mathfrak{M}$.

We usually treat the three different types of ballots separately. The case of party ballots in Section 4 is straightforward, and can be seen as a warm-up to the remaining sections with ordered and unordered ballots, where there are many reasonable choices of definitions of $\mathfrak{S}$. Our goal is to investigate some of these choices of $\mathfrak{S}$, and to calculate $\pi_{\mathfrak{S}}^{\mathfrak{M}}(\ell, S)$ for various election methods $\mathfrak{M}$. (In a few cases we give only some bounds.) Note that lower bounds always are found by giving examples, where the election method $\mathfrak{M}$ gives (or may give, in case of ties) a bad outcome for the chosen $\mathfrak{S}$.

There are many election methods, including many not mentioned in the present paper, and we define thresholds $\pi_{\mathfrak{S}}$ for several different scenarios (and further may be constructed), so there are many potential combinations and we will only give complete results for some of them. There are lots of cases remaining; these should be regarded as open problems (some are stated explicitly for emphasis), and we invite other researchers to continue this work.

The scenarios and thresholds that we consider are defined in Definitions 4.1 and 4.11 (π_{party}), 5.1 (π_{same}), 5.2 (π_{tactic}), 6.1 (π_{PJR}), 6.2 (π_{EJR}), 8.1 (π_{PSC}), 8.2 (π_{wPSC}).

The election methods that are studied in the paper are the following; each is listed after the abbreviation that we use. Brief definitions, and some alternative names, are given in Appendix A.

- (i) Party ballots
  - (a) D′H: D'Hondt's method.
  - (b) StL: Sainte-Laguë's method.
  - (c) Adams: Adams's method.
  - (d) Div(γ): Divisor method with divisors $d(n) = n - 1 + \gamma$, $\gamma \in [0, 1]$.
  - (e) LR: Method of Largest Remainder.
  - (f) Droop: Droop's method.
  - (g) Q(δ): Quota method with quota $V/(S + \delta)$, $\delta \in [0, 1]$.
- (ii) Unordered ballots
  - (a) BV: Block Vote.
  - (b) AV: Approval Vote.
  - (c) SNTV: Single Non-Transferable Vote.
  - (d) LV(L): Limited Vote, with L votes per voter.
  - (e) CV: Cumulative Vote.
  - (f) CV$^{=}$: Equal and Even Cumulative Vote.
  - (g) Phr-u: Phragmén's unordered method.
  - (h) Th-opt, Th-opt($\mathbf{w}$): Thiele's optimization method.
  - (i) Th-add, Th-add($\mathbf{w}$): Thiele's addition method.
  - (j) Th-elim: Thiele's elimination method.
- (iii) Ordered ballots
  - (a) STV: Single Transferable Vote. (This is really a family of methods, see Appendix A.3.1.)
  - (b) Phr-o: Phragmén's ordered method.

(c) Th-o: Thiele's ordered method.
(d) Borda($\mathbf{w}$): Borda method with weights $\mathbf{w}$.

Some numerical values for small S are given as illustrations and for comparisons in Appendix B.

For methods with ordered or unordered ballots, there are no formal parties in the definition of the methods. Nevertheless, we will sometimes, in particular in examples, talk about parties also when discussing such methods; we then mean a (formal or informal) group of voters with a common interest, typically voting with identical ballots, or possibly according to an organized strategy.

3. General properties

We assume always $1 \leqslant \ell \leqslant S$. By the definitions (2.1) and (2.2),

$$0 \leqslant \overline{\pi}(\ell, S) \leqslant \pi(\ell, S) \leqslant 1. \tag{3.1}$$

We state some further simple general results. These hold for any reasonable scenario $\mathfrak{S}$ and any reasonable election method, but since we do not formally define what we mean by 'reasonable', and we have not made any formal restrictions on the scenarios $\mathfrak{S}$ and what a bad outcome might be, we call these results "Theorem". They are real Theorems for the scenarios and the election methods considered in the present paper.[8]

"Theorem" 3.1.

$$\pi(1, S) \geqslant \frac{1}{S+1}. \tag{3.2}$$

Proof. Suppose that there are $S+1$ parties with one candidate each, and that each voter votes for only one of the candidates, with equally many voters ($V/(S+1)$) for each. By symmetry, none of the candidates is guaranteed a seat. (A typical election method would select S of the $S+1$ candidates by lot.) Hence, if $\mathcal{W}$ is the set of voters voting for candidate A, so $|\mathcal{W}|/|\mathcal{V}| = 1/(S+1)$, then a bad event can occur. □

"Theorem" 3.2. *If* $(S+1)/\ell$ *is an integer, then*

$$\pi(\ell, S) \geqslant \frac{\ell}{S+1}. \tag{3.3}$$

Proof. By the same proof as "Theorem" 3.1, now taking $(S+1)/\ell$ parties with exactly ℓ candidates each (the last requirement disappears for party ballots methods), and equally many voters each. □

Remark 3.3. Note that (3.3) is not valid in general for arbitrary ℓ and S, at least not in the case $\ell > (S+1)/2$. As is well-known, with the Block Vote method, a group of voters with a majority of the votes will get all seats if they vote for the same candidates. Hence $\pi_{\mathfrak{S}}^{\mathsf{BV}}(\ell, S) = 1/2$ for every $\ell \leqslant S$ and typical $\mathfrak{S}$, see Theorem 5.4.

[8] A scenario with stricter assumptions on the votes from $\mathcal{W}^{\mathsf{c}}$ than on the votes from $\mathcal{W}$ might fail some of these "Theorems", but we do not consider any such case.

"Theorem" 3.4. *If* $1 \leqslant \ell \leqslant S$, *then*

$$\pi(\ell, S) + \pi(S + 1 - \ell, S) \geqslant 1. \tag{3.4}$$

More generally, for any $\ell_1, \dots, \ell_m$ *with* $\ell_1 + \cdots + \ell_m > S$,

$$\pi(\ell_1, S) + \cdots + \pi(\ell_m, S) \geqslant 1. \tag{3.5}$$

Proof. Suppose that (3.4) fails. Then we can find a (large) $V = |\mathcal{V}|$ and a set of voters $\mathcal{W} \subset \mathcal{V}$ such that $|\mathcal{W}|/V > \pi(\ell, S)$ and $|\mathcal{W}^c|/V > \pi(S + \ell - 1, S)$. Thus $\mathcal{W}$ can get some candidates $A_1, \dots, A_\ell$ elected, and $\mathcal{W}^c$ some $S+1-\ell$ others $B_1, \dots, B_{S+1-\ell}$. But then at least $S+1$ candidates will be elected, a contradiction.

The same argument shows (3.5). □

Note that (3.5) implies both (3.2) and (3.3), by taking all $\ell_i := \ell$ and $m := (S + 1)/\ell$.

Remark 3.5. The "Theorems" above hold for $\overline{\pi}$ too.

Remark 3.6. Recall that the threshold π is the infimum that guarantees a good outcome. Thus, a small π is better (from the point of view of proportionality and representability), and proportionality criteria are always of the form $\pi(\ell, S) \leqslant p$ for some p (possibly with strict inequality, see Remark 2.8). Conversely, a large π is bad, and the extreme $\pi(\ell, S) = 1$ means that we need (almost) all votes in order to be sure to avoid a bad outcome.

Hence, the lower bounds in "Theorems" 3.2 and 3.4 give a bound on what can be achieved by any election method, and they support the idea that $\ell/(S+1)$ is the optimum, and therefore perhaps desirable; cf. (in related terms) Remark 1.1. (As shown in Remark 3.3, $\pi(\ell, S) < \ell/(S+1)$ is possible for some ℓ, but "Theorem" 3.4 implies that then $\pi(\ell', S) > \ell'/(S + 1)$ for $\ell' = S + 1 - \ell$, so it is impossible to have $\pi(\ell, S) < \ell/(S + 1)$ for all ℓ.)

Problem 3.7. What is $\pi_*^*(\ell, S) := \inf \pi_{\mathfrak{S}}^{\mathfrak{M}}(\ell, S)$ for any given ℓ and S (with $1 \leqslant \ell \leqslant S$), taking the infimum over all reasonable scenarios $\mathfrak{S}$ and election methods $\mathfrak{M}$? What is the infimum over $\mathfrak{M}$ for a given $\mathfrak{S}$?

The results above, together with simple examples, answer this problem in some cases. Note that, for all ℓ and S, using (4.3) below,

$$\pi_*^*(\ell, S) \leqslant \pi_{\mathsf{party}}^{\mathsf{D'H}}(\ell, S) = \frac{\ell}{S + 1}. \tag{3.6}$$

Hence, "Theorem" 3.2 shows that equality holds in (3.6) when $(S + 1)/\ell$ is an integer. Furthermore, if $\ell \geqslant (S + 1)/2$, then "Theorem" 3.4 implies $\pi(\ell, S) \geqslant 1/2$, which is attained by $\pi_{\mathsf{party}}^{\mathsf{BV}}$ (Theorem 5.4), and thus

$$\pi_*^*(\ell, S) = \frac{1}{2}, \qquad \ell \geqslant (S + 1)/2. \tag{3.7}$$

Remark 3.8. Let $\ell < m$. It might be argued that if $\mathcal{W}$ gets less than ℓ seats, then $\mathcal{W}$ gets less than m, and therefore $\pi(\ell, S) \leqslant \pi(m, S)$ ought to hold. However, this is not always true, since our scenarios often include restrictions on the votes that depend on ℓ. In fact, Theorem 6.10 below shows that $\pi^{\mathsf{BV}}_{\mathsf{EJR}}$ is a counterexample, see Remark 6.13 and Example 6.14.

Some proposed proportionality criteria state (for a specified scenario $\mathfrak{S}$) that a set of voters that form a fraction p of all voters should get at least the same proportion of seats, rounded down, i.e., at least $\lfloor pS \rfloor$ seats. This simply means that $\pi_{\mathfrak{S}}(\ell, S) < \ell/S$; we state this formally for easy reference.

"Theorem" 3.9. *The property that a set of voters $\mathcal{W}$ with a fraction $p = |\mathcal{W}|/V$ of the votes always gets at least $\lfloor pS \rfloor$ seats is equivalent to*

$$\pi_{\mathfrak{S}}(\ell, S) \leqslant \frac{\ell}{S}-, \qquad 1 \leqslant \ell \leqslant S, \tag{3.8}$$

and to

$$\pi_{\mathfrak{S}}(\ell, S) < \frac{\ell}{S}, \qquad 1 \leqslant \ell < S. \tag{3.9}$$

Proof. The property is equivalent to saying that for each integer $\ell \leqslant S$, if $|\mathcal{W}|/V \geqslant \ell/S$, then $\mathcal{W}$ will get at least ℓ seats. This is trivial for $\ell = 0$, and otherwise equivalent to (3.8).

Furthermore, (3.8) is, in reasonable cases, trivial for $\ell = S$, and otherwise equivalent to (3.9), see Remark 2.8. □

4. Party ballots

Proportional election methods are used for political elections in many countries, in most (but not all) cases using a list method with party lists. Thus, each ballot contains the name of one party, i.e., each voter votes for a party, and each party is given a number of seats decided by the numbers of votes for the parties.

Definition 4.1 (Party ballots, π_{party})**.** *Suppose that all voters in $\mathcal{W}$ vote for the same party A. A good outcome is when at least ℓ candidates from A are elected.*

π_{party} is the same as the *threshold of exclusion* defined by Rae, Hanby and Loosemore [40] ($\ell = 1$) and [39] ($\ell > 1$), see also Lijphart and Gibberd [30].[9]

Remark 4.2. In the literature, e.g. [30; 32; 29; 38], this threshold is often calculated (for various election methods) for a given number of parties. This too is covered by our formalism, by including it in the definition of the scenario $\mathfrak{S}$: we may define $\pi_{\mathsf{party}(n)}$, for an integer $n \geqslant 2$, as in Definition 4.1 but also assuming that there are n parties.[10]

For simplicity, we do not consider this version in the present paper; we let the number of parties be unspecified (or, equivalently, arbitrarily large), and refer to the references above for values of $\pi_{\mathsf{party}(n)}$.

[9] Later usage seems to follow [30] and use this name only for the case $\ell = 1$.

[10] Or at most n parties, which is equivalent since we can give some parties 0 votes. Hence, we can write $\pi_{\mathsf{party}} = \pi_{\mathsf{party}(\infty)}$.

There are two main classes of proportional list methods: divisor methods and quota methods, see Appendix A.1 for definitions.

For divisor methods, the threshold π_{party} was found for D'Hondt's method already by D'Hondt [11] ($\ell = 1$) and Hagenbach-Bischoff [21, 22],[11] and for Sainte-Laguë's method by Lijphart and Gibberd [30]. General divisor methods were treated by Palomares and Ramírez [32], with further results by Jones and Wilson [27], see also Pukelsheim [38, Sections 11.2–3]. In particular, [32] showed the following explicit result for linear divisor methods with divisors $d(n) = n - 1 + \gamma$, $\gamma \in [0, 1]$; recall that these include the important D'Hondt ($\mathsf{D'H}$) and Sainte-Laguë (StL) methods, with $\gamma = 1$ and $\gamma = 1/2$, respectively, as well as several less important methods. This result is also implicit in Kopfermann [29, Satz 6.2.11]. (In fact, [30; 32; 29; 38] more generally calculate the threshold $\pi_{\mathsf{party}(n)}(\ell, S)$ where the number of parties is given, see Remark 4.2.)

Theorem 4.3 (Palomares and Ramírez [32]; Kopfermann [29])**.** *Let* $\mathsf{Div}(\gamma)$ *be the divisor method with divisors* $d(n) = n - 1 + \gamma$ *for some* $\gamma \in [0, 1]$*. Then, for* $1 \leqslant \ell \leqslant S$*,*

$$\pi_{\mathsf{party}}^{\mathsf{Div}(\gamma)}(\ell, S) = \frac{\ell - 1 + \gamma}{\ell - 1 + \gamma(S + 2 - \ell)}. \tag{4.1}$$

In particular, for any $\gamma \in [0, 1]$*,*

$$\pi_{\mathsf{party}}^{\mathsf{Div}(\gamma)}(1, S) = \frac{1}{S + 1}. \tag{4.2}$$

As special cases [11; 21; 22; 30], we have, for $1 \leqslant \ell \leqslant S$*,*

$$\pi_{\mathsf{party}}^{\mathsf{D'H}}(\ell, S) = \frac{\ell}{S + 1}, \tag{4.3}$$

$$\pi_{\mathsf{party}}^{\mathsf{StL}}(\ell, S) = \frac{2\ell - 1}{S + \ell}. \tag{4.4}$$

Some numerical values of $\pi_{\mathsf{party}}^{\mathsf{D'H}}(\ell, S)$ and $\pi_{\mathsf{party}}^{\mathsf{StL}}(\ell, S)$ are given in Tables 2 and 3 in Appendix B.

Proof. $\pi_{\mathsf{party}}(\ell, S)$ equals $S_{\ell-1}$ in [32]. □

Example 4.4. Another special case is Adams's method Adams, which is the case $\gamma = 0$, i.e., $d(n) = n - 1$. In this case, for $\ell \geqslant 2$, (4.1) simply yields

$$\pi_{\mathsf{party}}^{\mathsf{Adams}}(\ell, S) = 1, \qquad 2 \leqslant \ell \leqslant S. \tag{4.5}$$

For $\ell = 1$, however, (4.1) yields the indeterminate $0/0$. The reason is that Adams's method gives every party at least one seat. Hence, if the number of parties is at most the number of seats, then the threshold $\pi(1, S) = 0$ (and (4.5) is obvious with S parties), but if there are more parties, then the method is ill-defined; for example, if the seats are distributed by lot then

[11] Hagenbach-Bischoff [21, 22] used this threshold to reduce the calculations needed for D'Hondt's method.

$\pi^{\mathsf{Adams}}_{\mathsf{party}}(1,S)=1$, but if they are distributed to the S largest parties, then $\pi^{\mathsf{Adams}}_{\mathsf{party}}(1,S)=1/(S+1)$ (in accordance with (4.2)).[12]

Remark 4.5. Lijphart and Gibberd [30] consider also the modified Sainte-Laguë method, with $d(1)=0.7$ and $d(n)=n-0.5$ for $n\geqslant 1$. (Or equivalently, and traditionally, $d(1)=1.4$ and $d(n)=2n-1$ for $n\geqslant 1$.) Then they find

$$\pi_{\mathsf{party}}(\ell,S)=\begin{cases}\frac{1}{S+1}, & \ell=1,\\ \frac{2\ell-1}{1.4S+0.6\ell+0.4}, & 2\leqslant\ell\leqslant S.\end{cases}\tag{4.6}$$

For the modification with $d(1)=0.6$ instead (or, equivalently, the sequence of divisors $1.2,3,5,\dots$), used in Sweden since 2018, similar calculations yield

$$\pi_{\mathsf{party}}(\ell,S)=\begin{cases}\frac{1}{S+1}, & \ell=1,\\ \frac{2\ell-1}{1.2S+0.8\ell+0.2}, & 2\leqslant\ell\leqslant S.\end{cases}\tag{4.7}$$

See [30] and [38, Section 11.4] for the version $\pi_{\mathsf{party}(n)}$ where the number of parties is given.

We next compute π_{party} for quota methods, including the method of Largest Remainder and Droop's method, at least assuming that the quota is defined without rounding. For the method of Largest Remainder, this was done by Lijphart and Gibberd [30]. The general case follows from Pukelsheim [38, Section 11.5] (by taking, in his notation, $\ell=h-x_j+1$, where x_j is our $\ell-1$) and is implicit in Kopfermann [29, Satz 6.2.3], see also [24, Sats 8.8]. (Again, [30; 29; 38] more generally treat the threshold $\pi_{\mathsf{party}(n)}(\ell,S)$ where the number of parties is given, see Remark 4.2.)

The general formula is a little involved, and for convenience we give three equivalent versions of it.

Theorem 4.6 (Pukelsheim [38]; Kopfermann [29])**.** *Let $\mathsf{Q}(\delta)$ be the quota method with quota $V/(S+\delta)$, where $\delta\in[0,1]$. Then, for $1\leqslant\ell\leqslant S$,*

$$\begin{aligned}\pi^{\mathsf{Q}(\delta)}_{\mathsf{party}}(\ell,S)&=\frac{\ell(S+2-\ell)-1+\delta}{(S+\delta)(S+2-\ell)}=\frac{\ell}{S+\delta}-\frac{1-\delta}{(S+\delta)(S+2-\ell)}\\&=\frac{\ell}{S+1}+\frac{(1-\delta)(\ell-1)(S+1-\ell)}{(S+\delta)(S+1)(S+2-\ell)}.\end{aligned}\tag{4.8}$$

In particular,

$$\pi^{\mathsf{Q}(\delta)}_{\mathsf{party}}(1,S)=\frac{1}{S+1}.\tag{4.9}$$

As special cases, for $\delta=0$ [30] and $\delta=1$, respectively,

$$\pi^{\mathsf{LR}}_{\mathsf{party}}(\ell,S)=\frac{\ell(S+2-\ell)-1}{S(S+2-\ell)}=\frac{\ell}{S}-\frac{1}{S(S+2-\ell)},\tag{4.10}$$

[12] The same applies to any method guaranteeing every party a seat, for example *Huntington–Hill's method* used for allocation in the US House of Representatives.

$$\pi_{\mathsf{party}}^{\mathsf{Droop}}(\ell, S) = \frac{\ell}{S+1}. \tag{4.11}$$

Some numerical values of $\pi_{\mathsf{party}}^{\mathsf{Droop}}(\ell, S)$ and $\pi_{\mathsf{party}}^{\mathsf{LR}}(\ell, S)$ are given in Tables 2 and 4 in Appendix B.

Proof. Suppose that party i gets v_i votes and s_i seats, with $\sum_i v_i = V$ and $\sum_i s_i = S$. Let $x_i := v_i/Q = \frac{v_i}{V}(S+\delta)$. Thus

$$\sum_i x_i = \frac{\sum_i v_i}{V}(S+\delta) = S+\delta. \tag{4.12}$$

The quota method means that for some $t \in [0,1]$, see (A.2),

$$s_i + t - 1 \leqslant x_i \leqslant s_i + t. \tag{4.13}$$

In particular, for a party i with $s_i > 0$,

$$x_i \geqslant s_i + t - 1 \geqslant ts_i, \tag{4.14}$$

and trivially $x_i \geqslant ts_i$ also if $s_i = 0$. Hence, using also (4.12) and $x_1 - s_1 \leqslant t$ from (4.13),

$$S + \delta = x_1 + \sum_{i\neq 1} x_i \geqslant x_1 + \sum_{i\neq 1} ts_i = x_1 + t(S - s_1) \tag{4.15}$$

$$\geqslant x_1 + (x_1 - s_1)(S - s_1) = x_1(S + 1 - s_1) - s_1(S - s_1). \tag{4.16}$$

Consequently,

$$x_1 \leqslant \frac{S + \delta + s_1(S - s_1)}{S - s_1 + 1} = s_1 + 1 - \frac{1-\delta}{S - s_1 + 1}. \tag{4.17}$$

Furthermore, we may have equality in (4.17), for any given S and $s_1 \leqslant S$, by taking

$$t = x_1 - s_1 = \frac{S - s_1 + \delta}{S - s_1 + 1} \in [0,1] \tag{4.18}$$

and $S - s_1$ other parties with $x_i = t$ and $s_i = 1$; note that this satisfies (4.12) and (4.13). We may then take $v_i := Nx_i$ for a suitable integer N in order to get integer numbers of votes. (Provided δ is rational; otherwise we take N large and approximate by rounding v_i suitably.)

Taking $s_1 = \ell - 1$, and noting that the bound in (4.17) is increasing in s_1, we see that if (4.17) does not hold, then party 1 has to get at least ℓ seats, and that this is best possible. Since the proportion of votes is $v_1/V = x_1/(S+\delta)$, cf. (4.12), it thus follows from (4.17) that

$$\pi_{\mathsf{party}}^{\mathsf{Q}(\delta)}(\ell, S) = \frac{1}{S+\delta}\Bigl(\ell - \frac{1-\delta}{S - \ell + 2}\Bigr). \tag{4.19}$$

This is equivalent to the three different forms in (4.8) by elementary algebraic manipulations. □

Remark 4.7. If the quota is defined by rounding $V/(S+\delta)$ up or down to an integer, the result still holds for $\overline{\pi}_{\mathsf{party}}$, see (2.2).

Remark 4.8. The assumption $\delta \in [0,1]$ means that the standard definition of the quota method works. The results extends (with the same proof) to any real $\delta \in (-S, 1)$, i.e. quota $Q \geqslant V/(S+1)$, with the interpretation in Appendix A.1.4; note that then still $t \leqslant 1$ in the proof. We have not investigated the case $\delta > 1$, e.g. the Imperiali quota $V/(S+2)$.

Remark 4.9. Note that for $\ell = 1$, all linear divisor methods and all quotient methods in Theorems 4.3 and 4.6 yield equality in (3.2).

Remark 4.10. Comparing (4.3) and (4.4), we see that

$$\pi^{\mathsf{D'H}}_{\mathsf{party}}(\ell, S) \leqslant \pi^{\mathsf{StL}}_{\mathsf{party}}(\ell, S), \qquad 1 \leqslant \ell \leqslant S, \tag{4.20}$$

with strict inequality for $\ell > 1$. This can be interpreted as meaning that D'Hondt's method has better proportionality than Sainte-Laguë, in a certain sense. On the other hand, it is well-known that Sainte-Laguë's method is (asymptotically) unbiased, in an average sense, while D'Hondt's method has a bias in favour of larger parties, see e.g. [25], so in this sense Sainte-Laguë's method is more proportional than D'Hondt. (The same results hold when comparing the method of Largest Remainder and Droop's method, see again e.g. [25].)

This just stresses that there is no single criterion that defines "proportionality".

4.1. **Unordered and ordered ballots.** We have here considered election methods with party ballots. Also in an election using unordered or ordered ballots, it may happen that the voters are so polarised that they do not use their freedom to mix candidates; instead they all belong to different parties (organised or not), with all voters in each party voting for the same party list. (These naturally being disjoint.) The election then really becomes an election with party ballots. We formally define, for use in later sections, a scenario for this special (but not unrealistic) case, using the same notation as in Definition 4.1.

Definition 4.11 (Party lists for unordered or ordered ballots, π_{party})**.** *Suppose that all voters vote for disjoint party lists, and that all voters in $\mathcal{W}$ vote for the same list σ, containing at least ℓ candidates. A good outcome is when at least ℓ candidates from σ are elected.*

5. Unordered ballots: Block Vote, SNTV, Limited Vote, . . .

In this section we begin the study of election methods using unordered ballots by studying some simple, classical election methods, viz. Block Vote, Approval Vote, Single Non-Transferable Vote (SNTV), Limited Vote, Cumulative Vote. The results below for them are not new; on the contrary, they were known already in the 19th century.

We begin by defining scenarios for elections with unordered ballots. For later use, we note that these definitions apply also to elections with ordered ballots. One interesting scenario that we consider is the party list case in

Definition 4.11. Another interesting scenario is the less restrictive assumption that the voters in $\mathcal{W}$ vote on a common list of candidates, while the other voters may be less organized and we do not assume anyting about their votes.

Definition 5.1 (Ordered or unordered ballots, π_{same}). *Suppose that all voters in $\mathcal{W}$ vote for the same list of candidates $\mathcal{A}$. (In the ordered case, all vote on the same ordered list.) The other voters may vote arbitrarily. We assume $|\mathcal{A}| \geqslant \ell$. A good outcome is when at least ℓ candidates from $\mathcal{A}$ are elected.*

It is obvious that for some election methods (e.g. SNTV), it is sometimes a bad strategy for a group of voters to vote on the same list, and that a party in order to be successful is forced to use more elaborate strategies. We therefore consider also a scenario where we assume that the set of voters $\mathcal{W}$ is organized in some sense, so that the voters in $\mathcal{W}$ vote as directed by a "leadership" or "party organization"; thus making tactical voting feasible. We also assume that the leadership is intelligent and uses the mathematically optimal strategy.[13]

Definition 5.2 (Ordered or unordered ballots, π_{tactic}). *Let a set $\mathcal{A}$ of candidates be given, with $|\mathcal{A}| \geqslant \ell$, and suppose that all voters in $\mathcal{W}$ vote as instructed (by some intelligent leader). A good outcome is when at least ℓ candidates from $\mathcal{A}$ are elected.*

Before considering specific election methods, we note two simple relations.

Theorem 5.3. *For any election method with unordered or ordered ballots, and $1 \leqslant \ell \leqslant S$,*

$$\pi_{\mathsf{party}}(\ell, S) \leqslant \pi_{\mathsf{same}}(\ell, S), \tag{5.1}$$

$$\pi_{\mathsf{tactic}}(\ell, S) \leqslant \pi_{\mathsf{same}}(\ell, S). \tag{5.2}$$

Proof. Each instance of the scenario party (Definition 4.11) is also an instance of the scenario same (Definition 5.1), and the definition of good outcome is the same. Hence, a bad outcome for party is also a bad outcome for same, and thus (2.1) implies (5.1), since the supremum is taken over a larger set for π_{same}.

Next, since π_{tactic} assumes that $\mathcal{W}$ uses an optimal strategy, a bad outcome for π_{tactic} with a given set $\mathcal{W}$ means that $\mathcal{W}$ has no strategy that guarantees a good outcome. In particular, the strategy that all voters in $\mathcal{W}$ vote for the same list $\mathcal{A}$ may fail, so there is a bad outcome for π_{same} too for this $\mathcal{W}$. Hence, (2.1) yields (5.2). □

[13] To find the optimal strategy might be a more or less difficult and perhaps interesting problem in game theory. We assume that the size $|\mathcal{W}|$ of $\mathcal{W}$ as well as the total number of voters V are known; otherwise further game theoretical complications arise. Note also that since the definition (2.1) consider the worst cases, we may assume that the strategy is known to an omniscient adversary.

Consider first the simple and important *Block Vote* method BV (*multi-member plurality*; *plurality-at-large*); recall that here every voter votes for (at most) S candidates. In this case, the simple strategy to vote on the same list is the best strategy. As is well-known, this method gives a majority all seats, so the next result is rather obvious.

Theorem 5.4. *For Block Vote:*

$$\pi^{\mathsf{BV}}_{\mathsf{party}}(\ell,S)=\pi^{\mathsf{BV}}_{\mathsf{same}}(\ell,S)=\pi^{\mathsf{BV}}_{\mathsf{tactic}}(\ell,S)=\frac{1}{2},\qquad 1\leqslant\ell\leqslant S. \tag{5.3}$$

Proof. If $|\mathcal{W}|>\frac12 V$, so $|\mathcal{W}|>|\mathcal{W}^{\mathsf{c}}|$, then $\mathcal{W}$ will get all S seats by voting on the same list. Conversely, if $|\mathcal{W}|<\frac12 V$, then $\mathcal{W}$ will not get any seat, regardless of how they vote, if all other voters vote on a common list (with at least S names) disjoint from $\mathcal{A}$. This argument shows all three cases. □

Remark 5.5. For our purposes, it does not matter whether a voter has to vote for exactly S candidates, or whether it only has to be at most S. This is seen from the proof above, but also because we (implicitly) assume that there is an unlimited number of potential candidates, so that a voter always has the option of throwing away some votes on dummy candidates with no hope of being elected. The same applies to Limited Vote below.

Approval Vote (AV) differs mathematically from Block Vote only in that each ballot may contain an arbitrary number of candidates. The proof above applies to this method to, giving the following result.

Theorem 5.6. *For Approval Vote:*

$$\pi^{\mathsf{AV}}_{\mathsf{party}}(\ell,S)=\pi^{\mathsf{AV}}_{\mathsf{same}}(\ell,S)=\pi^{\mathsf{AV}}_{\mathsf{tactic}}(\ell,S)=\frac{1}{2},\qquad 1\leqslant\ell\leqslant S. \tag{5.4}$$

Another related simple election method is *Single Non-Transferable Vote* (SNTV), where each voter only votes for one candidate. In this case, if the objective is to get more than one candidate elected, it is obviously a stupid strategy to let all voters in $\mathcal{W}$ vote in the same way, so more elaborate strategies are required, and π_{party} and π_{same} are not relevant. The following result was essentially shown by Droop [13, p. 174] (and may have been known earlier); it is a special case of Theorem 5.10 below, shown by Dodgson [12].

Theorem 5.7 (Droop [13])**.** *For Single Non-Transferable Vote:*

$$\overline{\pi}^{\mathsf{SNTV}}_{\mathsf{tactic}}(\ell,S)=\frac{\ell}{S+1},\qquad 1\leqslant\ell\leqslant S. \tag{5.5}$$

Proof. Let $\mathcal{W}$ use the strategy to divide its votes equally between ℓ candidates $A_1,\dots,A_\ell$. Of course, an exactly equal split is possible only if $|\mathcal{W}|$ is divisible by ℓ, but $\mathcal{W}$ can always assure that each candidate gets al least $\lfloor|\mathcal{W}|/\ell\rfloor>|\mathcal{W}|/\ell-1$ votes.

If the outcome is bad, then not all these ℓ candidates get elected; say that A_ℓ does not. Then, at least $S+1-\ell$ other candidates are elected. Each of these must have at least as many votes as the unsuccessful A_ℓ, and thus $> |\mathcal{W}|/\ell - 1$ votes. Hence, there are (at least) $S+1$ candidates with $> |\mathcal{W}|/\ell - 1$ votes, and consequently,

$$V > (S+1)\Bigl(\frac{|\mathcal{W}|}{\ell} - 1\Bigr) \tag{5.6}$$

or

$$|\mathcal{W}| < \frac{\ell}{S+1}V + \ell \tag{5.7}$$

and

$$\frac{|\mathcal{W}|}{V} < \frac{\ell}{S+1} + \frac{\ell}{V}. \tag{5.8}$$

Hence, the definition (2.2) yields $\overline{\pi}^{\mathsf{SNTV}}_{\mathsf{tactic}}(\ell, S) \leqslant \ell/(S+1)$.

Conversely, let V be divisible by $S+1$ and write $w := V/(S+1)$. If $|\mathcal{W}| = \frac{\ell}{S+1}V = \ell w$, then $|\mathcal{W}^{\mathsf{c}}| = V - |\mathcal{W}| = V - \ell w = (S+1-\ell)w$. Hence, $\mathcal{W}^{\mathsf{c}}$ may split their votes on $S+1-\ell$ candidates $B_1, \dots, B_{S+1-\ell}$ with w votes each. On the other hand, for any strategy used by $\mathcal{W}$, at least one of $A_1, \dots, A_\ell$ gets at most w votes. Hence a bad outcome is possible, and (2.2) yields $\overline{\pi}^{\mathsf{SNTV}}_{\mathsf{tactic}}(\ell, S) \geqslant |\mathcal{W}|/V = \ell/(S+1)$. □

In Theorem 5.7, we use $\overline{\pi}$ and not π, i.e., we consider the limit as $V \to \infty$. The reason is that the strategy used in the proof above, of splitting the votes equally between some candidates, in general can be followed only approximately due to divisibility issues. The following example shows that this really matters when V is small.

Example 5.8. Let $\ell = 2$, $S = 3$, $|\mathcal{W}| = 3$ and $V = 5$. Suppose that $\mathcal{W}$ wants A_1 and A_2 to be elected. Regardless of the strategy used by $\mathcal{W}$, either A_1 or A_2 will get only one vote, say A_2. Hence, if the two voters in $\mathcal{W}^{\mathsf{c}}$ vote with one vote each for B_1 and B_2, then there is a tie and it is possible that B_1 and B_2 are elected together with A_1, a bad outcome. Consequently,

$$\pi^{\mathsf{SNTV}}_{\mathsf{tactic}}(2,3) \geqslant \frac{3}{5} > \frac{1}{2} = \overline{\pi}^{\mathsf{SNTV}}_{\mathsf{tactic}}(2,3). \tag{5.9}$$

Remark 5.9. Theorem 5.7 shows that SNTV under ideal conditions can be regarded as a proportional method. However, as is well-known, there are obvious practical problems with using this strategy in, say, a general election. Moreover, the strategy assumes that the size of $\mathcal{W}$ and of its opponents are known; misjudgement can lead to disastrous results. The same applies to Limited Vote and Cumulative Vote below. (These and other problems were discussed already by Droop [13].)

More generally, *Limited Vote* is a version of Block Vote, where each ballot contains only (at most) L candidates, for some fixed $L \leqslant S$; we use the notation $\mathsf{LV}(L)$. Thus, Block Vote is the special case $L = S$, and SNTV is the special case $L = 1$.

Theorems 5.4 and 5.7 generalize to Limited Vote as follows. Again we consider $\overline{\pi}$ for the same reason as for SNTV. This result was proved by

Charles Dodgson[14] [12], who also gave a table of numerical values for all cases with $1 \leqslant S \leqslant 6$. (The result may have been known earlier; Droop [13] mentions a few values in his discussions, so it is perhaps likely that he knew the general formula.)

Theorem 5.10 (Dodgson [12])**.** *For Limited Vote: Let $1 \leqslant L \leqslant S$. Then, for $1 \leqslant \ell \leqslant S$,*

$$\overline{\pi}^{\mathsf{LV}(L)}_{\mathsf{tactic}}(\ell, S) = \frac{\min\bigl\{1, L/(S+1-\ell)\bigr\}}{\min\bigl\{1, L/(S+1-\ell)\bigr\} + \min\bigl\{1, L/\ell\bigr\}} \tag{5.10}$$

$$= \frac{\ell \min\bigl\{L, S+1-\ell\bigr\}}{\ell \min\bigl\{L, S+1-\ell\bigr\} + (S+1-\ell)\min\bigl\{L, \ell\bigr\}}. \tag{5.11}$$

Hence, if $1 \leqslant L \leqslant (S+1)/2$, then

$$\overline{\pi}^{\mathsf{LV}(L)}_{\mathsf{tactic}}(\ell, S) = \begin{cases} \frac{L}{S+1+L-\ell}, & 1 \leqslant \ell \leqslant L, \\ \frac{\ell}{S+1}, & L \leqslant \ell \leqslant S+1-L, \\ \frac{\ell}{\ell+L}, & S+1-L \leqslant \ell \leqslant S, \end{cases} \tag{5.12}$$

and if $(S+1)/2 \leqslant L \leqslant S$, then

$$\overline{\pi}^{\mathsf{LV}(L)}_{\mathsf{tactic}}(\ell, S) = \begin{cases} \frac{L}{S+1+L-\ell}, & 1 \leqslant \ell \leqslant S+1-L, \\ \frac{1}{2}, & S+1-L \leqslant \ell \leqslant L, \\ \frac{\ell}{L+\ell}, & L \leqslant \ell \leqslant S, \end{cases} \tag{5.13}$$

Proof. As for SNTV, the strategy of $\mathcal{W}$ is to divide its votes equally between its ℓ favoured candidates $A_1, \dots, A_\ell$. If $\ell \leqslant L$, then all voters in $\mathcal{W}$ vote for the same list $\{A_1, \dots, A_\ell\}$. (If required to vote for exactly $L > \ell$ candidates, they further vote for $L - \ell$ dummy candidates each; these have to come from at least two different sets for different voters. See Remark 5.5.)

On the other hand, if $\ell > L$, then the votes are split so that each of the ℓ candidates gets $\mathcal{W}L/\ell + O(1)$ votes. (E.g. by splitting the votes as equally as possible between all L-subsets of $\{A_1, \dots, A_\ell\}$, or, more practically, between the ℓ sets $\{A_i, \dots, A_{i+L-1}\}$, $1 \leqslant i \leqslant \ell$, with indices taken modulo ℓ.)

This strategy thus gives each candidate A_i at least

$$v_A := \min\Bigl(1, \frac{L}{\ell}\Bigr)|\mathcal{W}| + O(1) \tag{5.14}$$

votes. Conversely, for any strategy, at least one of $A_1, \dots, A_\ell$ gets at most v_A votes. (Assuming that they get no votes from $\mathcal{W}^{\mathsf{c}}$.)

Similarly, $\mathcal{W}^{\mathsf{c}}$ can give each of $S - \ell + 1$ candidates $B_1, \dots, B_{S-\ell+1}$ at least

$$v_B := \min\Bigl(1, \frac{L}{S+1-\ell}\Bigr)(V - |\mathcal{W}|) + O(1) \tag{5.15}$$

votes, but they cannot give all $S - \ell + 1$ candidates more than v_B votes.

[14]Better known by his pseudonym Lewis Carroll used when writing fiction.

It follows that the outcome is good if $v_A > v_B$, but bad, for any strategy of $\mathcal{W}$, if $v_A < v_B$. It follows that $\overline{\pi}$ satisfies the equation

$$\overline{\pi}\min\Bigl(1,\frac{L}{\ell}\Bigr) = \bigl(1-\overline{\pi}\bigr)\min\Bigl(1,\frac{L}{S+1-\ell}\Bigr). \tag{5.16}$$

This gives (5.10), and (5.11)–(5.12) follow. □

For $\pi^{\mathsf{LV}(L)}_{\mathsf{same}}$, only $\ell \leqslant L$ are relevant. In this case, the strategy in the proof of Theorem 5.10 is to vote on the same list, and the proof (without $O(1)$) shows also the following.

Theorem 5.11. *For Limited Vote: If* $1 \leqslant \ell \leqslant L \leqslant S$, *then*

$$\pi^{\mathsf{LV}(L)}_{\mathsf{same}}(\ell,S) = \pi^{\mathsf{LV}(L)}_{\mathsf{tactic}}(\ell,S) = \overline{\pi}^{\mathsf{LV}(L)}_{\mathsf{tactic}}(\ell,S), \tag{5.17}$$

given in (5.10)*–*(5.13). □

Example 5.12. For $\ell = 1$, Theorems 5.10 and 5.11 yield

$$\pi^{\mathsf{LV}(L)}_{\mathsf{same}}(1,S) = \pi^{\mathsf{LV}(L)}_{\mathsf{tactic}}(1,S) = \frac{L}{S+L}. \tag{5.18}$$

Furthermore, taking $L = 1$, the proof above applies also to $\pi_{\mathsf{party}}(1,S)$; thus,

$$\pi^{\mathsf{SNTV}}_{\mathsf{party}}(1,S) = \pi^{\mathsf{SNTV}}_{\mathsf{same}}(1,S) = \pi^{\mathsf{SNTV}}_{\mathsf{tactic}}(1,S) = \frac{1}{S+1}. \tag{5.19}$$

It is easy to see that the result in Theorem 5.7 above for SNTV applies also to Cumulative Vote (CV), see Appendix A.2.5, with the same proof. However, Cumulative Vote has the practical advantage that the strategy of splitting the vote equally between ℓ candidates may be easier to organize; in particular, if the version of Cumulative Vote used allows each voter to split the vote equally between ℓ candidates, then all voters in $\mathcal{W}$ can vote the same way. Hence we obtain the following result, again essentially shown by Droop [13, p. 174].

Theorem 5.13 (Droop [13])**.** *For Cumulative Vote:*

$$\overline{\pi}^{\mathsf{CV}}_{\mathsf{tactic}}(\ell,S) = \overline{\pi}^{\mathsf{SNTV}}_{\mathsf{tactic}}(\ell,S) = \frac{\ell}{S+1}, \qquad 1 \leqslant \ell \leqslant S. \tag{5.20}$$

Furthermore, in an ideal case where a voter may split the vote equally between ℓ *candidates,*

$$\pi^{\mathsf{CV}}_{\mathsf{tactic}}(\ell,S) = \frac{\ell}{S+1}. \tag{5.21}$$

The scenarios party and same are not very interesting for Cumulative Vote, since they allow the voters in $\mathcal{W}$ to spread their votes too thinly over too many candidates. This is seen formally in the following theorem, where we consider the ideal Equal and Even version $\mathsf{CV}^{=}$, allowing a vote to be split (equally) on an arbitrary number of candidates.[15]

[15] Also called *Satisfaction Approval Voting (SAV)* [7].

Theorem 5.14. *For (ideal) Equal and Even Cumulative Vote:*

$$\pi^{\mathsf{CV}=}_{\mathsf{party}}(\ell,S)=\pi^{\mathsf{CV}=}_{\mathsf{same}}(\ell,S)=1,\qquad 1\leqslant\ell\leqslant S. \tag{5.22}$$

Proof. Let $\mathcal{V}:=\mathcal{W}\cup\mathcal{U}$, where $W:=|\mathcal{W}|\geqslant\ell$ and $\mathcal{U}$ is disjoint from $\mathcal{W}$ with $|\mathcal{U}|=S$. Furthermore, let each voter be a candidate, let each voter in $\mathcal{W}$ vote for $\mathcal{W}$, and let each voter in $\mathcal{W}^{\mathsf{c}}=\mathcal{U}$ vote only for herself. This is an instance of both party and same. Furthermore, there is a tie between all candidates, and it is possible that the outcome is $\mathcal{E}=\mathcal{U}$, which is a bad outcome for both party and same. Hence, $\pi^{\mathsf{CV}=}_{\mathsf{party}}(\ell,S)\geqslant(W+S)/W$, for any $W\geqslant\ell$, and thus $\pi^{\mathsf{CV}=}_{\mathsf{party}}(\ell,S)=1$, and similarly, or by (5.1), $\pi^{\mathsf{CV}=}_{\mathsf{same}}(\ell,S)=1$. □

Remark 5.15. We might define a scenario $\mathsf{same}_=$ to be as same, but with the further restriction that $|\mathcal{A}|=\ell$. Then we would have $\pi^{\mathsf{CV}=}_{\mathsf{same}_=}(\ell,S)=l/(S+1)$. We leave this to the reader to explore further.

Problem 5.16. Another possibility is to consider the version of Cumulative Vote where each voter may split her vote equally on at most S candidates. What is $\pi^{\mathsf{CV}}_{\mathsf{same}}(\ell,S)$ for this version?

Finally we note that the strategy for SNTV in Theorem 5.7, i.e., to split $\mathcal{W}$ into subsets voting for one candidate each, can be used (more or less successfully) for any election method, This yields the following general results, which will be used later. (Note that the results include a rigorous version of "Theorem" 3.4 for π_{tactic}.)

Theorem 5.17. *Let $\mathfrak{M}$ be any election method for ordered or unordered ballots.*

(i) *For $1\leqslant\ell\leqslant S$,*

$$\overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(\ell,S)\leqslant\ell\overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(1,S)\leqslant\ell\pi^{\mathfrak{M}}_{\mathsf{tactic}}(1,S). \tag{5.23}$$

(ii) *If $\ell+m\leqslant S$, then*

$$\overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(\ell+m,S)\leqslant\overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(\ell,S)+\overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(m,S). \tag{5.24}$$

(iii) *If $1\leqslant\ell\leqslant S$, then*

$$\pi^{\mathfrak{M}}_{\mathsf{tactic}}(\ell,S)+\pi^{\mathfrak{M}}_{\mathsf{tactic}}(S+1-\ell,S)\geqslant\overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(\ell,S)+\overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(S+1-\ell,S)\geqslant 1. \tag{5.25}$$

(iv) *If $\overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(1,S)\leqslant 1/(S+1)$, then*

$$\overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(\ell,S)=\frac{\ell}{S+1},\qquad 1\leqslant\ell\leqslant S. \tag{5.26}$$

In particular, this holds if $\pi^{\mathfrak{M}}_{\mathsf{tactic}}(1,S)\leqslant 1/(S+1)$.

(v) *If $\pi^{\mathfrak{M}}_{\mathsf{tactic}}(\ell,S)\leqslant\ell/(S+1)$ for $1\leqslant\ell\leqslant S$, then this is an equality:*

$$\pi^{\mathfrak{M}}_{\mathsf{tactic}}(\ell,S)=\frac{\ell}{S+1},\qquad 1\leqslant\ell\leqslant S. \tag{5.27}$$

Proof. (i): The second inequality is (2.3). The first inequality follows from (ii) and induction, but we prefer to give a direct proof.

Fix $p > \overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(1,S)$. If $|\mathcal{W}| > p\ell V + \ell$, split $\mathcal{W}$, as equally as possible, into ℓ subsets $\mathcal{W}_i$ of sizes

$$|\mathcal{W}_i| \geqslant \lfloor \mathcal{W}/\ell \rfloor > \mathcal{W}/\ell - 1 > pV. \tag{5.28}$$

Let $A_1,\dots,A_\ell \in \mathcal{A}$ be ℓ desired candidates, and let $\mathcal{W}_i$ vote with the aim of electing A_i. By (5.28) and our choice of p, if V is large enough, there is a strategy for $\mathcal{W}_i$ that always will succeed to get A_i elected. Hence, if each $\mathcal{W}_i$ uses such a strategy, then $A_1,\dots,A_\ell$ are all elected, a good outcome for $\mathcal{W}$. Consequently,

$$\overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(\ell,S) \leqslant \limsup_{V\to\infty} \frac{p\ell V + \ell}{V} = p\ell, \tag{5.29}$$

and (5.23) follows because p is arbitrary with $p > \overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(1,S)$.

(ii),(iii): For (iii), let $m := S + 1 - \ell$. Note that the first inequality in (5.25) follows from (2.3).

Let $p := \overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(\ell,S) + \overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(m,S) + 2\varepsilon$ with $\varepsilon > 0$. If $|\mathcal{W}| > pV$, split $\mathcal{W}$ into two sets $\mathcal{W}_1$ and $\mathcal{W}_2$ with $|\mathcal{W}_1| > \bigl(\overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(\ell,S) + \varepsilon\bigr)V - 1$ and $|\mathcal{W}_2| > \bigl(\overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(m,S) + \varepsilon\bigr)V - 1$. Then, if V is large enough, given $\ell + m$ candidates $A_1,\dots,A_{\ell+m}$, there exists a strategy for $\mathcal{W}_1$ to get $A_1,\dots,A_\ell$ elected, and a strategy for $\mathcal{W}_2$ to get $A_{\ell+1},\dots,A_{\ell+m}$ elected. Combining these, we have a strategy that guarantees that $A_1,\dots,A_{\ell+m}$ are elected.

In (iii), with $\ell + m = S + 1$, this is impossible. Thus no set $\mathcal{W} \subseteq \mathcal{V}$ with $|\mathcal{W}| > pV$ can exist, and thus $p \geqslant 1$, which yields (5.25) since ε is arbitrary.

In (ii), we have shown that, for large V, we have a good outcome (with $\ell + m$ elected) whenever $|\mathcal{W}| > pV$, and thus $\overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(\ell + m, S) \leqslant p$. This yields (5.24), since ε is arbitrary.

(iv): Part (i) yields the inequality

$$\overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(\ell,S) \leqslant \frac{\ell}{S+1}. \tag{5.30}$$

Conversely, by (5.25) and (5.30) with ℓ replaced by $S + 1 - \ell$,

$$\overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(\ell,S) \geqslant 1 - \overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(S+1-\ell,S) \geqslant 1 - \frac{S+1-\ell}{S+1} = \frac{\ell}{S+1}. \tag{5.31}$$

The final sentence follows by (2.3).

(v): By the assumption, $\pi^{\mathfrak{M}}_{\mathsf{tactic}}(1,S) \leqslant 1/(S+1)$, and thus (iv) applies and yields (5.26). Hence, using the assumption again and (2.3),

$$\frac{\ell}{S+1} = \overline{\pi}^{\mathfrak{M}}_{\mathsf{tactic}}(\ell,S) \leqslant \pi^{\mathfrak{M}}_{\mathsf{tactic}}(\ell,S) \leqslant \frac{\ell}{S+1}, \tag{5.32}$$

showing (5.27). ∎

6. JR, PJR, EJR

We turn to properties and thresholds intended for situations without organised parties, where a group of voters have similar opinions but do not necessarily vote identically.

For election methods with unordered ballots, Aziz et al. [1] defined two properties *JR (justified representation)* and (stronger) *EJR (extended justified representation)*; Sánchez-Fernández et al. [42, 43] then defined a related property *PJR (proportional justified representation)* such that EJR $\Longrightarrow$ PJR $\Longrightarrow$ JR. Inspired by their definitions, we define more generally the corresponding thresholds. Recall that $\mathcal{E}$ denotes the set of elected candidates.

Definition 6.1 (unordered ballots, π_{PJR})**.** *Let $\mathcal{A}$ be a set of at least ℓ candidates, and assume that every voter in $\mathcal{W}$ votes for a set $\sigma \supseteq \mathcal{A}$, i.e., for all candidates in $\mathcal{A}$ and possibly also for some others. A good outcome is when at least ℓ candidates are elected that someone in $\mathcal{W}$ has voted for, i.e. $\left|\mathcal{E} \cap \bigcup_{\sigma\in\mathcal{W}} \sigma\right| \geqslant \ell$.*

Definition 6.2 (unordered ballots, π_{EJR})**.** *Let $\mathcal{A}$ be a set of at least ℓ candidates, and assume that every voter in $\mathcal{W}$ votes for a set $\sigma \supseteq \mathcal{A}$, i.e., for all candidates in $\mathcal{A}$ and possibly also for some others. A good outcome is when there exists a voter in $\mathcal{W}$ that has voted for at least ℓ candidates that are elected, i.e. $\left|\mathcal{E} \cap \sigma\right| \geqslant \ell$ for some $\sigma \in \mathcal{W}$.*

Note that the difference between PJR and EJR disappears for $\ell = 1$:

$$\pi_{\mathsf{EJR}}(1, S) = \pi_{\mathsf{PJR}}(1, S). \tag{6.1}$$

For larger ℓ, the scenarios EJR and PJR have the same instances; the only difference is the definition of a good outcome. This leads to the following inequality.

Theorem 6.3. *For any election method with unordered ballots:*

$$\pi_{\mathsf{PJR}}(\ell, S) \leqslant \pi_{\mathsf{EJR}}(\ell, S), \qquad 1 \leqslant \ell \leqslant S, \tag{6.2}$$

with equality when $\ell = 1$.

Proof. A good outcome for the scenario EJR is a good outcome for PJR too. Hence, an instance with a bad outcome for PJR is a bad outcome for EJR too, and thus the supremum in (2.1) is taken over a larger set of instances for π_{EJR}; hence (6.2) follows. The case $\ell = 1$ is (6.1). □

We also have simple relations with the thresholds defined in earlier for scenarios with more organized voters.

Theorem 6.4. *For any election method with unordered ballots:*

$$\pi_{\mathsf{party}}(\ell, S) \leqslant \pi_{\mathsf{same}}(\ell, S) \leqslant \pi_{\mathsf{PJR}}(\ell, S) \leqslant \pi_{\mathsf{EJR}}(\ell, S), \qquad 1 \leqslant \ell \leqslant S. \tag{6.3}$$

Proof. An instance of the scenario same is also an instance of the scenario PJR, and the outcome is good for one scenario if it is for the other. Hence, an instance with a bad outcome for π_{same} is also a bad outcome for π_{PJR}, and thus (2.1) implies $\pi_{\mathsf{same}}(\ell, S) \leqslant \pi_{\mathsf{PJR}}(\ell, S)$. The other inequalities in (6.3) are repeated from Theorems 5.3 and 6.3. □

Remark 6.5. The conditions EJR and PJR defined in [1] and [42; 43] require a good outcome, in the sense of our definitions above, for any $\mathcal{W}$ with $|\mathcal{W}| \geqslant \ell V/S$, for any $\ell \leqslant S$. The condition JR [1] is the special case $\ell = 1$ of both. Consequently, for any reasonable election method, see Remark 2.8 and "Theorem" 3.9,

$$\text{PJR} \iff \pi_{\mathsf{PJR}}(\ell, S) < \frac{\ell}{S}, \qquad 1 \leqslant \ell < S, \tag{6.4}$$

$$\text{EJR} \iff \pi_{\mathsf{EJR}}(\ell, S) < \frac{\ell}{S}, \qquad 1 \leqslant \ell < S, \tag{6.5}$$

$$\text{JR} \iff \pi_{\mathsf{PJR}}(1, S) < \frac{1}{S}, \qquad S > 1. \tag{6.6}$$

Let us first consider the classical election method discussed in Section 5. It is obvious that the thresholds above are relevant for SNTV only when $\ell = 1$. In this case we have the following. (In particular, SNTV satisfies JR by (6.6).)

Theorem 6.6. *For Single Non-Transferable Vote:*

$$\pi^{\mathsf{SNTV}}_{\mathsf{PJR}}(1, S) = \pi^{\mathsf{SNTV}}_{\mathsf{EJR}}(1, S) = \pi^{\mathsf{SNTV}}_{\mathsf{same}}(1, S) = \frac{1}{S+1}, \qquad S \geqslant 1. \tag{6.7}$$

Proof. For SNTV and $\ell = 1$, the scenarios PJR and EJR require that all voters in $\mathcal{W}$ vote on the same candidate A, and a good outcome is when A is elected. Thus, $\pi^{\mathsf{SNTV}}_{\mathsf{PJR}}(1, S) = \pi^{\mathsf{SNTV}}_{\mathsf{EJR}}(1, S) = \pi^{\mathsf{SNTV}}_{\mathsf{same}}(1, S)$, and the result follows from (5.19). □

For BV and AV, the results in Theorems 5.4 and 5.6 extend to π_{PJR}, but not to π_{EJR}. The same holds for LV(L), but in this case, similarly to SNTV above, only the case $\ell \leqslant L$ is relevant. We consider π_{PJR} first, and begin with a separate treatment of BV and AV, although this result also can be obtained as a corollary of the more complicated result for LV in Theorem 6.9.

Theorem 6.7. *For Block Vote and Approval Vote:*

$$\pi^{\mathsf{BV}}_{\mathsf{PJR}}(\ell, S) = \pi^{\mathsf{AV}}_{\mathsf{PJR}}(\ell, S) = \frac{1}{2}, \qquad 1 \leqslant \ell \leqslant S. \tag{6.8}$$

Proof. The lower bound follows by Theorems 5.4, 5.6 and 6.4.

To bound $\pi^{\mathsf{BV}}_{\mathsf{PJR}}(\ell, S)$ from above, suppose that $|\mathcal{W}| > \frac{1}{2}V$. Then the candidates in $\mathcal{A}$ have at least $|\mathcal{W}|$ votes each, while the candidates not in $\bigcup_{\sigma\in\mathcal{W}} \sigma$ have at most $|\mathcal{W}^{\mathsf{c}}| = V - |\mathcal{W}| < |\mathcal{W}|$ votes each. Hence either no candidate outside $\bigcup_{\sigma\in\mathcal{W}} \sigma$ is elected, or all candidates in $\mathcal{A}$ are; in both cases the outcome is good. Consequently, $\pi^{\mathsf{BV}}_{\mathsf{PJR}}(\ell, S) \leqslant 1/2$. The same argument applies to AV. □

Remark 6.8. Aziz et al. [1] showed that Approval Vote does not satisfy JR when $S \geqslant 3$, and that for $S = 2$, the answer depends on the tie-breaking rule. By (6.6), the negative result for $S \geqslant 3$ is a consequence of (6.8). For $S = 2$, we interpret their result using our refined notation in Remark 2.8: it is easy to see that $\pi^{\mathsf{AV}}_{\mathsf{PJR}}(1,2) = \frac{1}{2}+$ (JR does not hold) with standard random tie-breaking, but $\pi^{\mathsf{AV}}_{\mathsf{PJR}}(1,2) = \frac{1}{2}-$ (JR holds) if we use a tie-breaking rule that in the case of several candidates with exactly $V/2$ votes each, gives preference to a pair of candidates with disjoint voter support before a pair of candidates supported by the same voters.

Theorem 6.9. *For Limited Vote: If* $1 \leqslant \ell \leqslant L$, *then*

$$\pi^{\mathsf{LV}(L)}_{\mathsf{PJR}}(\ell,S) = \pi^{\mathsf{LV}(L)}_{\mathsf{same}}(\ell,S) = \pi^{\mathsf{LV}(L)}_{\mathsf{tactic}}(\ell,S) = \overline{\pi}^{\mathsf{LV}(L)}_{\mathsf{tactic}}(\ell,S), \tag{6.9}$$

given in Theorem 5.10.

Proof. Consider a bad outcome for a set of voters $\mathcal{W}$ as in Definition 6.1. Let $\mathcal{A}^* := \bigcup_{\sigma\in\mathcal{W}} \sigma \supseteq \mathcal{A}$, the set of candidates voted for by some voter in $\mathcal{W}$. Since the outcome is bad, $|\mathcal{A}^*\cap\mathcal{E}| \leqslant \ell-1$, and thus there are at least $S-\ell+1$ candidates $B_1,\dots,B_{S+1-\ell}$ not in $\mathcal{A}^*$ that are elected. For a candidate C, let $v(C)$ be her number of votes, and let $v_* := \min_j v(B_j)$. Thus $v(B_j) \geqslant v_*$ for $1 \leqslant j \leqslant S+1-\ell$. Furthermore, any candidate C with $v(C) > v_*$ is elected; in particular, there are at most $\ell-1$ such candidates in $\mathcal{A}^*$.

If we modify the election by eliminating all votes on any candidate in $\mathcal{A}^*\setminus\mathcal{A}$, then each voter in $\mathcal{W}$ votes for $\mathcal{A}$, so the new election is an instance of same. Furthermore, we still have $v(B_j) \geqslant v_*$ for $S+1-\ell$ candidates $B_j \notin \mathcal{A}$, and $v(A_i) > v_*$ for at most $\ell-1$ candidates $A_i \in \mathcal{A}$. Hence, even if there are ties, a possible outcome is that $B_1,\dots,B_{S+1-\ell}$ are elected, and thus at most $\ell-1$ from $\mathcal{A}$, a bad outcome for same.

We have shown that for every bad outcome for PJR, there is an election with the same $|\mathcal{W}|$ and V and a bad outcome for same. Hence, (2.1) implies $\pi^{\mathsf{LV}(L)}_{\mathsf{PJR}}(\ell,S) \leqslant \pi^{\mathsf{LV}(L)}_{\mathsf{same}}(\ell,S)$. Theorem 6.4 provides the opposite inequality, and thus equality holds. The proof is completed by Theorem 5.11. ☐

The results for EJR are more complicated. We state the results as three separate theorems, but prove them together.

Theorem 6.10. *For Block Vote:*

$$\pi^{\mathsf{BV}}_{\mathsf{EJR}}(\ell,S) = \begin{cases} \frac{S}{2S+1-\ell}, & 1 \leqslant \ell \leqslant (S+1)/2, \\ \frac{2S+1-2\ell}{3S+2-3\ell}, & (S+1)/2 \leqslant \ell \leqslant S. \end{cases} \tag{6.10}$$

Theorem 6.11. *For Approval Vote:*

$$\pi^{\mathsf{AV}}_{\mathsf{EJR}}(\ell,S) = \frac{S}{2S+1-\ell}, \qquad 1 \leqslant \ell \leqslant S. \tag{6.11}$$

Theorem 6.12. *For Limited Vote: Let* $1 \leqslant L \leqslant S$. *Then, for* $1 \leqslant \ell \leqslant L$,

$$\pi^{\mathsf{LV}(L)}_{\mathsf{EJR}}(\ell,S) = \begin{cases} \frac{L}{S+L+1-\ell}, & 1 \leqslant \ell \leqslant (S+1)/2, \\ \frac{S+L+1-2\ell}{2S+L+2-3\ell}, & (S+1)/2 \leqslant \ell \leqslant L. \end{cases} \tag{6.12}$$

Some numerical values of $\pi^{\mathsf{BV}}_{\mathsf{EJR}}(\ell,S)$ and $\pi^{\mathsf{AV}}_{\mathsf{EJR}}(\ell,S)$ are given in Tables 5 and 6 in Appendix B.

Proof of Theorems 6.10–6.12. Take $L := S$ for BV, and $L := \infty$ for AV. Thus we can treat BV, AV and LV(L) together.

Consider a bad outcome, and let $a := |\mathcal{W}|/V$ and $k := |\mathcal{E}\cap\mathcal{A}|$, the number of elected candidates from $\mathcal{A}$. Note that $k < \ell$, since otherwise the outcome is good, as witnessed by any $\sigma \in \mathcal{W}$. Similarly, no voter in $\mathcal{W}$ votes for more than $\ell - k - 1$ candidates in $\mathcal{E} \setminus \mathcal{A}$. Furthermore, no voter votes for more than L candidates, so a voter in $\mathcal{W}$ votes for at most $L - \ell$ candidates outside $\mathcal{A}$. Thus, let

$$m(k) := (\ell - k - 1) \wedge (L - \ell). \tag{6.13}$$

Then no voter in $\mathcal{W}$ votes for more than $m(k)$ candidates in $\mathcal{E}\setminus\mathcal{A}$. Furthermore, no voter in $\mathcal{W}^{\mathsf{c}}$ votes for more than

$$m'(k) := |\mathcal{E} \setminus \mathcal{A}| \wedge L = (S - k) \wedge L \tag{6.14}$$

candidates in $\mathcal{E} \setminus \mathcal{A}$.

Hence, if v is the total number of votes for the $S - k$ candidates in $\mathcal{E} \setminus \mathcal{A}$, then

$$v \leqslant |\mathcal{W}|m(k) + |\mathcal{W}^{\mathsf{c}}|m'(k) = m(k)aV + m'(k)(1 - a)V. \tag{6.15}$$

On the other hand, there is at least one non-elected candidate in $\mathcal{A}$; she has at least $|\mathcal{W}| = aV$ votes, and thus each elected candidate has at least aV votes, so $v \geqslant (S - k)aV$. Consequently, using (6.15),

$$(S - k)a \leqslant v/V \leqslant m(k)a + m'(k)(1 - a) \tag{6.16}$$

and thus

$$\frac{1}{a} - 1 = \frac{1-a}{a} \geqslant \frac{S - k - m(k)}{m'(k)}. \tag{6.17}$$

Let $k_1 := (2\ell - L - 1) \vee 0$, and $k_2 := S - L$. Then (6.13)–(6.14) yield

$$m(k) = \begin{cases} L - \ell, & 0 \leqslant k < k_1, \\ \ell - k - 1, & k \geqslant k_1, \end{cases} \tag{6.18}$$

and

$$m'(k) = \begin{cases} L, & k \leqslant k_2, \\ S - k, & k \geqslant k_2. \end{cases} \tag{6.19}$$

Using (6.18)–(6.19), it is easily verified that, regardless of the value of k_2, the right-hand side of (6.17) is, as a function of k, (weakly) decreasing on $[0, k_1]$ and (weakly) increasing on $[k_1, \infty)$; hence it has a minimum at $k = k_1$ and (6.17) implies, using $m(k_1) = \ell - k_1 - 1$ from (6.18),

$$\frac{1}{a} \geqslant 1 + \frac{S - k_1 - m(k_1)}{m'(k_1)} = \frac{S + 1 - \ell + m'(k_1)}{m'(k_1)} \tag{6.20}$$

and thus

$$a \leqslant \frac{m'(k_1)}{S+1-\ell+m'(k_1)}. \tag{6.21}$$

Conversely, we construct an example with equality in (6.21). Suppose that a is such that equality holds in (6.21), and thus in (6.20). Let $\mathcal{W}$ be a set of voters with $|\mathcal{W}| = aV$, for a suitable V (so that aV and other numbers in the construction are integers), let $k := k_1$ and let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ be disjoint sets of candidates with $|\mathcal{C}_1| = k$, $|\mathcal{C}_2| = \ell - k$, $|\mathcal{C}_3| = S - k$. Let each voter in $\mathcal{W}$ vote on $\mathcal{A} := \mathcal{C}_1 \cup \mathcal{C}_2$, and on $m(k)$ candidates from $\mathcal{C}_3$, in an organised way such that each candidate in $\mathcal{C}_3$ gets equally many votes from $\mathcal{W}$, viz. $|\mathcal{W}|m(k)/(S-k)$; furthermore, let each voter in $\mathcal{W}^{\mathsf{c}}$ vote for $m'(k)$ candidates from $\mathcal{C}_3$, again with evenly spead votes. Then there is equality in (6.15) and (6.17), and thus in (6.16), which implies that each candidate in $\mathcal{C}_3$ gets exactly aV votes. Thus all candidates in $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ tie with $aV = |\mathcal{W}|$ votes each, and a possible outcome is $\mathcal{E} = \mathcal{C}_1 \cup \mathcal{C}_3$, which is a bad outcome for π_{EJR}. Consequently, (6.21) is best possible, i.e.,

$$\pi_{\mathsf{EJR}}^{\mathsf{LV}(L)}(\ell, S) = \frac{m'(k_1)}{S+1-\ell+m'(k_1)}. \tag{6.22}$$

The result follows from (6.22) and (6.18)–(6.19). ∎

Remark 6.13. Note that $\pi_{\mathsf{EJR}}^{\mathsf{BV}}(\ell, S)$ is not monotone in ℓ, cf. Remark 3.8. In fact, (6.10) shows that $\pi_{\mathsf{EJR}}^{\mathsf{BV}}(\ell, S)$, as a function of ℓ, increases from $\pi_{\mathsf{EJR}}^{\mathsf{BV}}(1, S) = \frac{1}{2}$ to a maximum at $\ell = \lceil (S+1)/2 \rceil$, and then decreases back to $\pi_{\mathsf{EJR}}^{\mathsf{BV}}(S, S) = \frac{1}{2}$. The maximum value is $\approx \frac{2}{3}$ for large S.

The behaviour of $\pi_{\mathsf{PJR}}^{\mathsf{LV}(L)}(\ell, S)$ is similar when $L > \lceil (S+1)/2 \rceil$.

Example 6.14. For $S = 3$, $\pi_{\mathsf{EJR}}^{\mathsf{BV}}(1,3) = \frac{1}{2}$, $\pi_{\mathsf{EJR}}^{\mathsf{BV}}(2,3) = \frac{3}{5}$, $\pi_{\mathsf{EJR}}^{\mathsf{BV}}(3,3) = \frac{1}{2}$.

Remark 6.15. As noted above, $\pi_{\mathsf{EJR}}^{\mathsf{BV}}(S, S) = \frac{1}{2} = \pi_{\mathsf{same}}^{\mathsf{BV}}(S, S)$. More generally,

$$\pi_{\mathsf{PJR}}^{\mathsf{LV}(L)}(L, S) = \pi_{\mathsf{EJR}}^{\mathsf{LV}(L)}(L, S) = \pi_{\mathsf{same}}^{\mathsf{LV}(L)}(L, S), \tag{6.23}$$

which follows directly from Definitions 6.1 and 6.2, since when $\ell = L$, the assumptions require every voter in $\mathcal{W}$ to vote for $\mathcal{A}$.

For Cumulative Vote, the scenarios above are not very interesting, for the same reason as same; they allow the voters in $\mathcal{W}$ to spread their votes over too many candidates. We state this for the ideal Equal and Even version $\mathsf{CV}^{=}$, as in Theorem 5.14.

Theorem 6.16. *For (ideal) Equal and Even Cumulative Vote:*

$$\pi_{\mathsf{PJR}}^{\mathsf{CV}^{=}}(\ell, S) = \pi_{\mathsf{EJR}}^{\mathsf{CV}^{=}}(\ell, S) = 1, \qquad 1 \leqslant \ell \leqslant S. \tag{6.24}$$

Proof. An immediate consequence of Theorems 5.14 and 6.4. ∎

Remark 6.17. In Definition 6.2, it is not required that ℓ candidates from the common set $\bigcap_{\sigma\in\mathcal{W}}\sigma \supseteq \mathcal{A}$ are elected. That requirement for $\ell = 1$ is used in the definition of *Strong Justified Representation* in [1]; similarly, the weaker version that $|\sigma \cap \mathcal{E}| \geqslant \ell$ for every $\sigma \in \mathcal{W}$ is used in the definition of *Semi-Strong Justified Representation.* However, as remarked in [1], these conditions are too strong, and they cannot be required in general. In our setting, we might define scenarios SJR and SSJR as in Definition 6.2 but using these requirements; however, then $\pi^{\mathfrak{M}}_{\mathsf{SJR}}(\ell, S) = \pi^{\mathfrak{M}}_{\mathsf{SSJR}}(\ell, S) = 1$ for every election method $\mathfrak{M}$ and $1 \leqslant \ell \leqslant S$. To see this, let $n \geqslant 1$ and consider an election with $S + 1$ candidates $C_1, \dots, C_{S+1}$ and $V = n + S + 1$ votes: n votes on all candidates and 1 vote on $\mathcal{A}_i := \{C_i, \dots, C_{i+\ell-1}\}$ (with indices mod $S + 1$) for each $i \in [S + 1]$. Let $\mathcal{W}_i := \{\sigma : \sigma \supseteq \mathcal{A}_i\}$. If, for SJR or SSJR, the outcome is good for $\mathcal{W}_i$, then $\mathcal{E} \supseteq \mathcal{A}_i$. This cannot hold for all i, and thus the outcome is bad for at least one set of voters $\mathcal{W}_i$. Hence,

$$\pi^{\mathfrak{M}}_{\mathsf{SJR}}(\ell, S), \pi^{\mathfrak{M}}_{\mathsf{SSJR}}(\ell, S) \geqslant \frac{|\mathcal{W}_i|}{V} = \frac{n+1}{n+S+1}, \tag{6.25}$$

and the result follows since n is arbitrary. We therefore do not consider these scenarios.

7. Phragmén's and Thiele's unordered methods

In this section, we continue the study of the thresholds $\pi_{\mathsf{party}}, \pi_{\mathsf{tactic}}, \pi_{\mathsf{same}}, \pi_{\mathsf{PJR}}, \pi_{\mathsf{EJR}}$ for unordered ballots; we now consider Phragmén's and Thiele's election methods, defined in Appendices A.2.6–A.2.9.

Problem 7.1. Further election methods for unordered ballots are described and studied in e.g. [28; 1; 43; 8; 16]. Study $\pi_{\mathsf{party}}, \pi_{\mathsf{tactic}}, \pi_{\mathsf{same}}, \pi_{\mathsf{PJR}}, \pi_{\mathsf{EJR}}$ for them![16]

7.1. **The party list case.** We begin with a simple result, presumably known already to Phragmén and Thiele.

Theorem 7.2. *For Phragmén's and Thiele's unordered methods:*

$$\begin{aligned}\pi^{\mathsf{Phr\text{-}u}}_{\mathsf{party}}(\ell, S) &= \pi^{\mathsf{Th\text{-}opt}}_{\mathsf{party}}(\ell, S) = \pi^{\mathsf{Th\text{-}add}}_{\mathsf{party}}(\ell, S) = \pi^{\mathsf{Th\text{-}elim}}_{\mathsf{party}}(\ell, S) \\ &= \pi^{\mathsf{D'H}}_{\mathsf{party}}(\ell, S) = \frac{\ell}{S+1}, \qquad 1 \leqslant \ell \leqslant S. \end{aligned} \tag{7.1}$$

Proof. It is easy to see that in the party list case, all four methods reduce to D'Hondt's method [26, Theorem 11.1], [34], [45]. Hence, the result follows from (4.3). □

[16] Some inequalities for π_{PJR} and π_{EJR} follow by (6.4)–(6.6) from results in [1; 43; 8] showing whether or not certain methods satisfy JR, PJR or EJR.

7.2. **Phragmén's method.** For Phragmén's unordered method, we have a simple result. The result for $\pi^{\mathsf{Phr\text{-}u}}_{\mathsf{same}}$ is shown in [24; 26], and its extension to $\pi^{\mathsf{Phr\text{-}u}}_{\mathsf{PJR}}$ in [8].

Theorem 7.3 ([8])**.** *For Phragmén's unordered methods:*

$$\pi^{\mathsf{Phr\text{-}u}}_{\mathsf{tactic}}(\ell,S)=\pi^{\mathsf{Phr\text{-}u}}_{\mathsf{same}}(\ell,S)=\pi^{\mathsf{Phr\text{-}u}}_{\mathsf{PJR}}(\ell,S)=\frac{\ell}{S+1},\qquad 1\leqslant\ell\leqslant S. \tag{7.2}$$

Proof. Consider first PJR. Let $\mathcal{W}$ be a set of voters and $\mathcal{A}$ a set of candidates as in Definition 6.1, and suppose that the outcome is bad. Let $\mathcal{A}^*:=\bigcup_{\sigma\in\mathcal{W}}\sigma\supseteq\mathcal{A}$. Thus each voter in $\mathcal{W}$ votes for a set σ with $\mathcal{A}\subseteq\sigma\subseteq\mathcal{A}^*$, where $|\mathcal{A}|\geqslant\ell$, but $k:=|\mathcal{A}^*\cap\mathcal{E}|<\ell$. In particular, at least one candidate in $\mathcal{A}$ is not elected.

We use the formulation with loads in Appendix A.2.6, and let $t=t^{(S)}$ be the final maximum load of a ballot. Say that a ballot with load u has *free voting power* $t-u$; this is the additional load that the ballot may accept without raising the maximum load t.

The k elected candidates in $\mathcal{A}^*\cap\mathcal{E}$ together give load k, of which some part may fall on voters not in $\mathcal{W}$. Thus, if ballot i has final load x_i,

$$\sum_{i\in\mathcal{W}}x_i\leqslant k\leqslant\ell-1. \tag{7.3}$$

Furthermore, the total free voting power of the ballots in $\mathcal{W}$ is at most 1, since otherwise another candidate from $\mathcal{A}\setminus\mathcal{E}$ would have been elected to the last place, with a smaller $t^{(S)}$. (This total free voting power may equal 1 if there is a tie for the last seat.) Thus,

$$\sum_{i\in\mathcal{W}}(t-x_i)\leqslant 1. \tag{7.4}$$

Consequently, combining (7.3) and (7.4),

$$|\mathcal{W}|t=\sum_{i\in\mathcal{W}}x_i+\sum_{i\in\mathcal{W}}(t-x_i)\leqslant\ell. \tag{7.5}$$

On the other hand, $S-k$ candidates not in $\mathcal{A}^*$ have been elected, incurring a total load $S-k\geqslant S+1-\ell$. Since the ballots in $\mathcal{W}$ do not get any of this load, all of it falls on the $V-|\mathcal{W}|$ other ballots, and thus

$$S+1-\ell\leqslant(V-|\mathcal{W}|)t. \tag{7.6}$$

Combining (7.5) and (7.6) we find

$$\frac{|\mathcal{W}|}{V-|\mathcal{W}|}=\frac{|\mathcal{W}|t}{(V-|\mathcal{W}|)t}\leqslant\frac{\ell}{S+1-\ell}, \tag{7.7}$$

which is equivalent to

$$\frac{|\mathcal{W}|}{V}\leqslant\frac{\ell}{S+1}. \tag{7.8}$$

Thus,

$$\pi^{\mathsf{Phr\text{-}u}}_{\mathsf{PJR}}(\ell,S)\leqslant\frac{\ell}{S+1}. \tag{7.9}$$

Theorems 5.3 and 6.4 together with (7.9) yield

$$\pi^{\mathsf{Phr\text{-}u}}_{\mathsf{tactic}}(\ell,S)\leqslant\pi^{\mathsf{Phr\text{-}u}}_{\mathsf{same}}(\ell,S)\leqslant\pi^{\mathsf{Phr\text{-}u}}_{\mathsf{PJR}}(\ell,S)\leqslant\frac{\ell}{S+1}. \tag{7.10}$$

Hence, Theorem 5.17(v) applies and yields

$$\pi^{\mathsf{Phr\text{-}u}}_{\mathsf{tactic}}(\ell,S)=\frac{\ell}{S+1}. \tag{7.11}$$

The result follows by (7.10) and (7.11). □

However, it was shown in [8, Example 5] that Phragmén's unordered method does not satisfy EJR, i.e., $\pi^{\mathsf{Phr\text{-}u}}_{\mathsf{EJR}}\geqslant\ell/S>\ell/(S+1)$ is possible. We may tweak that example a little to the following (found by computer experiment rather than an analysis).

Example 7.4 (based on [8])**.** Consider an election by Phragmén's unordered method with $S=12$ seats, candidates $\mathcal{C}=\{A,B,C_1,\dots,C_{12}\}$, and 2409 voters voting

200 A,B,C_1
209 A,B,C_2
600 $C_1,C_2,C_3,\dots,C_{12}$
500 $C_2,C_3,\dots,C_{12}$
900 $C_3,\dots,C_{12}$

Then, a computer assisted calculation shows that the elected are, in order, $C_3,C_4,C_5,C_6,C_2,C_7,C_8,C_1,C_9,C_{10},C_{11},C_{12}$ (with $C_3,\dots,C_{12}$ tying throughout; here one possibility is chosen). Hence, if $\mathcal{W}$ is the set of the 409 voters on the first two lines, the conditions of EJR are satisfied for $\ell=2$ with $\mathcal{A}=\{A,B\}$, but the outcome is bad. Consequently,

$$\pi^{\mathsf{Phr\text{-}u}}_{\mathsf{EJR}}(2,12)\geqslant\frac{409}{2409}>\frac{2}{12}=\frac{\ell}{S}>\frac{\ell}{S+1}. \tag{7.12}$$

(This is not sharp, and can be improved at least a little.)

Problem 7.5. What is $\pi^{\mathsf{Phr\text{-}u}}_{\mathsf{EJR}}(\ell,S)$ in general? Even the case $\ell=2$ and a given (small) S seems far from trivial.

7.3. **Thiele's optimization method.** Aziz et al. [1] showed that Thiele's optimization method satisfies EJR; i.e., $\pi^{\mathsf{Th\text{-}opt}}_{\mathsf{EJR}}(\ell,S)<\ell/S$ by (6.5). In fact, the proof can be improved to show the optimal bound $\ell/(S+1)$.

Theorem 7.6 ([1], improved)**.** *For Thiele's optimization method: For* $1\leqslant\ell\leqslant S$*,*

$$\pi^{\mathsf{Th\text{-}opt}}_{\mathsf{tactic}}(\ell,S)=\pi^{\mathsf{Th\text{-}opt}}_{\mathsf{same}}(\ell,S)=\pi^{\mathsf{Th\text{-}opt}}_{\mathsf{PJR}}(\ell,S)=\pi^{\mathsf{Th\text{-}opt}}_{\mathsf{EJR}}(\ell,S)=\frac{\ell}{S+1}. \tag{7.13}$$

Proof. Consider an election with a bad outcome $\mathcal{E}$ for the scenario EJR in Definition 6.2. Then not every candidate in $\mathcal{A}$ is elected, since otherwise the outcome would be good, witnessed by any $\sigma \in \mathcal{W}$. Fix some $A \in \mathcal{A} \setminus \mathcal{E}$.

For every ballot σ and every elected candidate $C \in \mathcal{E}$, let $\delta(\sigma, C)$ be the change in the satisfaction of σ (see Appendix A.2.7) if A is elected instead of C, i.e., $\mathcal{E}$ is replaced by $\mathcal{E} \cup \{A\} \setminus \{C\}$. Let further $\Delta(\sigma) := \sum_{C\in\mathcal{E}} \delta(\sigma, C)$.

Since $\mathcal{E}$ maximizes the total satisfaction,

$$\sum_{\sigma\in\mathcal{V}} \delta(\sigma, C) \leqslant 0 \tag{7.14}$$

for every $C \in \mathcal{E}$, and thus

$$\sum_{\sigma\in\mathcal{V}} \Delta(\sigma) = \sum_{C\in\mathcal{E}} \sum_{\sigma\in\mathcal{V}} \delta(\sigma, C) \leqslant 0. \tag{7.15}$$

Consider a ballot σ, and let $k := |\sigma \cap \mathcal{E}|$. Then $\delta(\sigma, C) \geqslant -1/k$ if $C \in \sigma \cap \mathcal{E}$, and $\delta(\sigma, C) \geqslant 0$ if $C \notin \sigma \cap \mathcal{E}$. (In both cases with equality if $A \notin \sigma$.) Hence,

$$\Delta(\sigma) \geqslant k \cdot \left(-\frac{1}{k}\right) = -1. \tag{7.16}$$

Moreover, if $\sigma \in \mathcal{W}$, then $A \in \sigma$. Hence, $\delta(\sigma, C) = 0$ if $C \in \sigma \cap \mathcal{E}$, and $\delta(\sigma, C) = 1/(k+1)$ if $C \notin \sigma \cap \mathcal{E}$. Furthermore, $k \leqslant \ell - 1$, since otherwise the outcome would be good. Hence,

$$\Delta(\sigma) = \frac{S-k}{k+1} \geqslant \frac{S+1-\ell}{\ell}, \qquad \sigma \in \mathcal{W}. \tag{7.17}$$

Consequently, using (7.17) for $\sigma \in \mathcal{W}$ and (7.16) for $\sigma \in \mathcal{W}^{\mathsf{c}}$,

$$\sum_{\sigma\in\mathcal{V}} \Delta(\sigma) \geqslant |\mathcal{W}|\frac{S+1-\ell}{\ell} - |\mathcal{W}^{\mathsf{c}}| = |\mathcal{W}|\frac{S+1}{\ell} - V. \tag{7.18}$$

Combining (7.15) and (7.18) we find

$$0 \geqslant |\mathcal{W}|\frac{S+1}{\ell} - V \tag{7.19}$$

and thus

$$\frac{|\mathcal{W}|}{V} \leqslant \frac{\ell}{S+1}. \tag{7.20}$$

This holds for every bad outcome, and thus

$$\pi^{\mathsf{Th\text{-}opt}}_{\mathsf{EJR}}(\ell, S) \leqslant \frac{\ell}{S+1}. \tag{7.21}$$

Combining (7.21) with Theorems 6.4 and 7.2 yields

$$\pi^{\mathsf{Th\text{-}opt}}_{\mathsf{party}}(\ell, S) = \pi^{\mathsf{Th\text{-}opt}}_{\mathsf{same}}(\ell, S) = \pi^{\mathsf{Th\text{-}opt}}_{\mathsf{PJR}}(\ell, S) = \pi^{\mathsf{Th\text{-}opt}}_{\mathsf{EJR}}(\ell, S) = \frac{\ell}{S+1}. \tag{7.22}$$

Finally, (7.22) and (5.2) yield

$$\pi^{\mathsf{Th\text{-}opt}}_{\mathsf{tactic}}(\ell, S) \leqslant \pi^{\mathsf{Th\text{-}opt}}_{\mathsf{same}}(\ell, S) = \frac{\ell}{S+1}. \tag{7.23}$$

Hence, Theorem 5.17(v) applies and shows that the inequality (7.23) is an equality. □

7.4. **Thiele's addition method.** The results in Theorems 7.3 and 7.6 do not hold for Thiele's addition method. We first give a concrete example by Tenow [44].

Example 7.7. Consider an election by Thiele's addition method with $S = 3$ seats and 50 voters, divided into two parties. Suppose first that all vote along party lines:

37 ABC
13 KLM

Then Thiele's method reduces to D'Hondt's method and the larger party gets 2 seats and the smaller 1.

However, the larger party may cunningly split their votes on five different lists as follows:

1 A
9 AB
9 AC
9 B
9 C
13 KLM

Then A gets the first seat (19 votes), and the next two go to B and C (in some order) with 13.5 votes each, beating KLM with 13. Thus the large party gets all seats. (See [26, Example 13.13] for a further discussion.)

For same or PJR with $\ell = 1$, this is a bad outcome for the KLM party, and thus

$$\pi_{\mathsf{PJR}}^{\mathsf{Th\text{-}add}}(1,3) \geqslant \pi_{\mathsf{same}}^{\mathsf{Th\text{-}add}}(1,3) \geqslant \frac{13}{50} = 0.26 > \frac{1}{4} = \frac{\ell}{S+1}. \tag{7.24}$$

Moreover, it was shown in [1] that Thiele's addition method does not satisfy JR, i.e., by (6.6), that $\pi_{\mathsf{PJR}}^{\mathsf{Th\text{-}add}} \geqslant 1/S$ for some S. In [1], this was shown for $S \geqslant 10$, by an example; [42; 43] then found the sharp range to be $S \geqslant 6$, by solving a linear programming problem. The analysis in [42; 43] is easily modified to give a method for calculating $\pi_{\mathsf{PJR}}^{\mathsf{Th\text{-}add}}(1,S)$ for any given S.

For convenience, we let in the remainder of this subsection the "number of votes" be arbitrary positive real (or at least rational) numbers; see Remarks 2.2 and 2.3, and note that this does not affect the results since Thiele's addition method is homogeneous. This really means that we allow ballots with weights; in such cases all counts and sums over ballots should be interpreted accordingly; for convenience we omit this from the notation.

First, consider the problem of electing n candidates $C_1, \dots, C_n$ with a score of at least 1 each, and as few votes as possible, assuming that there are no other candidates. Denote the minimum total number of votes by α_n.

We may compute α_n as follows. We may assume that $C_1, \dots, C_n$ are elected in this order. For notational convenience, we identify $\{C_1, \dots, C_n\}$ with $[n]$; we thus regard the ballots as subsets of $[n]$, with $\sigma \subseteq [n]$ interpreted as a vote on $\{C_i : i \in \sigma\}$. Let x_σ be the number of votes σ. The condition that the n elected are $C_1, \dots, C_n$ in this order, with scores at least 1, can then be written as a number of linear inequalities in the $2^n - 1$ variables x_σ, $\sigma \neq \emptyset$. (We may ignore $\sigma = \emptyset$, which counts blank votes, since these can be deleted without affecting the outcome.) For example, for $n = 3$, we obtain the system

$$x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123} \geqslant 0 \tag{7.25}$$

$$x_1 + x_{12} + x_{13} + x_{123} \geqslant x_2 + x_{12} + x_{23} + x_{123} \tag{7.26}$$

$$x_1 + x_{12} + x_{13} + x_{123} \geqslant x_3 + x_{13} + x_{23} + x_{123} \tag{7.27}$$

$$x_2 + \tfrac{1}{2}x_{12} + x_{23} + \tfrac{1}{2}x_{123} \geqslant x_3 + \tfrac{1}{2}x_{13} + x_{23} + \tfrac{1}{2}x_{123} \tag{7.28}$$

$$x_1 + x_{12} + x_{13} + x_{123} \geqslant 1 \tag{7.29}$$

$$x_2 + \tfrac{1}{2}x_{12} + x_{23} + \tfrac{1}{2}x_{123} \geqslant 1 \tag{7.30}$$

$$x_3 + \tfrac{1}{2}x_{13} + \tfrac{1}{2}x_{23} + \tfrac{1}{3}x_{123} \geqslant 1 \tag{7.31}$$

where (7.26)–(7.27) say that C_1 wins over C_2 and C_3 and thus is elected first. (At least, in case of a tie, this is possible.) Similarly, (7.28) says that C_2 is elected before C_3. Finally, (7.29)–(7.31) say that C_1, C_2, C_3 all are elected with scores $\geqslant 1$.[17]

This leads to the linear programming problem

$$\text{Minimize } x := \sum_\sigma x_\sigma$$
$$\text{subject to (7.25)–(7.31), generalized to } n \text{ candidates.} \tag{7.32}$$

Thus, α_n equals the minimum in this linear programming problem.[18]

Theorem 7.8. *For Thiele's addition method and* $\ell = 1$*:*

$$\pi^{\mathsf{Th\text{-}add}}_{\mathsf{tactic}}(1, S) = \pi^{\mathsf{Th\text{-}add}}_{\mathsf{same}}(1, S) = \pi^{\mathsf{Th\text{-}add}}_{\mathsf{PJR}}(1, S) = \pi^{\mathsf{Th\text{-}add}}_{\mathsf{EJR}}(1, S) = \frac{1}{1 + \alpha_S}, \tag{7.33}$$

where α_S *is given by* (7.32).

Proof. This time we show the lower bound first. Let $(x_\sigma)_\sigma$ be a vector yielding the minimum α_S in (7.32), with $n = S$. Consider an election with candidates $A, B_1, \dots, B_S$, 1 vote on A (this is $\mathcal{W}$), and for each $\sigma \subseteq [S]$, x_σ votes on the corresponding set of candidates $\{B_i : i \in \sigma\}$. Thus the total number of votes is $V = 1 + \sum_\sigma x_\sigma = 1 + \alpha_S$. A possible outcome is

[17] In fact, it is easily seen that (7.29) and (7.30) are redundant, since the scores of the elected always are non-increasing.

[18] Sánchez-Fernández et al. [42, 43] give an equivalent linear programming problem, with $\sum_\sigma x_\sigma = 1$ and maximizing the score of C_n when elected. Their problem has the maximum α_n^{-1}.

$\{B_1,\dots,B_S\}$, which is a bad outcome for $\mathcal{W}$ for any of the scenarios same, PJR, EJR. Thus, using (6.3),

$$\pi^{\mathsf{Th\text{-}add}}_{\mathsf{EJR}}(1,S) \geqslant \pi^{\mathsf{Th\text{-}add}}_{\mathsf{PJR}}(1,S) \geqslant \pi^{\mathsf{Th\text{-}add}}_{\mathsf{same}}(1,S) \geqslant \frac{|\mathcal{W}|}{V} = \frac{1}{1+\alpha_S}. \tag{7.34}$$

To see that this is also a lower bound for $\pi^{\mathsf{Th\text{-}add}}_{\mathsf{tactic}}$, regard $\mathcal{W}$ as a set of voters with total weight 1. They may vote in any way, but if we add α_S further votes as above, choosing $\mathcal{B} := \{B_1,\dots,B_S\}$ disjoint from $\mathcal{A}^* := \bigcup_{\sigma\in\mathcal{W}}\sigma$, then the candidates in $\mathcal{A}^*$ will have scores $\leqslant 1$, and again the bad outcome $\{B_1,\dots,B_S\}$ is possible. Hence,

$$\pi^{\mathsf{Th\text{-}add}}_{\mathsf{tactic}}(1,S) \geqslant \frac{|\mathcal{W}|}{V} = \frac{1}{1+\alpha_S}. \tag{7.35}$$

Conversely, consider any election with bad outcome for EJR with $\ell = 1$ for a set $\mathcal{W}$ of voters. (Recall that EJR and PJR are the same for $\ell = 1$.) This means that there exists some candidate A that everyone in $\mathcal{W}$ has voted for. Furthermore, if $\mathcal{A}^* := \bigcup_{\sigma\in\mathcal{W}}\sigma$, then $\mathcal{A}^*\cap\mathcal{E} = \emptyset$, i.e., none from $\mathcal{A}^*$ is elected. Hence, if $\sigma\in\mathcal{W}$, then no candidate in σ is ever elected, so σ is counted with full value throughout the counting. Thus A has score at least $|\mathcal{W}|$ throughout the counting.

Let the elected, in order, be $B_1,\dots,B_S$. Since $B_i\in\mathcal{E}$, we have $B_i\notin\mathcal{A}^*$ for every i, i.e., no voter in $\mathcal{W}$ votes for any B_i. Furthermore, we may delete all votes from voters in $\mathcal{W}^{\mathsf{c}}$ on unelected candidates; this will not affect the scores of $B_1,\dots,B_S$, and not increase the score of anyone else, so the outcome $\mathcal{B} := \{B_1,\dots,B_S\}$ is still possible. Hence we may assume that if $\sigma\in\mathcal{W}^{\mathsf{c}}$, then $\sigma\subseteq\mathcal{B}$. Moreover, each B_i is elected with score at least the current score of A, and thus at least $|\mathcal{W}|$.

Consequently, considering only the votes from $\mathcal{W}^{\mathsf{c}}$, we have after scaling all votes by $1/|\mathcal{W}|$ an election of the type described by (7.32), and thus

$$\frac{|\mathcal{W}^{\mathsf{c}}|}{|\mathcal{W}|} \geqslant \alpha_S. \tag{7.36}$$

This yields

$$\frac{|\mathcal{W}|}{V} \leqslant \frac{1}{1+\alpha_S} \tag{7.37}$$

for any bad outcome, and thus $\pi^{\mathsf{Th\text{-}add}}_{\mathsf{EJR}}(1,S) \leqslant 1/(1+\alpha_S)$, which together with (7.34), (7.35) and (5.2) completes the proof. □

Before considering $\ell > 1$, let us study the sequence α_n. We begin with the first values.

Example 7.9. For $n = 1$, the system (7.25)–(7.31) becomes the trivial single equation $x_1 \geqslant 1$, and thus

$$\alpha_1 = 1. \tag{7.38}$$

Example 7.10. For $n = 2$, the system (7.25)–(7.31) becomes

$$x_1, x_2, x_{12} \geqslant 0 \tag{7.39}$$

$$x_1 + x_{12} \geqslant x_2 + x_{12} \tag{7.40}$$

$$x_1 + x_{12} \geqslant 1 \tag{7.41}$$

$$x_2 + \tfrac{1}{2}x_{12} \geqslant 1. \tag{7.42}$$

It follows from (7.40) and (7.42) that $x_1+x_2+x_{12} \geqslant 2x_2+x_{12} \geqslant 2$, and this is attained by, for example, $x_1 = x_2 = 1$, $x_{12} = 0$ or $x_1 = x_2 = 0$, $x_{12} = 2$. (Consequently, there is no way to split votes tactically when $n = 2$.) Hence,

$$\alpha_2 = 2. \tag{7.43}$$

Example 7.11. For $n = 3$, the linear programming problem (7.32) has (by Maple) the minimum

$$\alpha_3 = 8/3 \doteq 2.667, \tag{7.44}$$

with a solution $x_{12} = x_{13} = x_2 = x_3 = 2/3$ and all other $x_\sigma = 0$.[19] Consequently,

$$\pi^{\mathsf{Th\text{-}add}}_{\mathsf{same}}(1,3) = \pi^{\mathsf{Th\text{-}add}}_{\mathsf{PJR}}(1,3) = \pi^{\mathsf{Th\text{-}add}}_{\mathsf{EJR}}(1,3) = \frac{1}{1+8/3} = \frac{3}{11}. \tag{7.45}$$

Note that the solution to the linear programming problem corresponds to the following election. (Thiele's addition method, 3 seats.)

3 A
2 B_1B_2
2 B_1B_3
2 B_2
2 B_3

B_1, B_2, B_3 tie for the first seat, and if it goes to B_1, then A, B_2 and B_3 tie for the remaining two seats, and it is possible that A is not elected. (Note that the second election in Example 7.7 is an approximation of this example, with votes multiplied by 4.5, modified to avoid ties.)

Example 7.12. For $n = 4$, the linear programming problem (7.32) has (by Maple) the minimum

$$\alpha_4 = 24/7 \doteq 3.4286, \tag{7.46}$$

with one solution $x_{123} = x_{124} = x_{13} = x_{14} = x_{23} = x_{24} = x_3 = x_4 = 3/7$ and all other $x_\sigma = 0$.[20]

For $n \geqslant 5$, we do not know the values of α_n, and thus not of $\pi^{\mathsf{Th\text{-}add}}_{\mathsf{same}}(1,n)$; it should be easy to obtain more values by solving the linear programming

[19] In fact, this is the unique solution. This is easily seen by solving the dual problem, which leads to considering the linear combination $\frac{1}{3}\cdot(7.26)+\frac{1}{3}\cdot(7.27)+\frac{4}{3}\cdot(7.28)+\frac{8}{3}\cdot(7.31)$; this also verifies (7.44). We omit the details.

[20] This solution is not unique. For example, another is given by $x_{124} = x_{13} = 6/7$, $x_{23} = x_4 = 4/7$, $x_{24} = x_3 = 2/7$.

problem (7.32) by computer, but we have not done so.[21] We have the following general result.

Theorem 7.13. *The sequence α_n is weakly increasing and subadditive. In particular,*

$$\alpha_n \leqslant \alpha_{n+1} \leqslant \alpha_n + 1, \qquad n \geqslant 1. \tag{7.47}$$

Furthermore,

$$\frac{n}{H_n} \leqslant \alpha_n \leqslant n, \qquad n \geqslant 1, \tag{7.48}$$

and if $n \geqslant 3$, then $\alpha_n < n$.

Proof. Let $\mathfrak{E}_n$ be the set of all elections of the type used to define α_n above. Given an election in $\mathfrak{E}_{n+1}$, with $C_1, \dots, C_{n+1}$ elected in this order, remove C_{n+1} from all ballots. This gives an election in $\mathfrak{E}_n$, and it follows that $\alpha_n \leqslant \alpha_{n+1}$.

Similarly, given two elections in $\mathfrak{E}_m$ and $\mathfrak{E}_n$, we may relabel the candidates and assume that the two elections have disjoint sets of candidates; then their union is an election in $\mathfrak{E}_{m+n}$, and the subadditivity $\alpha_{m+n} \leqslant \alpha_m + \alpha_n$ follows.

In particular, since $\alpha_1 = 1$, we have $\alpha_{n+1} \leqslant \alpha_n + 1$. Induction yields $\alpha_n \leqslant n$ and, since $\alpha_3 < 3$ by Example 7.11, $\alpha_n < n$ for $n \geqslant 3$.

Finally, let $w(\sigma, i) \geqslant 0$ be the contribution of ballot σ to the score of C_i when elected. Thus $\sum_\sigma w(\sigma, i) \geqslant 1$. On the other hand, if σ is a ballot with k names, then $\sum_{i=1}^n w(\sigma, i) = H_k \leqslant H_n$. Consequently,

$$n \leqslant \sum_{i=1}^n \sum_\sigma w(\sigma, i) = \sum_\sigma \sum_{i=1}^n w(\sigma, i) \leqslant \sum_\sigma H_n = H_n |\mathcal{V}|. \tag{7.49}$$

Hence $|\mathcal{V}| \geqslant n/H_n$, and thus $\alpha_n \geqslant n/H_n$. □

Problem 7.14. Find a general formula for α_n, and thus for $\pi^{\mathsf{Th\text{-}add}}_{\mathsf{same}}(1, S)$.

Problem 7.15. Find an asymptotic formula for α_n as $n \to \infty$, and thus for $\pi^{\mathsf{Th\text{-}add}}_{\mathsf{same}}(1, S)$ as $S \to \infty$.

The limit $\lim_{n\to\infty} \alpha_n/n$ exists since α_n is subadditive. We conjecture that the limit is 0, i.e., $\alpha_n = o(n)$; by Theorem 7.8, this is equivalent to $\pi^{\mathsf{Th\text{-}add}}_{\mathsf{same}}(1, S) \gg 1/S$ as $S \to \infty$.

Problem 7.16. Prove (or disprove) the conjecture $\alpha_n/n \to 0$ as $n \to \infty$.

Remark 7.17. It follows from (7.33) and (6.6) that Thiele's addition method satisfies JR for a given S if and only if $\alpha_S > S - 1$. It was shown by Sánchez-Fernández et al. [42, 43] that this holds if and only if $S \leqslant 5$, see Footnote 21 and (7.47).

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[21] Sánchez-Fernández et al. [42] report, in our notation, $\alpha_5^{-1} \doteq 0.2389$ and $\alpha_6^{-1} \doteq 0.204$, and thus $\alpha_5 \doteq 4.186$ and $\alpha_6 \doteq 4.90$, but they do not give exact rational values. See also the example in Table 1 in the full (arXiv) version of [43], which shows that $\alpha_6 \leqslant 4992/(6103/6) = 29952/6103 \doteq 4.908$. (This bound is not sharp and can be improved further.)

We return to $\pi^{\mathsf{Th\text{-}add}}(\ell, S)$ for the scenarios above. For $\ell > 1$, we can only prove an inequality, which we conjecture is sharp.

Theorem 7.18. *For Thiele's addition method and* $1 \leqslant \ell \leqslant S$,

$$\pi_{\mathsf{EJR}}^{\mathsf{Th\text{-}add}}(\ell, S) \geqslant \pi_{\mathsf{PJR}}^{\mathsf{Th\text{-}add}}(\ell, S) \geqslant \pi_{\mathsf{same}}^{\mathsf{Th\text{-}add}}(\ell, S) \geqslant \frac{\ell}{\ell + \alpha_{S+1-\ell}}, \tag{7.50}$$

where α_n *is given by* (7.32).

Conjecture 7.19. *The inequalities in* (7.50) *are equalities for* PJR *and* same, *i.e.,* $\pi_{\mathsf{PJR}}^{\mathsf{Th\text{-}add}}(\ell, S) = \pi_{\mathsf{same}}^{\mathsf{Th\text{-}add}}(\ell, S) = \ell/(\ell + \alpha_{S+1-\ell})$.

Proof and discussion. We prove (7.50) by a simple extension of the example in the proof of Theorem 7.8. Let $(x_\sigma)_\sigma$ be a vector yielding the minimum α_n in (7.32), with $n = S + 1 - \ell$. Consider an election with candidates $A_1, \dots, A_\ell, B_1, \dots, B_n$; let $\mathcal{W}$ be a set of ℓ votes on $\mathcal{A} := \{A_1, \dots, A_\ell\}$, and let there be for each $\sigma \subseteq [n]$, x_σ votes on the corresponding set of candidates $\{B_i : i \in \sigma\}$. Thus the total number of votes is $V = \ell + \sum_\sigma x_\sigma = \ell + \alpha_{S+1-\ell}$. A possible outcome is $\{A_1, \dots, A_{\ell-1}, B_1, \dots, B_{S+1-\ell}\}$, which is a bad outcome for $\mathcal{W}$ for same (and for PJR, EJR). Thus (7.50) follows, using (6.3).

For the converse, for PJR (or same) we may try to argue in the same way as for Theorem 7.8. Thus, consider an election with a bad outcome. let again $\mathcal{A}^* := \bigcup_{\sigma \in \mathcal{W}} \sigma$, so $|\mathcal{E} \cap \mathcal{A}^*| \leqslant \ell - 1$. Let $\mathcal{B} := \mathcal{E} \setminus \mathcal{A}^*$, and $n := |\mathcal{B}| \geqslant S + 1 - \ell$.

Throughout the counting, there is at least one candidate in $\mathcal{A}$ that is not elected, and this candidate has score $\geqslant |\mathcal{W}|/\ell$; hence every elected candidate is elected with a score $\geqslant |\mathcal{W}|/\ell$; in particular, this holds for the n candidates in $\mathcal{B}$. We may now delete all candidates not in $\mathcal{E}$. By construction, the voters in $\mathcal{W}$ vote only for candidates in $\mathcal{A}^*$. The problem is that it is possible that voters in $\mathcal{W}^{\mathsf{c}}$ vote not only for candidates in $\mathcal{B}$ but also for one or several candidates in $\mathcal{A}^*$. If this does not happen, then the votes from $\mathcal{W}^{\mathsf{c}}$ yield, after scaling by $\ell/|\mathcal{W}|$, an election of the type in (7.32), and thus

$$|\mathcal{W}^{\mathsf{c}}| \geqslant \alpha_n \frac{\mathcal{W}}{\ell} \geqslant \alpha_{S+1-\ell} \frac{\mathcal{W}}{\ell}, \tag{7.51}$$

which yields the desired estimate

$$\frac{|\mathcal{W}|}{|\mathcal{V}|} \leqslant \frac{\ell}{\ell + \alpha_{S+1-\ell}}. \tag{7.52}$$

However, the voters in $\mathcal{W}^{\mathsf{c}}$ are free to also vote for candidates in $\mathcal{A}^*$. This may decrease the score for some of the candidates in $\mathcal{B}$, which seems like a bad idea, and we conjecture that the optimal strategy for $\mathcal{W}^{\mathsf{c}}$ does not use votes outside $\mathcal{B}$. However, such votes may change the order in which the candidates from $\mathcal{B}$ are elected, which may change their scores in a complicated way, so we have not been able to show this conjecture rigorously, and thus not Conjecture 7.19. Note that voting on candidates from another party may be a good strategy in some situations, see Example 7.24 below, so one should be careful, even if that example is of a different type. □

Problem 7.20. Prove (or disprove) Conjecture 7.19.

Example 7.21. Theorem 7.18 and Example 7.11 show that

$$\pi^{\mathsf{Th\text{-}add}}_{\mathsf{same}}(3,5) \geqslant \frac{3}{3+\alpha_3} = \frac{3}{3+8/3} = \frac{9}{17} > \frac{1}{2}. \tag{7.53}$$

Thus, with $S = 5$, there are bad outcomes where $\mathcal{A}$ has a majority of the votes but does not get a majority of the seats. We may construct a concrete example from the solution to the linear programming problem in Example 7.11 as follows; cf. the election in Example 7.11. (Thiele's addition method, 5 seats.)

9 $A_1A_2A_3$
2 B_1B_2
2 B_1B_3
2 B_2
2 B_3

A possible outcome is that the elected are, in order, A_1, A_2, B_1, B_2, B_3.

Example 7.22. If $\ell \leqslant S-2$, the $S+1-\ell \geqslant 3$ and Theorem 7.13 yields $\alpha_{S+1-\ell} < S+1-\ell$. Hence, Theorem 7.18 yields

$$\pi^{\mathsf{Th\text{-}add}}_{\mathsf{same}}(\ell,S) > \frac{\ell}{S+1}, \qquad 1 \leqslant \ell \leqslant S-2. \tag{7.54}$$

Similarly, since $\alpha_6 < 5$ as shown by Sánchez-Fernández et al. [42, 43], see Remark 7.17 and Footnote 21, it follows that if $S-\ell \geqslant 5$, then $\alpha_{S+1-\ell} < S-\ell$, and thus Theorem 7.18 yields

$$\pi^{\mathsf{Th\text{-}add}}_{\mathsf{same}}(\ell,S) > \frac{\ell}{S}, \qquad 1 \leqslant \ell \leqslant S-5. \tag{7.55}$$

To find $\pi^{\mathsf{Th\text{-}add}}_{\mathsf{EJR}}$ seems even more complicated than to find $\pi^{\mathsf{Th\text{-}add}}_{\mathsf{PJR}}$ or $\pi^{\mathsf{Th\text{-}add}}_{\mathsf{same}}$, and we have no non-trivial result.

Problem 7.23. Find $\pi^{\mathsf{Th\text{-}add}}_{\mathsf{EJR}}(\ell,S)$ for $\ell \geqslant 2$.

Finally, consider $\pi^{\mathsf{Th\text{-}add}}_{\mathsf{tactic}}$. The values x_σ in the solution to (7.32) give an optimal strategy for how a well-organized party should distribute its votes in order to get n candidates elected, provided the other voters vote for different candidates. However, Thiele's addition method is non-monotone, and the strategy is risky and may be bad if the other voters vote in some other way, and in particular if another party knows the strategy and may counteract as in the following example.

Example 7.24 ([26])**.** Consider again the second election in Example 7.7, and suppose that two voters in the KLM party also vote for B, by chance or by clever design because they know the plan of the other party:

1 A
9 AB
9 AC

9 B
9 C
11 KLM
2 $BKLM$

Then B is elected first, followed by C and finally K (or L or M). Hence the ABC party gets only 2 seats.

Example 7.24 suggests that the strategy in the linear programming problem (7.32) is not the best strategy for the scenario tactic, where we consider the worst-case behaviour of the voters not in $\mathcal{W}$, and thus need a fool-proof strategy. We do not know the best such strategy, and leave it an open problem to find it, and the resulting threshold.

Problem 7.25. Find $\pi^{\mathsf{Th\text{-}add}}_{\mathsf{tactic}}(\ell, S)$.

7.5. **Thiele's elimination method.** We consider briefly also Thiele's elimination method.

Theorem 7.26. *For Thiele's elimination method:*

$$\pi^{\mathsf{Th\text{-}elim}}_{\mathsf{tactic}}(\ell, S) = \pi^{\mathsf{Th\text{-}elim}}_{\mathsf{same}}(\ell, S) = \frac{\ell}{S+1}, \qquad 1 \leqslant \ell \leqslant S. \tag{7.56}$$

Proof. We first give an upper bound for $\pi^{\mathsf{Th\text{-}elim}}_{\mathsf{same}}(\ell, S)$. Thus suppose that $\mathcal{W} \subseteq \mathcal{V}$, and that every voter in $\mathcal{W}$ votes for the same list $\mathcal{A}$ with $|\mathcal{A}| \geqslant \ell$. Suppose that the outcome is bad for same. This means that less than ℓ candidates in $\mathcal{A}$ remain at the end. Hence, in some round $\mathcal{A}$ has ℓ remaining candidates and one of them, say A, is eliminated.

Consider this round. The eliminated candidate A has score $\geqslant |\mathcal{W}|/\ell$, and thus every other remaining candidate has at least this score. Let m be the number of remaining candidates in $\mathcal{C} \setminus \mathcal{A}$. Since the total number of remaining candidates is $> S$ (otherwise there would be no more elimination), $m \geqslant S + 1 - \ell$. Hence, the total score of the candidates in $\mathcal{C} \setminus \mathcal{A}$, T say, is $\geqslant (S + 1 - \ell)|\mathcal{W}|/\ell$.

A ballot with k remaining candidates contributes $1/k$ to the score of each of them, so the total contribution of the ballot is 1, for any $k \geqslant 1$. (And 0 if no candidate on the ballot remains.) Since no voter in $\mathcal{W}$ votes for any candidate in $\mathcal{C} \setminus \mathcal{A}$, it follows that

$$V - |\mathcal{W}| = |\mathcal{W}^{\mathsf{c}}| \geqslant T \geqslant \frac{S+1-\ell}{\ell}|\mathcal{W}|, \tag{7.57}$$

which implies

$$\frac{|\mathcal{W}|}{V} \leqslant \frac{\ell}{S+1}. \tag{7.58}$$

Consequently,

$$\pi^{\mathsf{Th\text{-}elim}}_{\mathsf{same}}(\ell, S) \leqslant \frac{\ell}{S+1}. \tag{7.59}$$

Finally, (7.59) and (5.2) yield $\pi^{\mathsf{Th\text{-}elim}}_{\mathsf{tactic}}(\ell,S) \leqslant \pi^{\mathsf{Th\text{-}elim}}_{\mathsf{same}}(\ell,S) \leqslant \frac{\ell}{S+1}$, and thus Theorem 5.17(v) yields the equalities (7.56). □

For PJR we give only an example, giving a lower bound for $\pi_{\mathsf{PJR}}(1,S)$ (and thus for $\pi_{\mathsf{EJR}}(1,S)$). We do not believe that this example is sharp. In fact, we do not know whether $\pi^{\mathsf{Th\text{-}elim}}_{\mathsf{PJR}}(\ell,S) < 1$ or not, even in the case $\pi^{\mathsf{Th\text{-}elim}}_{\mathsf{PJR}}(1,1)$.

Example 7.27. Let $S,m,n \geqslant 1$. Consider an election with $1+mn+S$ candidates A, C_{ij} and B_k, for $i\in[m]$, $j\in[n]$, $k\in[S]$, and the following votes:

$n+1$ $\{A\}\cup\{C_{ij} : j\in[n]\}$ for every $i\in[m]$
$m-1$ $\{C_{ij}\}$ for every $i\in[m]$, $j\in[n]$
$m+n$ $\{B_k\}$ for every $k\in[S]$.

The total number of votes is thus

$$V = m(n+1)+mn(m-1)+S(m+n) = m(mn+1)+S(m+n). \tag{7.60}$$

Initially, A has score $\frac{m(n+1)}{n+1} = m$, each C_{ij} has score $\frac{n+1}{n+1}+m-1 = m$ and each B_k has score $m+n$. Thus either A or some C_{ij} is eliminated; suppose that A is. In the sequel, every remaining C_{ij} has score $\leqslant n+1+m-1 = m+n$, and each remaining B_k still has score $m+n$. We may thus assume that in each round (after the first) some C_{ij} is eliminated, until only $B_1,\dots,B_S$ remain and are elected.

Consider the scenario PJR with $\ell = 1$, let $\mathcal{A} = \{A\}$, and let the voters $\mathcal{W}$ be the $m(n+1)$ voters in the first line above. Then the election above is a bad outcome. Hence, for any $S,m,n\geqslant 1$,

$$\pi^{\mathsf{Th\text{-}elim}}_{\mathsf{PJR}}(1,S) \geqslant \frac{|\mathcal{W}|}{V} = \frac{m(n+1)}{m(mn+1)+(m+n)S}. \tag{7.61}$$

Letting $n\to\infty$, we obtain

$$\pi^{\mathsf{Th\text{-}elim}}_{\mathsf{PJR}}(1,S) \geqslant \frac{m}{m^2+S}, \qquad m\geqslant 1. \tag{7.62}$$

For a numerical example, with $S=3$ and $m=2$, and comparing with (7.56),

$$\pi^{\mathsf{Th\text{-}elim}}_{\mathsf{PJR}}(1,3) \geqslant \frac{2}{7} > \pi^{\mathsf{Th\text{-}elim}}_{\mathsf{same}}(1,3) = \frac{1}{4}. \tag{7.63}$$

Given $S\geqslant 1$, we may thus maximize the right-hand side of (7.62) over $m\geqslant 1$; the maximum is obtained for $m = \lfloor\sqrt{S}\rfloor$ or $m = \lceil\sqrt{S}\rceil$. Taking $m = \lfloor\sqrt{S}\rfloor$ yields the lower bound

$$\pi^{\mathsf{Th\text{-}elim}}_{\mathsf{PJR}}(1,S) \geqslant \frac{\lfloor\sqrt{S}\rfloor}{2S} \sim \frac{1}{2\sqrt{S}}, \qquad \text{as } S\to\infty. \tag{7.64}$$

In particular, by (6.6), Thiele's elimination method does not satisfy JR.

Problem 7.28. What is $\pi^{\mathsf{Th\text{-}elim}}_{\mathsf{PJR}}(\ell,S)$?

7.6. **Thiele's optimization method with general weights.** We have in Sections 7.3–7.4 studied Thiele's optimization and addition methods with the standard weights $w_k = 1/k$. In this and the next subsection, we extend some of the results to general weights $\mathbf{w} = (w_k)_0^\infty$. Our results are inspired by, and extend considerably, results by Aziz et al. [1] on the criteria JR and EJR, see Remarks 7.33 and 7.39.

We assume throughout that $w_1 = 1$ and that $w_1 \geqslant w_2 \geqslant \ldots \geqslant 0$.

Theorem 7.29. *For Thiele's optimization method with general weights* $\mathbf{w} = (w_k)_0^\infty$*: For* $S \geqslant 1$*,*

$$\pi_{\mathsf{same}}^{\mathsf{Th\text{-}opt(w)}}(1,S) = \pi_{\mathsf{PJR}}^{\mathsf{Th\text{-}opt(w)}}(1,S) = \pi_{\mathsf{EJR}}^{\mathsf{Th\text{-}opt(w)}}(1,S) = \frac{1}{1 + S/\max_{k\leqslant S}(kw_k)} \tag{7.65}$$

Proof. Consider an election with a bad outcome for EJR. Let $\mathcal{A}^* := \bigcup_{\sigma\in\mathcal{W}} \sigma$. Then $\mathcal{E} \cap \mathcal{A}^* = \emptyset$; furthermore, there exists a candidate $A \in \mathcal{A}^*$ such that $A \in \sigma$ for every $\sigma \in \mathcal{W}$.

Let the elected be $B_1, \ldots, B_S \notin \mathcal{A}^*$. We may eliminate all candidates except A and $B_1, \ldots, B_S$; this yields an election with the same bad outcome $\mathcal{E} = \mathcal{B} := \{B_1, \ldots, B_S\}$. Furthermore, we may delete A from any ballot not in $\mathcal{W}$; this will not change the total satisfaction $\Psi(\mathcal{B})$, while the total satisfaction $\Psi(\mathcal{C}')$ for every other set $\mathcal{C}' \subseteq \{A\} \cup \mathcal{B}$ is decreased or remains the same. Hence, $\mathcal{B}$ still maximizes the total satisfaction, and is thus a possible outcome which is bad. Note that each voter in $\mathcal{W}$ now votes for $\{A\}$, and each voter in $\mathcal{W}^{\mathsf{c}}$ for some subset of $\mathcal{B}$.

Suppose that A is elected instead of B_j. This increases the satisfaction of every $\sigma \in \mathcal{W}$ from 0 to $w_1 = 1$. If $\sigma \in \mathcal{W}^{\mathsf{c}}$ and $B_j \in \sigma$, then the satisfaction of σ decreases from $\psi(|\sigma|)$ to $\psi(|\sigma| - 1)$, i.e., by $w_{|\sigma|}$. For all other $\sigma \in \mathcal{W}^{\mathsf{c}}$, the satisfaction remains the same. Hence, the change in total satisfaction (A.4) is

$$\Psi\bigl(\mathcal{B} \setminus \{B_j\} \cup \{A\}\bigr) - \Psi(\mathcal{B}) = |\mathcal{W}| - \sum_{\sigma\ni B_j} w_{|\sigma|}. \tag{7.66}$$

This change is $\leqslant 0$, since $\mathcal{B}$ is elected and thus maximizes the total satisfaction. Summing over $j \in [S]$ we thus obtain

$$0 \geqslant \sum_{j=1}^{S}\Bigl(|\mathcal{W}| - \sum_{\sigma\ni B_j} w_{|\sigma|}\Bigr) = S|\mathcal{W}| - \sum_{\sigma\in\mathcal{W}^{\mathsf{c}}} |\sigma| w_{|\sigma|}. \tag{7.67}$$

Thus,

$$|\mathcal{W}| \leqslant \frac{1}{S}\sum_{\sigma\in\mathcal{W}^{\mathsf{c}}} |\sigma| w_{|\sigma|} \leqslant \frac{|\mathcal{W}^{\mathsf{c}}|}{S} \max_{1\leqslant k\leqslant S}\bigl(|k| w_k\bigr). \tag{7.68}$$

which yields

$$\frac{|\mathcal{W}|}{V} \leqslant \frac{1}{1 + S/\max_{1\leqslant k\leqslant S}\bigl(|k| w_k\bigr)} \tag{7.69}$$

and thus an upper bound "$\leqslant$" in (7.65).

Conversely, for any $k \in [S]$, consider an election with $S+1$ candidates $A, B_1, \dots, B_S$. Let $W \geqslant 2$ and let there be $V = W + S$ votes: W votes on $\{A\}$ (this is the set $\mathcal{W}$) and 1 vote on each set $\{B_i, \dots, B_{i+k-1}\}$ (indices mod S) for each $i \in [S]$.

The sets of S candidates are $\mathcal{B} := \{B_1, \dots, B_S\}$ and $\mathcal{A}_i := \{A\} \cup \mathcal{B} \setminus \{B_i\}$. The total satisfaction if these sets are elected are

$$\Psi(\mathcal{B}) = S\psi(k), \tag{7.70}$$

$$\Psi(\mathcal{A}_i) = W + (S-k)\psi(k) + k\psi(k-1) = \Psi(\mathcal{B}) + W - kw_k. \tag{7.71}$$

Hence, if $W < kw_k$, then $\mathcal{B}$ is elected, a bad outcome for same. As usual, we may here let W be a rational number, since we may multiply the votes above by the denominator of W. Hence, we obtain

$$\pi_{\mathsf{same}}^{\mathsf{Th\text{-}opt}(\mathbf{w})}(1, S) \geqslant \frac{kw_k}{kw_k + S} = \frac{1}{1 + S/(kw_k)}. \tag{7.72}$$

We may now take the maximum over $k \in [S]$ to get the inequality "$\geqslant$" for $\pi_{\mathsf{same}}^{\mathsf{Th\text{-}opt}(\mathbf{w})}$ in (7.65), which by (6.3) completes the proof. □

Theorem 7.30. *For Thiele's optimization method with weights $(w_k)_0^\infty$ satisfying $w_k \leqslant 1/k$, $k \geqslant 1$: For $S \geqslant 1$,*

$$\pi_{\mathsf{same}}^{\mathsf{Th\text{-}opt}(\mathbf{w})}(1, S) = \pi_{\mathsf{PJR}}^{\mathsf{Th\text{-}opt}(\mathbf{w})}(1, S) = \pi_{\mathsf{EJR}}^{\mathsf{Th\text{-}opt}(\mathbf{w})}(1, S) = \frac{1}{S+1}. \tag{7.73}$$

Furthermore,

$$\overline{\pi}_{\mathsf{tactic}}^{\mathsf{Th\text{-}opt}(\mathbf{w})}(\ell, S) = \frac{\ell}{S+1}, \qquad 1 \leqslant \ell \leqslant S. \tag{7.74}$$

In particular, this applies to Thiele's weak optimization method.

Proof. By assumption, recalling also the standing assumption $w_1 = 1$, we have $\max_k(kw_k) = 1$. Hence, (7.73) follows from Theorem 7.29.

This implies, by (5.2), $\pi_{\mathsf{tactic}}^{\mathsf{Th\text{-}opt}(\mathbf{w})}(1, S) \leqslant \pi_{\mathsf{same}}^{\mathsf{Th\text{-}opt}(\mathbf{w})}(1, S) = 1/(S+1)$, and thus (7.74) follows from Theorem 5.17(iv). □

With the weak method, it is for $\ell \geqslant 2$ obviously a bad strategy to vote on the same list. One aspect of this is the following trivial result. (Recall that the proof of (7.74) by Theorem 5.17 uses the same strategy as for SNTV in Theorem 5.7, with the votes split on different candidates separately.)

Theorem 7.31. *For Thiele's weak optimization method: If $2 \leqslant \ell \leqslant S$, then*

$$\pi_{\mathsf{same}}^{\mathsf{Th\text{-}opt(weak)}}(\ell, S) = \pi_{\mathsf{PJR}}^{\mathsf{Th\text{-}opt(weak)}}(\ell, S) = \pi_{\mathsf{EJR}}^{\mathsf{Th\text{-}opt(weak)}}(\ell, S) = 1. \tag{7.75}$$

Proof. Let $W \geqslant 1$. Consider an election with $S+\ell-1$ candidates $A_1, \dots, A_\ell$, $B_1, \dots, B_{S-1}$ and $V = W + S - 1$ votes: W votes on $\{A_1, \dots, A_\ell\}$ (this is the set $\mathcal{W}$) and 1 vote on each B_i. With Thiele's weak optimization method e.g. $\{A_1, B_1, \dots, B_{S-1}\}$ is elected, a bad outcome for same. Consequently, $\pi_{\mathsf{same}}^{\mathsf{Th\text{-}opt(weak)}}(\ell, S) \geqslant W/(W+S-1)$, and the result follows, using (6.3), since W is arbitrary. □

We can extend Theorem 7.31 to other weights in the form of an inequality.

Theorem 7.32. *For Thiele's optimization method with weights* $(w_k)_0^\infty$*:*

$$\pi_{\mathsf{EJR}}^{\mathsf{Th\text{-}opt}(\mathbf{w})}(\ell,S) \geqslant \pi_{\mathsf{PJR}}^{\mathsf{Th\text{-}opt}(\mathbf{w})}(\ell,S) \geqslant \pi_{\mathsf{same}}^{\mathsf{Th\text{-}opt}(\mathbf{w})}(\ell,S) \geqslant \frac{w_\ell^{-1}}{w_\ell^{-1}+S+1-\ell} = \frac{1}{w_\ell(S+1-\ell)+1}. \tag{7.76}$$

Proof. Let $W \geqslant 1$. Consider an election with $S+1$ candidates $\mathcal{A} = \{A_1,\dots,A_\ell\}$ and $\mathcal{B} = \{B_1,\dots,B_{S+1-\ell}\}$, and $V = W+S+1-\ell$ votes: W votes on $\mathcal{A}$ (this is the set $\mathcal{W}$) and 1 vote on $\{B_j\}$ for each $j \in [S+1-\ell]$.

Let $\hat{\mathcal{E}}$ be a set of S candidates. Then, the total satisfaction (A.4) if $\hat{\mathcal{E}}$ is elected is

$$\Psi(\hat{\mathcal{E}}) = \begin{cases} W\psi(\ell)+S-\ell, & \hat{\mathcal{E}} \supseteq \mathcal{A}, \\ W\psi(\ell-1)+S+1-\ell, & \hat{\mathcal{E}} \not\supseteq \mathcal{A} \end{cases} \tag{7.77}$$

The difference between the two values is

$$\bigl(W\psi(\ell)+S-\ell\bigr) - \bigl(W\psi(\ell-1)+S+1-\ell\bigr) = Ww_\ell - 1. \tag{7.78}$$

Hence, if $Ww_\ell < 1$, then the elected set $\mathcal{E} \not\supseteq \mathcal{A}$; this is a bad outcome for same, and thus

$$\pi_{\mathsf{same}}^{\mathsf{Th\text{-}opt}(\mathbf{w})} \geqslant \frac{|\mathcal{W}|}{V} = \frac{W}{W+S+1-\ell}. \tag{7.79}$$

We may here (as usual) let W be any rational number $< w_\ell^{-1}$, since we may multiply all numbers of votes by its denominator to obtain integer values; hence, (7.79) holds also if we replace W by w_ℓ^{-1}, which yields (7.76), using also (6.3). ☐

Remark 7.33. It follows from (7.65) that, as $S \to \infty$,

$$S\pi_{\mathsf{PJR}}^{\mathsf{Th\text{-}opt}(\mathbf{w})}(1,S) = S\pi_{\mathsf{same}}^{\mathsf{Th\text{-}opt}(\mathbf{w})}(1,S) \to \max_{k\geqslant 1}(kw_k). \tag{7.80}$$

In particular, if $w_k > 1/k$ for some k, then (7.65) or (7.80) implies that $\pi_{\mathsf{PJR}}^{\mathsf{Th\text{-}opt}(\mathbf{w})}(1,S) \geqslant 1/S$ for large S; thus, by (6.6), JR does not hold.

On the other hand, Theorem 7.30 shows that if $w_k \leqslant 1/k$ for every k, then $\pi_{\mathsf{PJR}}^{\mathsf{Th\text{-}opt}(\mathbf{w})}(1,S) < 1/S$; thus, by (6.6), JR holds.

Hence, as shown by Aziz et al. [1], JR holds for $\mathsf{Th\text{-}opt}(\mathbf{w})$ if and only if $w_k \leqslant 1/k$ for every k.

Furthermore, if $w_k < 1/k$ for some k, then (7.76) implies $\pi_{\mathsf{EJR}}^{\mathsf{Th\text{-}opt}(\mathbf{w})}(k,S) \geqslant \pi_{\mathsf{PJR}}^{\mathsf{Th\text{-}opt}(\mathbf{w})}(k,S) > k/S$ for large S, and thus $\mathsf{Th\text{-}opt}(\mathbf{w})$ does not satisfy PJR or EJR.

Consequently, recalling Theorem 7.6, the standard weights $w_k = 1/k$ are the only weights that yield a method satisfying PJR or EJR. For EJR, this too was shown in [1]; for PJR it was then shown in [42; 43].

7.7. **Thiele's addition method with general weights.** We next study Thiele's addition method with general weights $\mathbf{w} = (w_k)_0^\infty$. We assume as above that $w_1 = 1$ and that $w_1 \geqslant w_2 \geqslant \dots \geqslant 0$.

Some of the analysis in Section 7.4 extends to this case. Define $\alpha_n(\mathbf{w})$ as in Section 7.4, i.e., as the minimum number of (real-valued) votes in an election with (exactly) n candidates such that all are elected with scores $\geqslant 1$. Thus, $\alpha_n(\mathbf{w})$ can be computed by a linear programming problem of the type in (7.25)–(7.31), but with the fractions $1/k$ replaced by the weights w_k as coefficients. Clearly, for any $\mathbf{w}$,

$$\alpha_1(\mathbf{w}) = 1. \tag{7.81}$$

First, we note that Theorem 7.8 extends, with the same proof.

Theorem 7.34. *For Thiele's addition method with weights* $\mathbf{w}$ *and* $\ell = 1$*:*

$$\begin{aligned}\pi_{\mathsf{tactic}}^{\mathsf{Th\text{-}add}(\mathbf{w})}(1,S) = \pi_{\mathsf{same}}^{\mathsf{Th\text{-}add}(\mathbf{w})}(1,S) = \pi_{\mathsf{PJR}}^{\mathsf{Th\text{-}add}(\mathbf{w})}(1,S) = \pi_{\mathsf{EJR}}^{\mathsf{Th\text{-}add}(\mathbf{w})}(1,S)\\ = \frac{1}{1+\alpha_S(\mathbf{w})}, \qquad S \geqslant 1,\end{aligned} \tag{7.82}$$

where $\alpha_S(\mathbf{w})$ *is given by the version of* (7.32) *with weights* $\mathbf{w}$*.*

Also the proof of Theorem 7.13 extends, and yields the following.

Theorem 7.35. *The sequence* $\alpha_n(\mathbf{w})$ *is weakly increasing and subadditive. In particular,*

$$\alpha_n(\mathbf{w}) \leqslant \alpha_{n+1}(\mathbf{w}) \leqslant \alpha_n(\mathbf{w}) + 1, \qquad n \geqslant 1. \tag{7.83}$$

Furthermore,

$$\frac{n}{\psi(n)} \leqslant \alpha_n(\mathbf{w}) \leqslant n, \qquad n \geqslant 1, \tag{7.84}$$

where $\psi(n) = \sum_{k=1}^n w_k$*.* □

Corollary 7.36. *For Thiele's addition method with weights* $\mathbf{w}$ *and* $\ell = 1$*: For any scenario* $\mathfrak{S} = \mathsf{tactic}, \mathsf{same}, \mathsf{PJR}, \mathsf{EJR}$*,*

$$\frac{1}{S+1} \leqslant \pi_{\mathfrak{S}}^{\mathsf{Th\text{-}add}(\mathbf{w})}(1,S) \leqslant \frac{1}{1+S/\psi(S)}, \qquad S \geqslant 1. \tag{7.85}$$

Proof. By Theorem 7.34 and the inequalities (7.84). □

For the weak method, we have a simple result.

Theorem 7.37. *For Thiele's weak addition method and* $\ell = 1$*: For any scenario* $\mathfrak{S} = \mathsf{tactic}, \mathsf{same}, \mathsf{PJR}, \mathsf{EJR}$*,*

$$\pi_{\mathfrak{S}}^{\mathsf{Th\text{-}add(weak)}}(1,S) = \frac{1}{S+1}, \qquad S \geqslant 1. \tag{7.86}$$

Proof. For the weak method, $\psi(n) = 1$ for $n \geqslant 1$, and thus (7.84) yields

$$\alpha_n(\mathsf{weak}) = n, \qquad n \geqslant 1. \tag{7.87}$$

Thus the result follows by Theorem 7.34 (or Corollary 7.36). □

Example 7.38. For $n = 2$, the system (7.25)–(7.31) with weights becomes, after deleting a redundant inequality, cf. Footnote 17,

$$x_1, x_2, x_{12} \geqslant 0 \tag{7.88}$$

$$x_1 + x_{12} \geqslant x_2 + x_{12} \tag{7.89}$$

$$x_2 + w_2 x_{12} \geqslant 1. \tag{7.90}$$

Here (7.89) simplifies to $x_1 \geqslant x_2$. It is clear that the minimum $\alpha_2(\mathbf{w})$ of $x_1 + x_2 + x_{12}$ is obtained with $0 \leqslant w_2 x_{12} \leqslant 1$ (otherwise we could decrease x_{12}), and that we have equalities in (7.89) and (7.90) (otherwise we could decrease x_1 or x_2, respectively). Hence, the optimum is given by $x_1 = x_2 = 1 - w_2 x_{12}$, and we obtain

$$\alpha_2(\mathbf{w}) = \min_{0 \leqslant x \leqslant w_2^{-1}} \bigl(2(1 - w_2 x) + x\bigr) = \min\bigl(2, w_2^{-1}\bigr) = \begin{cases} 2, & 0 \leqslant w_2 \leqslant \frac{1}{2}, \\ w_2^{-1}, & \frac{1}{2} \leqslant w_2 \leqslant 1. \end{cases} \tag{7.91}$$

Theorem 7.34 and (7.91) yield, for $\mathfrak{S} = \mathsf{tactic}, \mathsf{same}, \mathsf{PJR}, \mathsf{EJR}$,

$$\pi_{\mathfrak{S}}^{\mathsf{Th\text{-}add}(\mathbf{w})}(1,2) = \begin{cases} \frac{1}{3}, & 0 \leqslant w_2 \leqslant \frac{1}{2}, \\ \frac{w_2}{1+w_2}, & \frac{1}{2} \leqslant w_2 \leqslant 1. \end{cases} \tag{7.92}$$

Remark 7.39. Theorem 7.37 shows in particular, by (6.6), that JR holds for Thiele's weak addition method. This was shown by Aziz et al. [1], who also proved that JR does not hold for any other weights. We show this in Example 7.40 below.

Example 7.40. Suppose $\mathbf{w} \neq \mathsf{weak}$; in other words (by our assumptions on $\mathbf{w}$) that $w_2 > 0$.

Consider an election with $n \geqslant 2$ candidates $C_0, \dots, C_{n-1}$, n seats, and $(n-1)^2$ votes: 1 vote for $\{C_0, C_i\}$ and $n-2$ votes for $\{C_i\}$ for each $i \in [n-1]$. All candidates tie with $n-1$ votes each. If C_0 is elected first, then the others are all elected with scores $n - 2 + w_2$. Hence,

$$\alpha_n(\mathbf{w}) \leqslant \frac{(n-1)^2}{n-2+w_2} = n - \frac{nw_2 - 1}{n-2+w_2}. \tag{7.93}$$

In particular, $\alpha_n(w) < n$ if $n > 1/w_2$. (Cf. the precise Example 7.11 and the final claim in Theorem 7.13 for the standard weights.)

Furthermore, by subadditivity (Theorem 7.35), for any $m \geqslant 1$,

$$\alpha_{mn}(\mathbf{w}) \leqslant m\alpha_n(\mathbf{w}) \leqslant mn - m\frac{nw_2 - 1}{n-2+w_2}. \tag{7.94}$$

In particular, for any $\mathbf{w} \neq \mathsf{weak}$, we can choose first n such that $\alpha_n(\mathbf{w}) < n$, and then m such that $\alpha_{mn}(\mathbf{w}) < mn - 1$. Then, with $S := mn$, Theorem 7.34 shows that

$$\pi_{\mathsf{PJR}}^{\mathsf{Th\text{-}add}(\mathbf{w})}(1,S) > \frac{1}{S}. \tag{7.95}$$

Consequently, by (6.6), JR does not hold for Thiele's addition method with weights $\mathbf{w} \neq \mathsf{weak}$.

For $\ell > 1$, we have even for the standard weights in Section 7.4 only partial results. Theorem 7.18 extends to general weights.

Theorem 7.41. *For Thiele's addition method with weights* $\mathbf{w}$ *and* $1 \leqslant \ell \leqslant S$,

$$\pi_{\mathsf{EJR}}^{\mathsf{Th\text{-}add}(\mathbf{w})}(\ell,S) \geqslant \pi_{\mathsf{PJR}}^{\mathsf{Th\text{-}add}(\mathbf{w})}(\ell,S) \geqslant \pi_{\mathsf{same}}^{\mathsf{Th\text{-}add}(\mathbf{w})}(\ell,S) \geqslant \frac{w_\ell^{-1}}{w_\ell^{-1}+\alpha_{S+1-\ell}(\mathbf{w})}. \tag{7.96}$$

If $w_\ell = 0$, *this is interpreted as*

$$\pi_{\mathsf{EJR}}^{\mathsf{Th\text{-}add}(\mathbf{w})}(\ell,S) = \pi_{\mathsf{PJR}}^{\mathsf{Th\text{-}add}(\mathbf{w})}(\ell,S) = \pi_{\mathsf{same}}^{\mathsf{Th\text{-}add}(\mathbf{w})}(\ell,S) = 1. \tag{7.97}$$

□

Proof. The same as for Theorem 7.18, taking w_ℓ^{-1} votes on $\mathcal{A}$ if $w_\ell > 0$, and otherwise an arbitrary number. □

In particular, (7.97) holds for Thiele's weak addition method. For the weak method, we have also a simple result for $\overline{\pi}_{\mathsf{tactic}}$.

Theorem 7.42. *For Thiele's weak addition method:*

$$\overline{\pi}_{\mathsf{tactic}}^{\mathsf{Th\text{-}add(weak)}}(\ell,S) = \frac{\ell}{S+1}, \qquad 1 \leqslant \ell \leqslant S. \tag{7.98}$$

Furthermore, if $2 \leqslant \ell \leqslant S$, *then*

$$\pi_{\mathsf{same}}^{\mathsf{Th\text{-}add(weak)}}(\ell,S) = \pi_{\mathsf{PJR}}^{\mathsf{Th\text{-}add(weak)}}(\ell,S) = \pi_{\mathsf{EJR}}^{\mathsf{Th\text{-}add(weak)}}(\ell,S) = 1. \tag{7.99}$$

Proof. First, (7.99) is a special case of (7.97).

Furthermore, (7.98) follows from Theorems 7.37 and 5.17(iv). □

8. Ordered ballots: PSC and STV

In this section we consider election methods with ordered ballots. First, recall that π_{party}, π_{same} and π_{tactic} in Definitions 4.11, 5.1 and 5.2 apply to ordered ballots too, and that the inequalities in Theorem 5.3 hold.

Furthermore, let us return to DPC and PSC, used as an example in Section 1, and define the corresponding threshold. (We use PSC to denote it, since PSC is used in recent literature, e.g. [15] and [2].)

Definition 8.1 (PSC, ordered ballots: π_{PSC})**.** *Suppose that all voters in* $\mathcal{W}$ *put the same* $m \geqslant \ell$ *candidates (not necessarily in the same order) as the top* m *candidates in their preference listings. A good outcome is when at least* ℓ *of these candidates are elected.*

As said in Section 1, the Droop Proportionality Criterion (DPC) formulated by Woodall [48] is equivalent to $\pi_{\mathsf{PSC}}(\ell, S) \leqslant \ell/(S+1)$.

Furthermore, Tideman [46] formulated what he called "proportionality for solid coalitions" (PSC), which easily is seen to be equivalent to $\pi_{\mathsf{PSC}}(\ell, S) \leqslant \ell/S-$, and thus in practice to $\pi_{\mathsf{PSC}}(\ell, S) < \ell/S$ for $\ell < S$, cf. "Theorem" 3.9.[22]

DPC and PSC are properties satisfied by STV, see Theorem 8.6 below, but more or less only by STV; the properties depend on the possibility of eliminations in the election method, so that, in Definition 8.1 (for sufficiently large $|\mathcal{W}|$), even if the votes of $\mathcal{W}$ are split on a large number m of candidates, they will by eliminations be concentrated on (at least) ℓ candidates that become elected. For election methods without eliminations, for example Phragmén's and Thiele's, this does not happen, and even a large set of voters $\mathcal{W}$ may fail to be represented by splitting their votes on too many candidates, even in the scenario PSC in Definition 8.1 above, see the trivial result in Theorem 9.3 below. Therefore, also a weak version of PSC requiring $m = \ell$ has been studied, e.g. by Aziz and Lee [2]. This leads to the following definition.

Definition 8.2 (weak PSC, ordered ballots: π_{wPSC})**.** *Suppose that all voters in $\mathcal{W}$ put the same ℓ candidates (not necessarily in the same order) as the top ℓ candidates in their preference listings. A good outcome is when all these candidates are elected.*

Remark 8.3. The property called "weak q-PSC" in Aziz and Lee [2] is, with $q = \kappa V$, equivalent to $\pi_{\mathsf{wPSC}}(\ell, S) < \ell\kappa$, or, more precicely, see Remark 2.8, $\pi_{\mathsf{wPSC}}(\ell, S) \leqslant (\ell\kappa)-$. Furthermore, the property called "Proportionality for Solid Coalitions" in [15] uses this (weak) version (and is the same as weak (V/S)-PSC in [2]); hence it is in our notation $\pi_{\mathsf{wPSC}}(\ell, S) \leqslant (\ell/S)-$ (or, in practice, $\pi_{\mathsf{wPSC}}(\ell, S) < \ell/S$, $\ell < S$).[23] Moreover, three further definitions in [15] also fit the weak version: "Solid coalitions" is the special case $\ell = 1$: $\pi_{\mathsf{wPSC}}(1, S) < 1/S$; "Unanimity" is the case $\ell = S$: $\pi_{\mathsf{wPSC}}(S, S) \leqslant 1-$; "Fixed Majority" is $\pi_{\mathsf{wPSC}}(S, S) \leqslant 1/2$.

Theorem 8.4. *For any election method with ordered ballots, and $1 \leqslant \ell \leqslant S$,*

$$\pi_{\mathsf{party}}(\ell, S) \leqslant \pi_{\mathsf{same}}(\ell, S) \leqslant \pi_{\mathsf{wPSC}}(\ell, S) \leqslant \pi_{\mathsf{PSC}}(\ell, S). \tag{8.1}$$

[22] Tideman [46] refers to Dummett [14, p. 282], and attributes both the property and the name "proportionality for solid coalitions" to him. This seems inaccurate. Dummett [14, pp. 282–283] discusses this type of property, but he does not give it a name (and does not use the term "solid coalition" used by Tideman [46]). Moreover, Dummett [14] talks about (in our notation) a set of ℓQ voters, where Q is the quota, so ignoring effects of rounding the quota, Dummett's property is the same as DPC in Woodall [48], and is thus essentially equivalent to $\pi_{\mathsf{PSC}}(\ell, S) \leqslant \ell/(S+1)$. See also Theorem 8.6.

[23] As remarked by [2], [15] inaccurately attributes this weak version of PSC to Dummett [14]; furthermore, [15] says that Dummett's original proposal is, in our notation, $\pi_{\mathsf{wPSC}}(\ell, S) \leqslant (\ell/S)-$. This seems to be doubly inaccurate, since Dummett [14, p. 283] uses the stronger PSC and the threshold $\ell Q/V \approx \ell/(S+1)$, see Footnote 22.

Proof. The first inequality is (5.1) in Theorem 5.3.

The second inequality follows since an instance of the scenario same also is an instance of wPSC, and if it is bad for same then it is bad for wPSC.

Similarly, the third inequality follows because an instance of wPSC with bad outcome is an instance of PSC with bad outcome. ☐

Dummett [14] and Woodall [48] both stress that STV satisfies the DPC type of property. In our notation, we can state their results as

$$\pi^{\mathsf{STV}}_{\mathsf{PSC}}(\ell,S) \leqslant \frac{\ell}{S+1}, \tag{8.2}$$

if the Droop quota is used and we ignore rounding. More precisely, [48] assumes that the unrounded Droop quota $V/(S+1)$ is used, and then (8.2) holds. On the other hand, [14] assumes that the quota Q is larger (as it usually is in practice), and states the result that ℓQ voters are enough to guarantee ℓ seats. This, in fact, holds for any quota $Q > V/(S+1)$, and can be stated as follows.

Theorem 8.5 (Dummett [14])**.** *Any (reasonable) version of STV with quota $Q = rV$, where $r > 1/(S+1)$, satisfies*

$$\pi^{\mathsf{STV}}_{\mathsf{PSC}}(\ell,S) < r\ell, \qquad 1 \leqslant \ell \leqslant S. \tag{8.3}$$

Thus, a set $\mathcal{W}$ of at least ℓQ voters forming a "solid coalition" in the sense of PSC *will get at least ℓ of its candidates elected.*

This is easy to see directly, but we instead prove the following more precise result, yielding Theorem 8.5 as a corollary.

Theorem 8.6. *Consider any (reasonable) version of STV with quota $Q = V/(S+\delta)$, where $\delta \in [0,1]$, Then, for $1 \leqslant \ell \leqslant S$,*

$$\begin{aligned}
\pi^{\mathsf{STV}}_{\mathsf{party}}(\ell,S) &= \pi^{\mathsf{STV}}_{\mathsf{same}}(\ell,S) = \pi^{\mathsf{STV}}_{\mathsf{wPSC}}(\ell,S) = \pi^{\mathsf{STV}}_{\mathsf{PSC}}(\ell,S)\\
&= \frac{\ell(S+2-\ell)-1+\delta}{(S+\delta)(S+2-\ell)} = \frac{\ell}{S+\delta} - \frac{1-\delta}{(S+\delta)(S+2-\ell)}\\
&= \frac{\ell}{S+1} + \frac{(1-\delta)(\ell-1)(S+1-\ell)}{(S+\delta)(S+1)(S+2-\ell)},
\end{aligned} \tag{8.4}$$

$$\overline{\pi}^{\mathsf{STV}}_{\mathsf{tactic}}(\ell,S) = \frac{\ell}{S+1}. \tag{8.5}$$

As a special case,

$$\pi^{\mathsf{STV}}_{\mathsf{party}}(1,S) = \pi^{\mathsf{STV}}_{\mathsf{tactic}}(1,S) = \pi^{\mathsf{STV}}_{\mathsf{same}}(1,S) = \pi^{\mathsf{STV}}_{\mathsf{wPSC}}(1,S) = \pi^{\mathsf{STV}}_{\mathsf{PSC}}(1,S) = \frac{1}{S+1}. \tag{8.6}$$

In particular, with the (unrounded) Droop quota $Q = V/(S+1)$,

$$\pi^{\mathsf{STV}}_{\mathsf{party}}(\ell,S) = \pi^{\mathsf{STV}}_{\mathsf{tactic}}(\ell,S) = \pi^{\mathsf{STV}}_{\mathsf{same}}(\ell,S) = \pi^{\mathsf{STV}}_{\mathsf{wPSC}}(\ell,S) = \pi^{\mathsf{STV}}_{\mathsf{PSC}}(\ell,S) = \frac{\ell}{S+1}. \tag{8.7}$$

Furthermore, also with a rounded Droop quota $Q = V/(S+1) + O(1)$ *with* $Q \geqslant V/(S+1)$,

$$\bar{\pi}^{\mathsf{STV}}_{\mathsf{party}}(\ell,S) = \bar{\pi}^{\mathsf{STV}}_{\mathsf{tactic}}(\ell,S) = \bar{\pi}^{\mathsf{STV}}_{\mathsf{same}}(\ell,S) = \bar{\pi}^{\mathsf{STV}}_{\mathsf{wPSC}}(\ell,S) = \bar{\pi}^{\mathsf{STV}}_{\mathsf{PSC}}(\ell,S) = \frac{\ell}{S+1}. \tag{8.8}$$

Proof. In the party list case, it is easy to see that STV becomes equivalent to the quota method with quota Q, which we denote by $\mathsf{Q}(\delta)$. Hence $\pi^{\mathsf{STV}}_{\mathsf{party}}(\ell,S) = \pi^{\mathsf{Q}(\delta)}_{\mathsf{party}}(\ell,S)$, which is given by (4.8). Consequently, recalling also the general inequalities (8.1), to prove (8.4), it remains only to show that $\pi^{\mathsf{STV}}_{\mathsf{PSC}}(\ell,S) \leqslant \pi^{\mathsf{STV}}_{\mathsf{party}}(\ell,S)$. In other words, somewhat informally, we want to show that the worst case in the party list case (or, equivalently, for the quota method), also is the worst case for PSC.

Thus, consider an instance of PSC with a bad outcome, and let $\mathcal{A}$ be the set of $m = |\mathcal{A}| \geqslant \ell$ common top candidates for $\mathcal{W}$; thus, at most $\ell - 1$ of the candidates in $\mathcal{A}$ are elected. Let $\mathcal{D} := \mathcal{E} \setminus \mathcal{A}$, so $|\mathcal{D}| \geqslant S + 1 - \ell$.

Suppose that during some stage of the counting process, there are exactly ℓ candidates in $\mathcal{A}$ that have not yet been eliminated, and that the next event is that one of these, say A_1, is eliminated. Let $k \leqslant \ell - 1$ be the number of candidates from $\mathcal{A}$ elected so far; thus there are $\ell - k > 0$ candidates from $\mathcal{A}$ still remaining (including A_1). Let further m be the number of candidates in $\mathcal{C} \setminus \mathcal{A}$ that have been elected so far. There are also at least $S + 1 - \ell - m$ remaining candidates in $\mathcal{C}\setminus\mathcal{A}$, since otherwise all remaining candidates would be elected, and thus $|\mathcal{A} \cap \mathcal{E}| = \ell$, contrary to our assumption.

Let x be the current number of votes counted for A_1. Thus $x < Q$, since A_1 is not elected. Since there is still at least one remaining candidate from $\mathcal{A}$, the (remaining) ballots in $\mathcal{W}$ are all counted for some candidate in $\mathcal{A}$ (by the assumption in PSC). Each elected candidate accounts for Q votes, and each of the $\ell - k - 1$ remaining candidate besides A_1 has currently less than Q votes; hence

$$|\mathcal{W}| \leqslant kQ + (\ell - k - 1)Q + x = (\ell - 1)Q + x \tag{8.9}$$

and thus

$$x \geqslant |\mathcal{W}| - (\ell - 1)Q. \tag{8.10}$$

Similarly, the m elected candidates in $\mathcal{C} \setminus \mathcal{A}$ account for mQ votes, and the at least $S + 1 - \ell - m$ remaining candidates in $\mathcal{C} \setminus \mathcal{A}$ have at least x votes each (otherwise A_1 would not be the next to be eliminated). Thus, since so far only votes from $\mathcal{W}^{\mathsf{c}}$ are counted for candidates in $\mathcal{C} \setminus \mathcal{A}$,

$$\begin{aligned} |\mathcal{W}^{\mathsf{c}}| &\geqslant mQ + (S + 1 - \ell - m)x \geqslant (S + 1 - \ell)x \\ &\geqslant (S + 1 - \ell)\bigl(|\mathcal{W}| - (\ell - 1)Q\bigr). \end{aligned} \tag{8.11}$$

Consequently,

$$V = |\mathcal{W}| + |\mathcal{W}^{\mathsf{c}}| \geqslant (S + 2 - \ell)|\mathcal{W}| - (S + 1 - \ell)(\ell - 1)Q. \tag{8.12}$$

and thus, recalling $Q = V/(S+\delta)$,

$$\frac{|\mathcal{W}|}{V} \leqslant \frac{1}{S+2-\ell} + \frac{(\ell-1)(S+1-\ell)}{(S+\delta)(S+2-\ell)} \tag{8.13}$$

Note that the right-hand side of (8.13) equals, by simple algebra, $\pi^{\mathsf{STV}}_{\mathsf{party}}(\ell, S) = \pi^{\mathsf{Q}(\delta)}_{\mathsf{party}}(\ell, S)$ given by (4.8).

We have shown that if the number of elected + remaining candidates from $\mathcal{A}$ drops below ℓ, then (8.13) must hold. Suppose now that there is a bad outcome such that this does not happen. Then there is at least one remaining candidate from $\mathcal{A}$ throughout the counting, so votes from $\mathcal{W}$ are only counted for candidates in $\mathcal{A}$. Furthermore, all S elected have to reach the quota, since otherwise at the end all remaining candidates would have been elected. In particular, at least $S+1-\ell$ candidates from $\mathcal{C} \setminus \mathcal{A}$ are elected with Q votes each, all coming from $\mathcal{W}^{\mathsf{c}}$, and thus

$$|\mathcal{W}^{\mathsf{c}}| \geqslant (S+1-\ell)Q. \tag{8.14}$$

Hence,

$$\begin{aligned} |\mathcal{W}| = V - |\mathcal{W}^{\mathsf{c}}| \leqslant V - (S+1-\ell)Q = (S+\delta)Q - (S+1-\ell)Q \\ = (\ell+\delta-1)Q \leqslant \ell Q. \end{aligned} \tag{8.15}$$

It follows from (8.15) and (8.14) that (8.10) and (8.11) hold with $x = Q$; thus (8.12) and (8.13) hold in this case too.

We have shown that (8.13) holds for every bad outcome. Consequently, $\pi^{\mathsf{STV}}_{\mathsf{PSC}}(\ell, S) \leqslant \pi^{\mathsf{STV}}_{\mathsf{party}}(\ell, S)$, which completes the proof of (8.4).

For $\overline{\pi}^{\mathsf{STV}}_{\mathsf{tactic}}$, consider first the case $\ell = 1$, where by (5.2) and (8.4),

$$\pi^{\mathsf{STV}}_{\mathsf{tactic}}(1, S) \leqslant \pi^{\mathsf{STV}}_{\mathsf{same}}(1, S) = \frac{1}{S+1}. \tag{8.16}$$

Hence, (8.5) follows by Theorem 5.17(iv).

We immediately obtain as special cases, taking $\ell = 1$ or $\delta = 1$, (8.6) and (8.7) with π_{tactic} replaced by $\overline{\pi}_{\mathsf{tactic}}$. The equalities for π_{tactic} in (8.6) and (8.7) then follow from (2.3) and (5.2), cf. (8.16).

Finally, if $Q = V/(S+1) + O(1)$, define $\delta := V/Q - S$, so that $Q = V/(S+\delta)$, and note that $\delta \leqslant 1$ and $\delta = 1 + o(1)$ as $V \to \infty$. Hence, (8.4) implies, for any of the scenarios there, that for a bad outcome, $|\mathcal{W}|/V \leqslant \ell/(S+1) + o(1)$ as $V \to \infty$, and thus $\overline{\pi}^{\mathsf{STV}}(\ell, S) \leqslant \ell/(S+1)$. A corresponding lower bound follows easily by the party case, and thus (8.8) follows. □

Proof of Theorem 8.5. Let $\delta := 1/r - S < 1$, so $r = 1/(S+\delta)$. Then, (8.4) holds and implies (8.3). □

9. Phragmén's and Thiele's ordered methods

We continue the study of the thresholds $\pi_{\mathsf{party}}, \pi_{\mathsf{tactic}}, \pi_{\mathsf{same}}, \pi_{\mathsf{wPSC}}, \pi_{\mathsf{PSC}}$ for election methods with ordered ballots. In this section we consider Phragmén's and Thiele's election methods, see Appendices A.3.2 and A.3.3; in Section 10 we study Borda methods.

Problem 9.1. Further election methods for ordered ballots are described and studied in e.g. [15; 16; 2]. Study $\pi_{\mathsf{party}}, \pi_{\mathsf{tactic}}, \pi_{\mathsf{same}}, \pi_{\mathsf{wPSC}}, \pi_{\mathsf{PSC}}$ for them![24]

9.1. **Two simple cases.** We note first that in the party list case, we have the same result as for Phragmén's and Thiele's methods for unordered ballots, Theorem 7.2.

Theorem 9.2. *For Phragmén's and Thiele's ordered methods:*

$$\pi^{\mathsf{Phr\text{-}o}}_{\mathsf{party}}(\ell,S) = \pi^{\mathsf{Th\text{-}o}}_{\mathsf{party}}(\ell,S) = \pi^{\mathsf{D'H}}_{\mathsf{party}}(\ell,S) = \frac{\ell}{S+1}. \qquad 1 \leqslant \ell \leqslant S. \tag{9.1}$$

Proof. It is easy to see that in the party list case, both methods reduce to D'Hondt's method [26, Theorem 11.1]. Hence, the result follows from (4.3). □

Next we note that, as said in Section 8, Phragmén's and Thiele's methods do not satisfy any PSC condition, since they do not eliminate candidates.

Theorem 9.3. *For Phragmén's and Thiele's ordered methods:*

$$\pi^{\mathsf{Phr\text{-}o}}_{\mathsf{PSC}}(\ell,S) = \pi^{\mathsf{Th\text{-}o}}_{\mathsf{PSC}}(\ell,S) = 1, \qquad 1 \leqslant \ell \leqslant S. \tag{9.2}$$

Proof. Consider an election where each voter in $\mathcal{W}$ also is a candidate, and votes for herself first, followed by all others in $\mathcal{W}$ (in any order). Suppose also that there are S other candidates $\mathcal{B}$, and $2S$ other voters $\mathcal{W}^{\mathsf{c}}$, with each candidate in $\mathcal{B}$ being the first and only name on 2 ballots from $\mathcal{W}^{\mathsf{c}}$. This fits the scenario PSC, provided $|\mathcal{W}| \geqslant \ell$, and obviously both Phragmén's and Thiele's methods will elect $\mathcal{B}$, which is a bad outcome. Hence,

$$\pi^{\mathsf{Phr\text{-}o}}_{\mathsf{PSC}}(\ell,S), \pi^{\mathsf{Th\text{-}o}}_{\mathsf{PSC}}(\ell,S) \geqslant \frac{|\mathcal{W}|}{|\mathcal{W}|+2S} \tag{9.3}$$

for any $|\mathcal{W}| \geqslant \ell$, and the result follows. □

We will see below (Theorem 9.4 and Corollary 9.12) that the weak PSC $\pi_{\mathsf{wPSC}}(\ell,S) < \ell/S$ is satisfied by Phragmén's method but not by Thiele's.

[24]Some inequalities for π_{PSC} and π_{wPSC} follow from results in [15; 2] showing whether or not certain methods satisfy (weak) PSC and some related criteria, see Remark 8.3.

9.2. **Phragmén's ordered method.** Phragmén's method has the optimal result for weak PSC, as well as for the scenarios same and tactic.

Theorem 9.4 ([26])**.** *For Phragmén's ordered method: For* $1 \leqslant \ell \leqslant S$*,*

$$\pi^{\mathsf{Phr\text{-}o}}_{\mathsf{party}}(\ell, S) = \pi^{\mathsf{Phr\text{-}o}}_{\mathsf{tactic}}(\ell, S) = \pi^{\mathsf{Phr\text{-}o}}_{\mathsf{same}}(\ell, S) = \pi^{\mathsf{Phr\text{-}o}}_{\mathsf{wPSC}}(\ell, S) = \frac{\ell}{S+1}. \tag{9.4}$$

Proof. The result for $\pi^{\mathsf{Phr\text{-}o}}_{\mathsf{party}}$ is in Theorem 9.2.

We next show the upper bound for $\pi^{\mathsf{Phr\text{-}o}}_{\mathsf{wPSC}}$. This follows by almost the same proof as for Theorem 7.3.

Let $\mathcal{W}$ be a set of voters and $\mathcal{A}$ a set of candidates as in Definition 8.2. Thus $|\mathcal{A}| = \ell$ and each voter in $\mathcal{W}$ votes for the set $\mathcal{A}$ in some order, possibly followed by some other candidates. Suppose that the outcome is bad, i.e., $k := |\mathcal{A} \cap \mathcal{E}| < \ell$; in other words, at least one candidate in $\mathcal{A}$ is not elected.

We use the formulation with loads in Appendix A.3.2, and let $t = t^{(S)}$ be the final maximum load of a ballot. Let x_i be the final load on ballot i, and let, as in the proof of Theorem 7.3, the free voting power of the ballot be $t - x_i$.

The k elected candidates in $\mathcal{A} \cap \mathcal{E}$ together give load k. Voters in $\mathcal{W}$ have not contributed to the election of any other candidate, and thus has no load from any other candidate. Thus (7.3) holds.

Since at least one candidate in $\mathcal{A}$ is not elected, each ballot in $\mathcal{A}$ has when the election finishes a current top candidate that belongs to $\mathcal{A} \setminus \mathcal{E}$. The total free voting power assigned to each candidate is at most 1, since otherwise this candidate would have been elected in the last step (if not earlier), with a lower $t^{(S)}$. Hence, we have, instead of (7.4),

$$\sum_{i \in \mathcal{W}} (t - x_i) \leqslant |\mathcal{A} \setminus \mathcal{E}| = \ell - k, \tag{9.5}$$

but this is, combined with (7.3), enough to yield (7.5).

The same argument as in the proof of Theorem 7.3 now gives (7.6)–(7.8), and thus

$$\pi^{\mathsf{Phr\text{-}o}}_{\mathsf{wPSC}}(\ell, S) \leqslant \frac{\ell}{S+1}. \tag{9.6}$$

Next, (9.6), the result for $\pi^{\mathsf{Phr\text{-}o}}_{\mathsf{party}}$, and (8.1) imply the result in (9.4) for $\pi^{\mathsf{Phr\text{-}o}}_{\mathsf{same}}$ and $\pi^{\mathsf{Phr\text{-}o}}_{\mathsf{wPSC}}$.

Finally, this shows by (5.2) that $\pi^{\mathsf{Phr\text{-}o}}_{\mathsf{tactic}}(\ell, S) \leqslant \ell/(S+1)$. Equality holds by Theorem 5.17(v). ∎

9.3. **Thiele's ordered method.** Although Theorem 9.2 for party lists holds for Thiele's method as well as for Phragmén's, Theorem 9.4 does not; the reason is that Thiele's method invites to tactical voting, where a party may gain seats by a splitting their votes on different (carefully chosen) lists. Cf. Section 7.4 where the same phenomen is seen for Thiele's addition method for unordered ballots. We give first one example from the commission report [19].

Example 9.5. Thiele's ordered method with $S = 2$ seats, and 100 voters voting

61 AB
39 CD

It is easy to see that AC are elected. (This is a party list case.)

However, suppose that instead the larger party split their votes as follows:

41 AB
20 B (or BA)
39 CD

Then, the first seat goes to A; for the second seat, B has $41/2 + 20 = 40.5$ votes, and beats C. Elected: AB.

Thus, if $\mathcal{W}$ is the set of 39 CD voters, $\mathcal{W}$ gets no candidate elected. This is an instance of same (and thus of wPSC) for $\ell = 1$ and $S = 2$ with a bad outcome. Hence,

$$\pi^{\mathsf{Th\text{-}o}}_{\mathsf{wPSC}}(1, 2) \geqslant \pi^{\mathsf{Th\text{-}o}}_{\mathsf{same}}(1, 2) \geqslant \frac{39}{100} = 0.39 > \frac{\ell}{S+1}. \tag{9.7}$$

We can calculate the thresholds exactly for Thiele's ordered method. We begin with some preliminaries. For convenience, we let in the remainder of this section, as in Section 7.4, the "number of votes" be arbitrary positive real numbers; see Remarks 2.2 and 2.3, and note that this does not affect the results since the method is homogeneous.

When, as in the next lemma, we do not specify the number of seats S, we just assume that it is large enough.

Lemma 9.6. *Let a_n, $n \geqslant 1$, be the smallest total number of (real-values) votes in an election by Thiele's ordered method where there are n candidates $A_1, \dots, A_n$, and no others, and each of them is elected with score at least 1.*

(i) *Consider an election by Thiele's ordered method where there are n candidates $A_1, \dots, A_n$, and perhaps others, and each A_i is elected with score at least $t \geqslant 0$. Let W be the number of voters that have voted for at least one of $A_1, \dots, A_n$. Then $W \geqslant a_n t$.*

(ii) *Define also a sequence b_n, $n \geqslant 1$, by the recursion*

$$b_n := 1 - \sum_{i=1}^{n-1} \frac{b_i}{n+1-i}, \qquad n \geqslant 1, \tag{9.8}$$

or, equivalently,

$$\sum_{i=1}^{n} \frac{b_i}{n+1-i} = 1, \qquad n \geqslant 1. \tag{9.9}$$

Then

$$a_n = \sum_{i=1}^{n} b_i. \tag{9.10}$$

Furthermore, a_n and b_n are given by the Taylor coefficients

$$a_n = [z^n]\frac{z^2}{-(1-z)^2\log(1-z)}, \tag{9.11}$$

$$b_n = [z^n]\frac{z^2}{-(1-z)\log(1-z)}. \tag{9.12}$$

Asymptotically, as $n \to \infty$,

$$a_n \sim \frac{n}{\log n}, \tag{9.13}$$

$$b_n \sim \frac{1}{\log n}. \tag{9.14}$$

The first numbers a_n and b_n are given in Table 1.

n	1	2	3	4	5	6
a_n	1	$\frac{3}{2}$	$\frac{23}{12}$	$\frac{55}{24}$	$\frac{1901}{720}$	$\frac{4277}{1440}$
b_n	1	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{3}{8}$	$\frac{251}{720}$	$\frac{95}{288}$
c_n	1	2	4	6	9	12

TABLE 1. The numbers a_n, b_n and c_n in Lemmas 9.6 and 9.9.

Before proving Lemma 9.6, we prove a technical result.

Lemma 9.7. *The numbers b_n defined by* (9.8) *form a strictly decreasing sequence with* $0 < b_n \leqslant 1$.

Proof. We first give a simple proof by induction that $b_n > 0$ (although this also follows from the argument below). Thus, assume $n \geqslant 1$ and $b_1, \dots, b_n > 0$. Then, by (9.9),

$$\sum_{i=1}^{n} \frac{b_i}{n+2-i} < \sum_{i=1}^{n} \frac{b_i}{n+1-i} = 1, \tag{9.15}$$

and thus (9.8) yields $b_{n+1} > 0$, completing the induction proof.

Thus $b_n > 0$, and hence $b_n \leqslant 1$ by (9.8).

To show that b_n is decreasing is (as far as we know) less elementary. Let

$$B(z) := \sum_{n=1}^{\infty} b_n z^n \tag{9.16}$$

be the generating function of b_n. Since $-\log(1-z) = \sum_{m\geqslant 1} z^m/m$, (9.9) is equivalent to

$$-\log(1-z)B(z) = \sum_{n=2}^{\infty} z^n = \frac{z^2}{1-z} \tag{9.17}$$

and thus

$$B(z) = -\frac{z^2}{(1-z)\log(1-z)}. \tag{9.18}$$

Let $b_0 := 0$ and

$$C(z) := \sum_{n=1}^{\infty}\bigl(b_{n-1}-b_n\bigr)z^n = (z-1)B(z) = \frac{z^2}{\log(1-z)}. \tag{9.19}$$

Note that $C(z)$ is (extends to) an analytic function in $\mathcal{D} := \mathbb{C}\setminus[1,\infty)$. Thus, Cauchy's integral formula yields, for any simple closed curve γ in $\mathcal{D}$ that encircles 0 in the positive direction,

$$b_{n-1}-b_n = \frac{1}{2\pi\mathrm{i}}\oint_\gamma C(z)\frac{\mathrm{d}z}{z^{n+1}} = \frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{z^{1-n}}{\log(1-z)}\,\mathrm{d}z. \tag{9.20}$$

Let $\varepsilon>0$ and $R>1$, and let γ consist of: an arc on the circle $|z|=\sqrt{R^2+\varepsilon^2}$, going from $R+\varepsilon\mathrm{i}$ to $R-\varepsilon\mathrm{i}$ in the positive direction; the straight line from $R-\varepsilon\mathrm{i}$ to $1-\varepsilon\mathrm{i}$; the semicircle $\{1+\varepsilon e^{-\mathrm{i}t} : \frac{\pi}{2}\leqslant t\leqslant\frac{3\pi}{2}\}$; the straight line from $1+\varepsilon\mathrm{i}$ to $R+\varepsilon\mathrm{i}$. For $n\geqslant 2$, we then let first $\varepsilon\to 0$ and then $R\to\infty$, and obtain, using simple estimates for the circular parts,

$$\begin{aligned} b_{n-1}-b_n &= \frac{1}{2\pi\mathrm{i}}\biggl(\int_\infty^1 \frac{x^{1-n}\,\mathrm{d}x}{\log(x-1)+\pi\mathrm{i}} + \int_1^\infty \frac{x^{1-n}\,\mathrm{d}x}{\log(x-1)-\pi\mathrm{i}}\biggr) \\ &= \int_1^\infty \frac{x^{1-n}\,\mathrm{d}x}{\log^2(x-1)+\pi^2}. \end{aligned} \tag{9.21}$$

This proves that $b_{n-1}-b_n>0$ for $n\geqslant 2$, which completes the proof. ∎

Proof of Lemma 9.6. We begin by giving a good strategy; we will later show that it is optimal. (The strategy used by the AB party in Example 9.5 is an example, apart from rounding to integers.)

Consider an election with b_i votes for $A_iA_{i+1}\cdots A_n$, for each $i\leqslant n$. (Note that $b_i>0$ by Lemma 9.7.) The total number of votes is $\widehat{a}_n := \sum_{i=1}^n b_i$. We claim that, by induction, $A_1,\dots,A_n$ will be elected in this order with score 1 each. (Assuming $S\geqslant n$.) In fact, if $A_1,\dots,A_{k-1}$ already have been elected, then A_k has score, using (9.9),

$$\sum_{i=1}^{k}\frac{b_i}{k-i+1} = 1, \tag{9.22}$$

while if $j>k$, then A_j has score $b_j<1$. Hence A_k is elected next, which verifies the induction step. Thus $a_n\leqslant\widehat{a}_n$.

Conversely, consider an arbitrary election as in (i). Let $\mathcal{W}$ be the set of voters that have voted for some A_i. Consider a ballot σ. Suppose that there are m candidates $A_i\in\mathcal{A}$ that are elected with the help of σ, i.e. the candidates $A_i\in\sigma\cap\mathcal{E}$ such that all candidates before A_i on σ are elected before A_i. Let these candidates by $A_{i_1},\dots,A_{i_m}$, with $1\leqslant i_1<i_2<\dots<i_m$, and let their positions on the ballot σ be $j_1,\dots,j_m$. Note that $1\leqslant j_1<\dots<j_m$, since $A_1,\dots,A_n$ are elected in order. Define the weight $w(\sigma,i)$ for

$i \in [n]$ by

$$w(\sigma, i_k) := \frac{1}{j_k} b_{n+1-i_k}, \tag{9.23}$$

$$w(\sigma, i) := 0, \qquad i \notin \{i_1, \dots, i_k\}. \tag{9.24}$$

If $i = i_k$, then the ballot σ is worth $1/j_k$ votes when A_i is elected. Thus, for each $i \in [n]$, $w(\sigma, i)$ equals b_{n+1-i} times the score contributed by σ to the election of A_i; hence, $\sum_\sigma w(\sigma, i)$ is b_{n+1-i} times the score of A_i when elected, and thus, noting that $w(\sigma, i) = 0$ unless $\sigma \in \mathcal{W}$,

$$\sum_{\sigma \in \mathcal{W}} w(\sigma, i) = \sum_{\sigma \in \mathcal{V}} w(\sigma, i) \geqslant b_{n+1-i}. \tag{9.25}$$

On the other hand, returning to the ballot σ, since $i_1 < \dots < i_m \leqslant n$, we have $i_k \leqslant n-m+k$ and thus $n+1-i_k \geqslant m+1-k$. Hence, using Lemma 9.7,

$$b_{n+1-i_k} \leqslant b_{m+1-k}. \tag{9.26}$$

Furthermore, $j_k \geqslant k$. Hence, the total weight on σ is, using also (9.9),

$$\sum_{i=1}^{n} w(\sigma, i) = \sum_{k=1}^{m} w(\sigma, i_k) = \sum_{k=1}^{m} \frac{1}{j_k} b_{n+1-i_k} \leqslant \sum_{k=1}^{m} \frac{1}{k} b_{m+1-k} = 1. \tag{9.27}$$

Consequently, summing (9.27) over $\sigma \in \mathcal{W}$ and (9.25) over i,

$$W = |\mathcal{W}| = \sum_{\sigma \in \mathcal{W}} 1 \geqslant \sum_{\sigma \in \mathcal{W}} \sum_{i=1}^{n} w(\sigma, i) \geqslant \sum_{i=1}^{n} b_{n+1-i} = \widehat{a}_n. \tag{9.28}$$

In particular, the definition of a_n shows $a_n \geqslant \widehat{a}_n$, and thus $a_n = \widehat{a}_n$, which is (9.10). (Thus, the strategy given above is optimal.) Hence, (9.28) also shows (i), using homogeneity.

Finally, (9.12) follows by (9.16) and (9.18), and (9.11) by (9.10) and (9.18). The asymptotic formulas (9.13)–(9.14) follow by (9.11)–(9.12) and singularity analysis, see [18, Theorem VI.2]. □

Remark 9.8. By summing (9.21) for $n \geqslant i+1$, and then summing again for $i = 1, \dots k$, we obain the integral formulas

$$b_n = \int_1^\infty \frac{x^{1-n}\,\mathrm{d}x}{(x-1)\bigl(\log^2(x-1)+\pi^2\bigr)}, \tag{9.29}$$

$$a_n = \int_1^\infty \frac{x(1-x^{-n})\,\mathrm{d}x}{(x-1)^2\bigl(\log^2(x-1)+\pi^2\bigr)}. \tag{9.30}$$

It is also possible to derive (9.13)–(9.14) from these.

Lemma 9.9. *Let c_n, $n \geqslant 1$, be the smallest real number such that in any election by Thiele's ordered method where there are n candidates $A_1, \dots, A_n$,* *and each voter votes for all of them in some order, each A_i is elected with score at least* 1*. Then*

$$c_n = \Big\lfloor \frac{n+1}{2} \Big\rfloor \Big\lceil \frac{n+1}{2} \Big\rceil = \begin{cases} m^2, & n = 2m-1, \\ m(m+1), & n = 2m. \end{cases} \tag{9.31}$$

The first numbers c_n are given in Table 1.

Proof. Consider any election with n candidates $A_1, \dots, A_n$, and each of the V voters voting for all of them in some order. When $k \geqslant 0$ of the candidates have been elected, there are $n-k$ candidates left, and each ballot is worth at least $1/(k+1)$, so at least one candidate has score $\geqslant V/\big((k+1)(n-k)\big)$.

Conversely, for a given $k \in \{0, \dots, n-1\}$, consider an election where each voter votes for $A_1, \dots, A_k$ in order followed by the remaining candidates in some order, with each of the remaining candidates in place $k+1$ by $V/(n-k)$ voters. Then, in round $k+1$, each remaining candidate has score $V/\big((k+1)(n-k)\big)$.

It follows that c_n is given by

$$\min_{0 \leqslant k < n} \frac{c_n}{(k+1)(n-k)} = 1, \tag{9.32}$$

and thus

$$c_n = \max_{0 \leqslant k < n} (k+1)(n-k). \tag{9.33}$$

The maximum is attained for $k = \lfloor (n-1)/2 \rfloor$, and (9.31) follows. □

Lemma 9.10. *Consider an election by Thiele's ordered method where there are n candidates $A_1, \dots, A_n$, and perhaps others, and each A_i is elected with score at least $t \geqslant 0$. Add an arbitrary set on new votes (and perhaps new candidates). Then, throughout the counting, as long as not all candidates A_i have been elected, at least one of them has score $\geqslant t$.*

Proof. This is perhaps not completely obvious since the order in which $A_1, \dots, A_n$ are elected might be changed. Thus, consider some round in the new election (after adding ballots). If A_j is the first candidate in $\mathcal{A} := \{A_i\}$ not yet elected, so $A_1, \dots, A_{j-1}$ (and possibly some others) already are elected, then A_j has at least the same score as when A_j was elected in the original election (without extra ballots), which is $\geqslant t$. Hence, until all of $\mathcal{A}$ have been elected, there is always at least one of them with score $\geqslant t$. □

Theorem 9.11. *Let a_n be as in Lemma 9.6 and c_n as in Lemma 9.9. Then, for Thiele's ordered method: For $1 \leqslant \ell \leqslant S$,*

$$\pi^{\mathsf{Th\text{-}o}}_{\mathsf{tactic}}(\ell, S) = \frac{a_\ell}{a_\ell + a_{S+1-\ell}}, \tag{9.34}$$

$$\pi^{\mathsf{Th\text{-}o}}_{\mathsf{same}}(\ell, S) = \frac{\ell}{\ell + a_{S+1-\ell}}, \tag{9.35}$$

$$\pi^{\mathsf{Th\text{-}o}}_{\mathsf{wPSC}}(\ell, S) = \frac{c_\ell}{c_\ell + a_{S+1-\ell}}. \tag{9.36}$$

Some numerical values are given in Tables 8–10 in Appendix B.

Proof. We show the three equations using separate but similar arguments. Let $\mathcal{A} = \{A_1, \dots, A_\ell\}$ be a set of ℓ candidates, and let $\mathcal{W}$ be a set of $W := |\mathcal{W}|$ voters.

π_{same}: Suppose that the set $\mathcal{W}$ of voters vote on the same list σ, beginning with $A_1 \cdots A_\ell$. Suppose also that the outcome is bad; then not all of $A_1, \dots, A_\ell$ are elected. We may then assume that $\sigma = A_1 \cdots A_\ell$, since deleting later names will not affect the counting. Furthermore, at least $S-\ell+1$ other candidates $B_1, \dots, B_{S+1-\ell}$ are elected. Throughout the counting, since not all of $A_1, \dots, A_\ell$ are elected, at least one of them (the first not yet elected) has a score of at least W/ℓ. Consequently, $B_1, \dots, B_{S+1-\ell}$ are all elected with a score of $\geqslant W/\ell$. Since only voters from $\mathcal{W}^{\mathsf{c}}$ vote for any B_j, Lemma 9.6(i) applied to $\{B_j\}_1^{S+1-\ell}$ yields

$$V - W = |\mathcal{W}^{\mathsf{c}}| \geqslant a_{S+1-\ell}\frac{W}{\ell}, \tag{9.37}$$

which is equivalent to

$$\frac{W}{V} \leqslant \frac{\ell}{\ell + a_{S+1-\ell}}. \tag{9.38}$$

Conversely, consider an election with ℓ votes on the list $A_1 \cdots A_\ell$, and $a_{S+1-\ell}$ votes on (only) $B_1, \dots, B_{S+1-\ell}$, distributed (using the strategy in the proof of Lemma 9.6) such that that each B_j is elected with score $\geqslant 1$ if enough seats are distributed. Then, as long as not all $B_1, \dots, B_{S+1-\ell}$ have been elected, some B_j will have score $\geqslant 1$ (in fact, exactly 1). If $A_1, \dots, A_{\ell-1}$ have been elected, then A_ℓ will have score 1; thus there will be a tie for each of the remaining seats and it is possible that $B_1, \dots, B_{S+1-\ell}$ are elected and A_ℓ not, a bad outcome. This example shows that equality may hold in (9.38) for a bad outcome, and thus (9.35) holds.

$\overline{\pi}_{\mathsf{tactic}}$: Let the W voters in $\mathcal{W}$ vote according to the strategy in Lemma 9.6 (and its proof), scaling the number of votes by W/a_ℓ. Then, using Lemma 9.10, as long as not all A_i are elected, at least one of them has score $\geqslant W/a_\ell$. Suppose that the outcome is bad. Then, at least $S-\ell+1$ other candidates $B_1, \dots, B_{S+1-\ell}$ are elected. Furthermore, throughout the counting, some A_i has score $\geqslant W/a_\ell$; thus every elected candidate, and in particular every B_j, is elected with score $\geqslant W/a_\ell$. Similarly to (9.37), it now follows by Lemma 9.6(i) that

$$V - W = |\mathcal{W}^{\mathsf{c}}| \geqslant a_{S+1-\ell}\frac{W}{a_\ell}, \tag{9.39}$$

which is equivalent to

$$\frac{W}{V} \leqslant \frac{a_\ell}{a_\ell + a_{S+1-\ell}}. \tag{9.40}$$

Conversely, suppose that $W < a_\ell$ and $V = a_\ell + a_{S+1-\ell}$, so $|\mathcal{W}^{\mathsf{c}}| > a_{S+1-\ell}$. Suppose that the voters in $\mathcal{W}^{\mathsf{c}}$ vote on $B_1, \dots, B_{S+1-\ell}$ (and no others) using the strategy in Lemma 9.6 and its proof; thus, using also Lemma 9.10, no matter how the voters in $\mathcal{W}$ vote, as long as not $B_1, \dots, B_{S+1-\ell}$ are not all elected, at least one of them will have score $\geqslant 1$. Hence, if the outcome is good, so $A_1, \dots, A_\ell$ all are elected, then each of them has to be elected with score $\geqslant 1$. However, this contradicts Lemma 9.6(i), since $W < a_\ell$. Consequently, for such W, the outcome may be bad for any strategy chosen by $\mathcal{W}$. This and (9.40) yield (9.34).

π_{wPSC}: Suppose that all voters in V vote on lists beginning with $\mathcal{A}$ in some order. Then, by Lemmas 9.9 and 9.10 (and homogeneity), as long as not all A_i are elected, at least one of them has score $\geqslant W/c_\ell$. By the same argument as just given for $\overline{\pi}_{\mathsf{tactic}}$ (with a_ℓ replaced by c_ℓ), it follows that if the outcome is bad, then

$$\frac{W}{V} \leqslant \frac{c_\ell}{c_\ell + a_{S+1-\ell}}. \tag{9.41}$$

Conversely, consider an election with $V = c_\ell + a_{S+1-\ell}$ votes; c_ℓ votes on $A_1, \dots, A_\ell$ distributed as in the extremal case in Lemma 9.9 and its proof, and $a_{S+1-\ell}$ votes on $B_1, \dots, B_{S+1-\ell}$ distributed according to the strategy in Lemma 9.6 and its proof. Then, in some round during the counting, not all A_i have been elected and the leading A_i has score 1; hence all remaining candidates in $\mathcal{A}$ have scores $\leqslant 1$. Furthermore, unless all B_j already are elected (in which case the outcome is bad), at least one of them has score $\geqslant 1$. Hence, assuming that $\mathcal{A}$ loses every time there is a tie, a possible outcome is that all $B_1, \dots, B_{S+1-\ell}$ are elected before any further A_i, and thus the outcome is bad. Hence, equality may hold in (9.41) for a bad outcome, and thus (9.36) holds. ☐

Corollary 9.12. *For Thiele's ordered method and $\ell = 1$: With a_S as in Lemma 9.6,*

$$\overline{\pi}^{\mathsf{Th\text{-}o}}_{\mathsf{tactic}}(1,S) = \pi^{\mathsf{Th\text{-}o}}_{\mathsf{tactic}}(1,S) = \pi^{\mathsf{Th\text{-}o}}_{\mathsf{same}}(1,S) = \pi^{\mathsf{Th\text{-}o}}_{\mathsf{wPSC}}(1,S) = \frac{1}{1+a_S}, \qquad S \geqslant 1. \tag{9.42}$$

Asymptotically, this is $\sim \log S/S$ as $S \to \infty$.

Proof. By (9.34)–(9.36), since $a_1 = c_1 = 1$; the result for π_{tactic} follows using (2.3) and (5.2). The asymptotic formula follows from (9.13). ☐

In particular, Thiele's ordered method does not satisfy the weak PSC $\pi_{\mathsf{wPSC}}(\ell, S) < \ell/S$, see Section 8, not even for $\ell = 1$ and S large. (In fact, not for $S \geqslant 3$, as easily follows from (9.36) and Lemma 9.6.) Consequently, Thiele's method does not satisfy "Solid coalitions" in [15], see Remark 8.3.

Example 9.13. Since $a_1 = 1$ and $a_2 = 3/2$, Theorem 9.11 yields $\overline{\pi}^{\mathsf{Th\text{-}o}}_{\mathsf{tactic}}(1,2) = 2/5$ and $\overline{\pi}^{\mathsf{Th\text{-}o}}_{\mathsf{tactic}}(2,2) = 3/5$. See Example 9.5, which is the extremal case with a perturbation to avoid ties. This also shows that no strategy can help the smaller party in Example 9.5.

Example 9.14. Theorem 9.11 yields $\pi^{\mathsf{Th\text{-}o}}_{\mathsf{same}}(2,3) = 4/7 \doteq 0.571 > 0.5$. Hence, in an election with 3 seats, even if a majority votes for the same list, they can fail to obtain a majority of the seats. A concrete example (modified from [26]), which essentially uses the strategy above, is:

55 ABC
30 XYZ
15 YZX

A gets the first seat, but then the second and third seats go to X and Y with 30 votes each. Thus A, X, Y are elected, and the ABC party gets only 1 seat, in spite of a majority of the votes.

10. Borda (scoring) methods

Finally, we consider Borda methods, Appendix A.3.4. Recall than $H_n := \sum_{i=1}^n 1/i$, the harmonic number.

Theorem 10.1. *Let* $\mathsf{Borda}(\mathbf{w})$ *be the Borda method with weights* $\mathbf{w} = (w_k)_1^\infty$, *and let*

$$\overline{w}_k := \frac{1}{k}\sum_{i=1}^k w_i. \tag{10.1}$$

Then, for $1 \leqslant \ell \leqslant S$,

$$\overline{\pi}_{\mathsf{tactic}}^{\mathsf{Borda}(\mathbf{w})}(\ell, S) = \frac{\overline{w}_{S+1-\ell}}{\overline{w}_\ell + \overline{w}_{S+1-\ell}}, \tag{10.2}$$

$$\pi_{\mathsf{same}}^{\mathsf{Borda}(\mathbf{w})}(\ell, S) = \pi_{\mathsf{wPSC}}^{\mathsf{Borda}(\mathbf{w})}(\ell, S) = \frac{\overline{w}_{S+1-\ell}}{w_\ell + \overline{w}_{S+1-\ell}}. \tag{10.3}$$

In particular, for the harmonic Borda method $\mathsf{Borda}(1/k)$: *For* $1 \leqslant \ell \leqslant S$,

$$\overline{\pi}_{\mathsf{tactic}}^{\mathsf{Borda}(1/k)}(\ell, S) = \frac{\ell H_{S+1-\ell}}{(S+1-\ell)H_\ell + \ell H_{S+1-\ell}}, \tag{10.4}$$

$$\pi_{\mathsf{same}}^{\mathsf{Borda}(1/k)}(\ell, S) = \pi_{\mathsf{wPSC}}^{\mathsf{Borda}(1/k)}(\ell, S) = \frac{\ell H_{S+1-\ell}}{S+1-\ell+\ell H_{S+1-\ell}}. \tag{10.5}$$

Furthermore,

$$\pi_{\mathsf{party}}^{\mathsf{Borda}(1/k)}(\ell, S) = \frac{\ell}{S+1}. \tag{10.6}$$

Some numerical values for the harmonic Borda method are given in Tables 11 and 12 in Appendix B.

Proof. It is easy (more or less trivial) to see that Lemmas 9.6(i), 9.9 and 9.10 hold also for $\mathsf{Borda}(\mathbf{w})$ with a_n replaced by $1/\overline{w}_n$ and c_n is replaced by $1/w_n$.

The result (10.2)–(10.3) then follows by the same proof as for Theorem 9.11, where for π_{same} we also replace ℓ by $1/w_\ell$.

For the harmonic case $w_k = 1/k$, we have $\overline{w}_k = H_k/k$, and (10.4)–(10.5) follow. Finally, in the party list case, the harmonic Borda method is equivalent to D'Hondt's method, and thus (10.6) follows from Theorem 4.3. □

Corollary 10.2. *For the harmonic Borda method with $\ell = 1$:*

$$\overline{\pi}_{\mathsf{tactic}}^{\mathsf{Borda}(1/k)}(1,S) = \pi_{\mathsf{tactic}}^{\mathsf{Borda}(1/k)}(1,S) = \pi_{\mathsf{same}}^{\mathsf{Borda}(1/k)}(1,S) = \pi_{\mathsf{wPSC}}^{\mathsf{Borda}(1/k)}(1,S)$$
$$= \frac{H_S}{S+H_S}, \qquad S \geqslant 1. \tag{10.7}$$

Asymptotically, this is $\sim \log S/S$ as $S \to \infty$.

Proof. By (10.4)–(10.5), since $H_1 = 1$; the result for π_{tactic} follows using (2.3) and (5.2). The asymptotic formula follows since $H_S \sim \log S$ as $S \to \infty$. □

Note that $a_n \sim H_n/n$ as $n \to \infty$, and thus the asymptotics in, for example, Corollaries 9.12 and 10.2 are the same, but exact values differ.

Example 10.3. Since $H_1 = 1$ and $H_2 = 3/2$, $\overline{\pi}_{\mathsf{tactic}}^{\mathsf{Borda}(1/k)}(1,2) = 3/7$ and $\overline{\pi}_{\mathsf{tactic}}^{\mathsf{Borda}(1/k)}(2,2) = 4/7$. Cf. Example 9.13.

Note that the Borda method with weights $\mathbf{w} = (1,0,0,\dots)$ is equivalent to SNTV; more generally, the Borda method with weights $\mathbf{w} = (1,\dots,1,0,\dots)$ with L 1's is equivalent to LV(L); in particular, $L = S$ yields BV. Consequently, the results in Theorems 5.4, 5.7 and 5.10 follow from Theorem 10.1.

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Appendix A. Election methods

We give here brief descriptions of the election methods discussed above. For further details, other election methods, political aspects and examples of actual use, see e.g. [17; 23; 24; 26; 37; 38]. We give also the abbreviation used in formulas in the present paper.

Note that many election methods are known under several different names. We give a few synonyms, but omit many others.

A.1. **Election methods with party lists.** Among the proportional election methods, the ones that are most often used are party list methods, where the voter votes for a party and the seats are distributed among the parties according to their numbers of votes. (The seats obtained by a party then are distributed to candidates by some method; many versions are used in practice.)

Conversely, most (but not all) party list methods that are used in practice, including the ones below, are proportional, so that each party gets a proportion of the seats that approximates its proportion of votes.[25]

Many different list methods are described in detail in e.g. Pukelsheim [38]; see also Balinski and Young [3] for the mathematically (but not politically) equivalent problem of allocating the seats in the US House of Representatives proportionally among the states.[26]

There are two major types of party list methods: *divisor methods* and *quota methods*.

A.1.1. *Divisor methods.* A divisor method is defined by a sequence $d(1), d(2), d(3)\dots$ of *divisors*.[27] In one traditional formulation, seats are allocated sequentially. For each seat, a party that so far has obtained s seats gets its number of votes divided by $d(s+1)$; the party with the highest quotient gets the seat.[28] The two most important divisor methods are described in the next subsubsections. They are both of the linear form

$$d(n) = n - 1 + \gamma \tag{A.1}$$

for some constant γ; several other choices of γ (usually but not always with $\gamma \in [0,1]$) have also been used or suggested, for example *Adams's method* with $\gamma = 0$, and there are also divisor methods with other (non-linear) sequences of divisors, see e.g. Footnote 25. We let $\mathsf{Div}(\gamma)$ denote the linear divisor method given by (A.1); for simplicity we do not consider other divisor methods (except in Remark 4.5).

A.1.2. *D'Hondt's method* ($\mathsf{D'H}$). This is the divisor method with the sequence of divisors $1, 2, 3, \dots$, i.e. $d(n) = n$; thus this is the method (A.1) with $\gamma = 1$. It was proposed by Victor D'Hondt in 1882 [10]. D'Hondt's method is (almost) equivalent to *Jefferson's method*, proposed by Thomas

[25]Examples that are not proportional are the divisor methods with divisors $d(n) = n^{0.9}$ used in Estonia, and $d(n) = 2^{n-1}$ used in Macau, see [24].

[26]There, until 1941, the method was decided after each census, so the choice of method was heavily influenced by its result. Moreover, the number of seats was not fixed in advance and thus also open to negotiations.

[27]Warning: Some authors denote our $d(n+1)$ by $d(n)$, thus starting the sequence with $d(0)$.

[28]A different, but equivalent, formulation is that the number of seats for each party is obtained by selecting a number D and then giving a party with v_i votes $s_i \geqslant 0$ seats where $d(s_i) \leqslant v_i/D \leqslant d(s_i+1)$ (with $d(0) := 0$), where D is chosen such that the total number of seats given to the parties is S. (This can be regarded as a special rounding rule used to round v_i/D to an integer s_i.) In this version, the number D is called *divisor*. See Pukelsheim [38] for a details and examples. This formulation (with rounding downwards) was used already by D'Hondt when proposing his method [10]. Similar formulations have also been used in the United States for allocating the seats in the House of Representatives to the states [3].

Jefferson in 1792 for allocating the seats in the US House of Representatives, see [3].[29]

A.1.3. *Sainte-Laguë's method* (StL). This is the divisor method with the sequence of divisors $1, 3, 5, \dots$, i.e., $d(n) = 2n - 1$. Equivalently, we may take $d(n) = n - 1/2$; thus this is the method (A.1) with $\gamma = 1/2$. The method was proposed in 1910 by Sainte-Laguë [41]. It is (almost) equivalent to *Webster's method*, proposed by Daniel Webster in 1832 for allocating the seats in the US House of Representatives, see [3].[30]

A.1.4. *Quota methods.* In a quota method, first a *quota* Q is calculated; this is roughly the number of votes required for each seat, and different quota methods differ in the choice of Q. The two main quotas that are used are the *Hare quota* V/S and the *Droop quota* $V/(S+1)$ [13], where V is the total number of (valid) votes and S is the number of seats (cf. Remark 1.1); in practice, the quota is often rounded to an integer, either up, or down, or to the nearest integer (see e.g. [38] and [24] for various examples with different, or no, rounding). Rounding is mathematically a nuisance, leading to various problems, but has in practice usually no effect. In the present paper, we consider the ideal versions without rounding, but see Remark 4.7.

The quota method gives a party with v_i votes first $\lfloor v_i/Q \rfloor$ seats; the remaining seats, if any, are given to the parties with largest remainder in these divisions. In other words, we find a threshold $t \in [0,1]$ such that we round v_i/Q upwards if the remainder is larger than t. Mathematically, this means that we find t and integers s_i (the number of seats for party i) such that

$$s_i - 1 + t \leqslant \frac{v_i}{Q} \leqslant s_i + t. \tag{A.2}$$

The (traditional) description above with remainders assumes that

$$\sum_i \lfloor v_i/Q \rfloor \leqslant S \leqslant \sum_i \lceil v_i/Q \rceil, \tag{A.3}$$

and thus that we can find $t \in [0,1]$ such that (A.2) holds and $\sum_i s_i = S$. This is always the case with the Hare and Droop quotas, and more generally with any quota in the interval $[V/(S+1), V/S]$. (Except a case with ties for the Droop quota, but then (A.2) still works with $t = 1$.) Mathematically, the method works with any quota $Q > 0$, taking (A.2) as the definition together with $\sum_i s_i = S$ and letting t be arbitrary real (and requiring $s_i \geqslant 0$ with a

[29]Jefferson's method is formulated as in Footnote 28, with rounding downwards, which is the same as D'Hondt's method, and nowadays "Jefferson's method" is used for this divisor method, equivalent to D'Hondt's. However, historically this is not quite accurate: for the allocation of seats in the House of Representatives, the total number was not fixed in advance but decided by the outcome once a suitable divisor D was chosen by Congress, which makes the method as used by Jefferson different from D'Hondt's, where the number of seats is given in advance.

[30]Webster's method is formulated as in Footnote 28, with rounding to the nearest integer. However, Footnote 29 applies here too.

modification of (A.2) if $t > 1$). It is easily shown that if $Q \geqslant V/(S+1)$, then we can take $t \leqslant 1$, and if $Q \leqslant V/S$, then we can take $t \geqslant 0$.

As said above, there are two main quotas used, and thus two main quota methods.

A.1.5. *Method of largest remainder (Hare's method)* (LR). The quota method using the Hare quota V/S.

A.1.6. *Droop's method* (Droop). The quota method using the Droop quota $V/(S+1)$.

A.2. **Election methods with unordered ballots.** In these methods, each ballot contains a list of names, but their order is ignored. The number of candidates on each ballot is, depending on the method, either arbitrary or limited to at most (or exactly) a given number, for example S, the number to be elected.

A.2.1. *Block Vote* (BV). Each voter votes for at most S candidates. The S candidates with the largest numbers of votes are elected.[31] Also called e.g. *Multi-Member Plurality*; *Plurality-at-Large*.

The special case $S = 1$ is the widely used *Single-Member Plurality* method, also called *First-Past-The-Post*, where each voter votes for one candidate, and the candidate with the largest number of votes wins; this simple method has probably been used since pre-historic times. Block Vote usually means the multi-member case ($S \geqslant 2$); this too is an ancient method, and it is widely used in non-political elections. Block Vote is well-known for not being proportional; when there are organized parties, the largest party will get all seats, cf. Theorem 5.4.

A.2.2. *Approval Vote* (AV). Each voter votes for an arbitrary number of candidates. The S candidates with the largest numbers of votes are elected.

This is also an old method, and has been used, even for public elections, since the 19th century, although much less frequently than Block Vote.

The method is one of the methods proposed by Thiele [45] in 1895, viz. his "strong method", see Appendix A.2.7 below. The method was reinvented again (and given the name Approval Voting) by Weber c. 1976; see further e.g. [47; 6; 28].

For our problems, Approval Vote behaves often similarly to Block Vote, see e.g. Theorem 5.6, but note also the difference in Theorems 6.10 and 6.11.

A.2.3. *Single Non-Transferable Vote* (SNTV). Each voter votes for one candidate. The S candidates with the largest numbers of votes are elected.

The idea is that a set $\mathcal{W}$ of voters can concentrate their votes on a suitable number of candidates (one or several), according to the size of $\mathcal{W}$, and thus get these candidates elected. In an ideal situation where all voters belong

[31] One version requires each voter to vote for exactly S candidates. For our purposes this makes no difference, see Remark 5.5.

to parties and vote as instructed by their party, and, moreover, the parties accurately know in advance the number of votes for each party, optimal strategies lead to the same result as D'Hondt's method, see Theorem 5.7 and [24, Appendix A.8].

However, in practice, there are several practical problems. In particular, the outcome depends heavily on how the votes are distributed inside the parties; note that a party can get hurt by too much concentration of votes as well as by too little concentration. Hence, parties are more or less forced to use schemes of tactical voting.

A.2.4. *Limited Vote* (LV, LV(L)). Each voter votes for at most L candidates, where L is some given number. The S candidates with the largest numbers of votes are elected.[32]

Here $L \leqslant S$. (Although, formally, this is not necessary, and Approval Voting in Appendix A.2.2 could be seen as the case $L = \infty$.) The case $L = S$ is Block Vote, and $L = 1$ is SNTV, so usually only the case $1 < L < S$ is called Limited Vote.

The idea behind Limited Vote is the same as for SNTV, and its problems are more or less the same, see e.g. [9, p. 193], [12], [17, p. 27].

A.2.5. *Cumulative Vote* (CV). Each voter can choose between voting for a single candidate or splitting the vote between two or several. Usually there are, for practical reasons, limitations on how the votes may be split.[33] In one common version, each voter has a fixed number L votes, and can give distribute them arbitrarily between one or several (up to L) candidates. In another version, sometimes called *Equal and Even Cumulative Voting*, the voter has one vote that may be given to a single candidate or split between two or several candidates; this version uses unordered ballots, and a ballot with n names is counted as $1/n$ vote for each of them. We let $\mathsf{CV}^=$ denote the mathematically ideal version of this, where there is no limitation on the number of candidates on each ballot. (This version has been reinvented and called *Satisfaction Approval Voting (SAV)* [7].) Another version of Equal and Even Cumulative Voting, used in practice, allows at most S names on each ballot.

A.2.6. *Phragmén's unordered method* (Phr-u). Phragmén's method for unordered ballots was proposed in 1894 by the Swedish mathematician Phragmén[33; 34; 35; 36], using several different (but equivalent) formulations. See [26] for a detailed discussion, including an algorithmic definition. We use here a description from [36].

The candidates are elected sequentially. When a candidate is elected, the participating ballots (i.e., the ballots that include this candidate) incur a

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[32] As for Block Vote, one version requires each voter to vote for exactly L candidates.

[33] An ideal mathematical version, probably never used, is that each voter may give each candidate i a vote with weight $w_i \geqslant 0$, where these weights are arbitrary real numbers with $\sum_i w_i = 1$.

total *load* of 1 unit, which has to be distributed between them. This load is distributed such that the maximum load on each ballot (inluding loads from earlier elected candidates) is a small as possible; furthermore, in each round, the candidate is elected that makes the resulting maximal load on a ballot as small as possible. (For example, the first elected candidate is the one that appears on the largest number of ballots; if she appears on m ballots, these ballots get a load $1/m$ each.)

The total load when k candidates have been elected is thus k. Note that the load on a given ballot increases when someone on the ballot is elected, and otherwise stays the same; it never decreases. We denote the maximum load when k candidates have been elected by $t^{(k)}$. Thus $0 = t^{(0)} < t^{(1)} \leqslant t^{(2)} \dots$ (equalities are possible when there are ties). When k candidates have been elected, say $A_1, \dots, A_k$ in this order, the load on each ballot that includes A_k is exactly $t^{(k)}$. More generally, the load on a ballot σ is $t^{(j)}$, where $j := \max\{i \leqslant k : A_i \in \sigma\}$.

A.2.7. *Thiele's optimization method* (Th-opt). Let $\psi(n)$ be a given function, which represents the "satisfication" of a voter that sees n of her candidates elected. Thus the total satisfication if a set $\mathcal{E}$ is elected is

$$\Psi(\mathcal{E}) := \sum_{\sigma \in \mathcal{V}} \psi\bigl(|\sigma \cap \mathcal{E}|\bigr), \tag{A.4}$$

The elected set $\mathcal{E}$ is the set with $|\mathcal{E}| = S$ that maximizes $\Psi(\mathcal{E})$.

Of course, the method depends on the choice of the satisfaction function ψ. It is convenient to write $\psi(n) = \sum_{k=1}^{n} w_k$ for a non-negative sequence $\mathbf{w} = (w_k)_1^\infty$ of weights. The standard choice is $w_k = 1/k$ and thus $\psi(n) = H_n$; this is assumed unless another choice is explicitly mentioned. In particular, Th-opt always denotes the standard version; we denote the version defined by another weight sequence $\mathbf{w}$ by Th-opt($\mathbf{w}$).

This method was proposed in 1895 by Thiele [45]. Thiele considered apart from the standard choice $w_k = 1/k$ also the *strong method* with $w_k = 1$ and thus $\psi(n) = n$ (which is the same as appoval voting, see Appendix A.2.2) and the *weak method* with $\psi(n) = 1$, $n \geqslant 1$, and thus $w_k = 0$ for $k \geqslant 2$. We denote the weak method by Th-opt(weak).

Thiele's optimization method was reinvented in 2001 under the name *Proportional Approval Voting (PAV)* [28].

Thiele [45] realized that is was computationally difficult to find the optimal set $\mathcal{E}$, and thus proposed also two sequential methods as approximations, the *addition method* and the *elimination method*,[34] described in the following subsubsections.

A.2.8. *Thiele's addition method* (Th-add). Candidates are elected one by one. In each round, the candidate A is elected that maximizes the increase

[34]In Danish: *Tilføjelsesreglen* and *Udskydelsesreglen*, respectively. Thiele [45] gave also the French names *règle d'addition* and *règle de rejet.*

of the satisfaction (A.4), i.e., $\Psi(\mathcal{E}\cup\{A\})-\Psi(\mathcal{E})$, where $\mathcal{E}$ is the set of already elected. (Equivalently, $\Psi(\mathcal{E}\cup\{A\})$ is maximized.)

This is thus the greedy version of Thiele's optimization method. As in Appendix A.2.7, we assume the standard choice $w_k = 1/k$ unless we say otherwise. Thus Th-add denotes the standard version, the weak version is denoted Th-add(weak), and the general version with weights $\mathbf{w}$ is denoted Th-add($\mathbf{w}$).

The standard version can, equivalently but more concretely, be described by the following rule. (For general weights, $1/(k+1)$ is replaced by w_{k+1}.)

Seats are awarded one by one. For each seat, a ballot where $k \geqslant 0$ names already have been elected is counted as $1/(k+1)$ votes for each remaining candidate on the ballot. The score *of a (not yet elected) candidate is the number of votes for her counted with these weights. The candidate with the largest score is elected.*

Note that the score may change (decrease) from one round to another.

Thiele's addition method was used in Sweden 1909–1921 for the distribution of seats within parties, see [26, Appendix D] for details.

A.2.9. *Thiele's elimination method* (Th-elim). Candidates are eliminated one by one until only S remain; these are elected. In each round, the candidate is eliminated that minimized the decrease in total satisfaction (A.4). In other words, if the set of candidates is $\mathcal{C}$, we define by backwards recursion a sequence of subsets $\mathcal{E}_i$ for $i = |\mathcal{C}|, \dots, S$ (in decreasing order) by $\mathcal{E}_{|\mathcal{C}|} = \mathcal{C}$ and then letting $\mathcal{E}_i$ be the subset of $\mathcal{E}_{i+1}$ with $|\mathcal{E}_i| = i$ that maximizes $\Psi(\mathcal{E}_i)$.

We consider only the standard version with weights $w_k = 1/k$. We then have an equivalent, more explicit, description:

Candidates are eliminated one by one until only S remain; these are elected. In each round, a ballot where $k \geqslant 1$ names remain is counted as $1/k$ votes for each remaining candidate on the ballot. The score *of a candidate is the number of votes for her counted with these weights. The candidate with the smallest score is eliminated.*

The elimination method seems to have been ignored after Thiele's paper. It was recently reinvented under the name *Harmonic Weighting* as a method for ordering alternatives for the electronic voting system *LiquidFeedback* [4].

A.3. **Election methods with ordered ballots.** In these method, each ballot contains an ordered list of names. The number of candidates on each ballot is usually arbitrary, but may be limited to at most, or at least, or exactly some given number, for example S, the number to be elected; some versions require each voter to rank all candidates. (In the latter case, each ballot can be seen as a permutation of the set of candidates.)

A.3.1. *Single Transferable Vote (STV)* (STV). First, a quota is calculated, nowadays almost always the Droop quota $V/(S+1)$ (usually but not always rounded to the nearest higher integer), cf. Appendix A.1.4. Each ballot is counted for its first name only (at later stages, ignoring candidates that have been elected or eliminated). A candidate whose number of votes is at least the quota is elected; the surplus, i.e., the ballots exceeding the quota, are transferred to the next (remaining) name on the ballot. This is repeated as long as some unelected candidate reaches the quota. If there is no such candidate, and not enough candidates have been elected, then the candidate with the least number of votes is eliminated, and the votes for that candidate are transferred to the next name on the ballot. At the end, if the number of remaining candidates equals the number of remaining seats, then all these are elected, even if they have not reached the quota.

This description is far from a complete definition; many details are omitted, and can be filled in in different ways, see e.g. [46], [24], [26, Appendix E.2]. In particular, several quite different methods are used to transfer the surplus,[35] but some other details can also vary (which might influence the outcome). Thus, STV is a family of election methods rather than a single method.

A.3.2. *Phragmén's ordered method* (Phr-o). Phragmén's method for ordered ballots was proposed in 1913 by a Swedish Royal Commission (where Phragmén was one of the members) [19].[36] As the version for unordered ballots, Appendix A.2.6, it can be defined in several different, equivalent ways; see again [26] for a detailed discussion, including an algorithmic definition.

We use here, as for the unordered version, a definition using *loads*; the method is defined as Phragmén's unordered method in Appendix A.2.6, with the added rule that in each round, a ballot is only counted for the first name on it, ignoring candidates already elected. (For example, the first elected candidate is the one that is first on the largest number of ballots.)

A.3.3. *Thiele's ordered method* (Th-o). Seats are distributed sequentially. In each round, each ballot is only counted for the first name on it that has not already been elected; if this is name number k (so the $k-1$ preceding have been elected), then this ballot is counted as a vote with weight $1/k$. To avoid confusion, we use "score" for the total number of votes for a given candidate, counted with these weights. (Thus the score may change from one round to another.) In each round, the candidate with the largest score is elected.

This version of Thiele's method is not due to Thiele [45], who only considered unordered ballots (Appendix A.2.7–A.2.8).[37] The method was proposed in 1912 in the Swedish parliament by Nilson in Örebro, for distribution of

[35]Some versions even involve randomness, e.g. the version of STV used for the lower house, Dáil, of the Irish parliament, where the outcome may depend on the random order in which ballots are counted.

[36]Phragmén's ordered method has since 1921 been part of the Swedish election law (for distribution of seats within parties; nowadays only in a minor role).

[37]Thus our name for the method is historically incorrect, but we regard it as a version of Thiele's addition method for unordered ballots.

seats within parties in parliamentary elections; it is discussed (and rejected) in [19].[38]

Remark A.1. As for Thiele's unordered methods (Appendix A.2.7–A.2.8), an arbitrary given sequence of weights w_k could be used instead of $1/k$. We do not consider this possibility in the present paper.

A.3.4. *Borda (scoring) methods* ($\mathsf{Borda}(\mathbf{w})$). Each ballot is counted as w_1 votes for the first name, w_2 for the second name, and so on, according to some given non-increasing sequence $\mathbf{w} = (w_k)_{k=1}^{\infty}$ of non-negative numbers. The S candidates with highest scores are elected.

Several different sequences $(w_k)_k$ are used. The method was suggested (for the election of a single person) by Jean-Charles de Borda in 1770 [5] with $w_k = n - k + 1$ where n is the number of candidates, i.e., the sequence $n, n-1, \dots, 1$. (His proposal required each voter to rank all candidates.)[39] This scoring rule is called the *Borda method*; more generally, any scoring rule is sometimes called a Borda method.

Another common choice of scoring rule uses the *harmonic series*, $w_k = 1/k$; we call this the *harmonic Borda method*. This was proposed in 1857 by Thomas Hare (who later instead developed and proposed STV, Section A.3.1). The harmonic Borda method gives in the party-list case (see Section 4.1) the same result as D'Hondt's method. However, tactical voting where the voters of a party vote on the same name in different orders may give quite different results, cf. Theorem 10.1.

The choice $\mathbf{w} = (1, \dots, 1, 0, \dots)$ with $w_k = 1$ for $k \leqslant L$ and $w_k = 0$ for $k > L$, for some given $L \geqslant 1$, means that only the first L names on each ballot matter, and that their order is irrelevant; this Borda method is equivalent to Limited Vote with L names on each ballot, see Appendix A.2.4. In particular, the weights $(1, 0, 0, \dots)$ yield SNTV, Appendix A.2.3.

[38] The method was not adopted for general elections, but it was later chosen, still within parties, for elections of committees inside city and county councils, and it is still (formally, at least) used for that purpose, see [24; 26].

[39] The same method was proposed already in 1433 by Nicolas Cusanus for election of the king (and future emperor, after being crowned by the pope) of the Holy Roman Empire [31; 20].

Appendix B. Some numerical values

The tables below give, as illustrations and to faciliate comparisons, numerical values for $1 \leqslant S \leqslant 5$ for many of the thresholds discussed in the paper.

	$\ell = 1$	2	3	4	5
$S = 1$	$\frac{1}{2} = 0.5$				
2	$\frac{1}{3} \doteq 0.333$	$\frac{2}{3} \doteq 0.667$			
3	$\frac{1}{4} = 0.25$	$\frac{1}{2} = 0.5$	$\frac{3}{4} = 0.75$		
4	$\frac{1}{5} = 0.2$	$\frac{2}{5} = 0.4$	$\frac{3}{5} = 0.6$	$\frac{4}{5} = 0.8$	
5	$\frac{1}{6} \doteq 0.167$	$\frac{1}{3} \doteq 0.333$	$\frac{1}{2} = 0.5$	$\frac{2}{3} \doteq 0.667$	$\frac{5}{6} \doteq 0.833$

Table 2. The "optimal" threshold $\ell/(S+1)$, attained by e.g. $\pi^{\mathsf{D'H}}_{\mathsf{party}}$, $\pi^{\mathsf{Droop}}_{\mathsf{party}}$, $\overline{\pi}^{\mathsf{SNTV}}_{\mathsf{tactic}}$, $\pi^{\mathsf{CV}}_{\mathsf{tactic}}$, $\pi^{\mathsf{Phr\text{-}u}}_{\mathsf{PJR}}$, $\pi^{\mathsf{Th\text{-}opt}}_{\mathsf{EJR}}$, $\pi^{\mathsf{Th\text{-}elim}}_{\mathsf{same}}$, $\pi^{\mathsf{STV}}_{\mathsf{PSC}}$ (with the Droop quota), $\pi^{\mathsf{Phr\text{-}o}}_{\mathsf{wPSC}}$. (Theorems 4.3, 4.6, 5.7, 5.13, 7.3, 7.6, 7.26, 8.6, 9.4)

	$\ell = 1$	2	3	4	5
$S = 1$	$\frac{1}{2} = 0.5$				
2	$\frac{1}{3} \doteq 0.333$	$\frac{3}{4} = 0.75$			
3	$\frac{1}{4} = 0.25$	$\frac{3}{5} = 0.6$	$\frac{5}{6} \doteq 0.833$		
4	$\frac{1}{5} = 0.2$	$\frac{1}{2} = 0.5$	$\frac{5}{7} \doteq 0.714$	$\frac{7}{8} = 0.875$	
5	$\frac{1}{6} \doteq 0.167$	$\frac{3}{7} \doteq 0.429$	$\frac{5}{8} = 0.625$	$\frac{7}{9} \doteq 0.778$	$\frac{9}{10} = 0.9$

Table 3. $\pi^{\mathsf{StL}}_{\mathsf{party}}(\ell, S)$ for $1 \leqslant \ell \leqslant S \leqslant 5$. (Theorem 4.3)

	$\ell = 1$	2	3	4	5
$S = 1$	$\frac{1}{2} = 0.5$				
2	$\frac{1}{3} \doteq 0.333$	$\frac{3}{4} = 0.75$			
3	$\frac{1}{4} = 0.25$	$\frac{5}{9} \doteq 0.556$	$\frac{5}{6} \doteq 0.833$		
4	$\frac{1}{5} = 0.2$	$\frac{7}{16} \doteq 0.438$	$\frac{2}{3} \doteq 0.667$	$\frac{7}{8} = 0.875$	
5	$\frac{1}{6} \doteq 0.167$	$\frac{9}{25} = 0.36$	$\frac{11}{20} = 0.55$	$\frac{11}{15} \doteq 0.733$	$\frac{9}{10} = 0.9$

Table 4. $\pi^{\mathsf{LR}}_{\mathsf{party}}(\ell, S)$ for $1 \leqslant \ell \leqslant S \leqslant 5$. (Theorem 4.6)

	$\ell = 1$	2	3	4	5
$S = 1$	$\frac{1}{2} = 0.5$				
2	$\frac{1}{2} = 0.5$	$\frac{1}{2} = 0.5$			
3	$\frac{1}{2} = 0.5$	$\frac{3}{5} = 0.6$	$\frac{1}{2} = 0.5$		
4	$\frac{1}{2} = 0.5$	$\frac{4}{7} \doteq 0.571$	$\frac{3}{5} = 0.6$	$\frac{1}{2} = 0.5$	
5	$\frac{1}{2} = 0.5$	$\frac{5}{9} \doteq 0.555$	$\frac{5}{8} = 0.625$	$\frac{3}{5} = 0.6$	$\frac{1}{2} = 0.5$

TABLE 5. The non-monotone $\pi^{\mathsf{BV}}_{\mathsf{EJR}}(\ell, S)$ for $1 \leqslant \ell \leqslant S \leqslant 5$. (Theorem 6.10)

	$\ell = 1$	2	3	4	5
$S = 1$	$\frac{1}{2} = 0.5$				
2	$\frac{1}{2} = 0.5$	$\frac{2}{3} \doteq 0.667$			
3	$\frac{1}{2} = 0.5$	$\frac{3}{5} = 0.6$	$\frac{3}{4} = 0.75$		
4	$\frac{1}{2} = 0.5$	$\frac{4}{7} \doteq 0.571$	$\frac{2}{3} \doteq 0.667$	$\frac{4}{5} = 0.8$	
5	$\frac{1}{2} = 0.5$	$\frac{5}{9} \doteq 0.555$	$\frac{5}{8} = 0.625$	$\frac{5}{7} \doteq 0.714$	$\frac{5}{6} \doteq 0.833$

TABLE 6. $\pi^{\mathsf{AV}}_{\mathsf{EJR}}(\ell, S)$ for $1 \leqslant \ell \leqslant S \leqslant 5$. (Theorem 6.11)

	$\ell = 1$	2	3	4	5
$S = 1$	$\frac{1}{2} = 0.5$				
2	$\frac{1}{3} \doteq 0.333$	$\frac{2}{3} \doteq 0.667$?			
3	$\frac{3}{11} \doteq 0.273$	$\frac{1}{2} = 0.5$?	$\frac{3}{4} = 0.75$?		
4	$\frac{7}{31} \doteq 0.226$	$\frac{3}{7} \doteq 0.429$?	$\frac{3}{5} = 0.6$?	$\frac{4}{5} = 0.8$?	
5	?? $\doteq 0.193$	$\frac{7}{19} \doteq 0.368$?	$\frac{9}{17} \doteq 0.529$?	$\frac{2}{3} \doteq 0.667$?	$\frac{5}{6} \doteq 0.833$?

TABLE 7. Known and conjectured values of $\pi^{\mathsf{Th\text{-}add}}_{\mathsf{same}}(\ell, S)$ and $\pi^{\mathsf{Th\text{-}add}}_{\mathsf{PJR}}(\ell, S)$ for $1 \leqslant \ell \leqslant S \leqslant 5$. (Theorem 7.8, Conjecture 7.19)

	$\ell=1$	2	3	4	5
$S=1$	$\frac{1}{2}=0.5$				
2	$\frac{2}{5}=0.4$	$\frac{3}{5}=0.6$			
3	$\frac{12}{35}\doteq 0.343$	$\frac{1}{2}=0.5$	$\frac{23}{35}\doteq 0.657$		
4	$\frac{24}{79}\doteq 0.304$	$\frac{18}{41}\doteq 0.439$	$\frac{23}{41}\doteq 0.561$	$\frac{55}{79}\doteq 0.696$	
5	$\frac{720}{2621}\doteq 0.275$	$\frac{36}{91}\doteq 0.396$	$\frac{1}{2}=0.5$	$\frac{55}{91}\doteq 0.604$	$\frac{1901}{2621}\doteq 0.725$

TABLE 8. $\pi^{\mathsf{Th\text{-}o}}_{\mathsf{tactic}}(\ell,S)$ for $1\leqslant\ell\leqslant S\leqslant 5$. (Theorem 9.11)

	$\ell=1$	2	3	4	5
$S=1$	$\frac{1}{2}=0.5$				
2	$\frac{2}{5}=0.4$	$\frac{2}{3}\doteq 0.667$			
3	$\frac{12}{35}\doteq 0.343$	$\frac{4}{7}\doteq 0.571$	$\frac{3}{4}=0.75$		
4	$\frac{24}{79}\doteq 0.304$	$\frac{24}{47}\doteq 0.511$	$\frac{2}{3}\doteq 0.667$	$\frac{4}{5}=0.8$	
5	$\frac{720}{2621}\doteq 0.275$	$\frac{48}{103}\doteq 0.466$	$\frac{36}{59}\doteq 0.610$	$\frac{8}{11}\doteq 0.727$	$\frac{5}{6}\doteq 0.833$

TABLE 9. $\pi^{\mathsf{Th\text{-}o}}_{\mathsf{same}}(\ell,S)$ for $1\leqslant\ell\leqslant S\leqslant 5$. (Theorem 9.11)

	$\ell=1$	2	3	4	5
$S=1$	$\frac{1}{2}=0.5$				
2	$\frac{2}{5}=0.4$	$\frac{2}{3}\doteq 0.667$			
3	$\frac{12}{35}\doteq 0.343$	$\frac{4}{7}\doteq 0.571$	$\frac{4}{5}=0.8$		
4	$\frac{24}{79}\doteq 0.304$	$\frac{24}{47}\doteq 0.511$	$\frac{8}{11}\doteq 0.727$	$\frac{6}{7}\doteq 0.857$	
5	$\frac{720}{2621}\doteq 0.275$	$\frac{48}{103}\doteq 0.466$	$\frac{48}{71}\doteq 0.676$	$\frac{4}{5}=0.8$	$\frac{9}{10}=0.9$

TABLE 10. $\pi^{\mathsf{Th\text{-}o}}_{\mathsf{wPSC}}(\ell,S)$ for $1\leqslant\ell\leqslant S\leqslant 5$. (Theorem 9.11)

	$\ell=1$	2	3	4	5
$S=1$	$\frac{1}{2}=0.5$				
2	$\frac{3}{7}\doteq 0.429$	$\frac{4}{7}\doteq 0.571$			
3	$\frac{11}{29}\doteq 0.379$	$\frac{1}{2}=0.5$	$\frac{18}{29}=0.621$		
4	$\frac{25}{73}\doteq 0.342$	$\frac{22}{49}\doteq 0.449$	$\frac{27}{49}\doteq 0.551$	$\frac{48}{73}\doteq 0.657$	
5	$\frac{137}{437}\doteq 0.314$	$\frac{25}{61}\doteq 0.410$	$\frac{1}{2}=0.5$	$\frac{36}{61}\doteq 0.59$	$\frac{300}{437}\doteq 0.686$

TABLE 11. $\pi^{\mathsf{Borda}(1/k)}_{\mathsf{tactic}}(\ell,S)$ for $1\leqslant\ell\leqslant S\leqslant 5$. (Theorem 10.1)

	$\ell=1$	2	3	4	5
$S=1$	$\frac{1}{2}=0.5$				
2	$\frac{3}{7}\doteq 0.429$	$\frac{2}{3}\doteq 0.667$			
3	$\frac{11}{29}\doteq 0.379$	$\frac{3}{5}=0.6$	$\frac{3}{4}=0.75$		
4	$\frac{25}{73}\doteq 0.342$	$\frac{11}{20}=0.55$	$\frac{9}{13}\doteq 0.692$	$\frac{4}{5}=0.8$	
5	$\frac{137}{437}\doteq 0.314$	$\frac{25}{49}\doteq 0.510$	$\frac{11}{17}\doteq 0.647$	$\frac{3}{4}=0.75$	$\frac{5}{6}\doteq 0.833$

TABLE 12. $\pi_{\mathsf{same}}^{\mathsf{Borda}(1/k)}(\ell,S)=\pi_{\mathsf{wPSC}}^{\mathsf{Borda}(1/k)}(\ell,S)$ for $1\leqslant\ell\leqslant S\leqslant 5$. (Theorem 10.1)

References

[1] Haris Aziz, Markus Brill, Vincent Conitzer, Edith Elkind, Rupert Freeman & Toby Walsh: Justified representation in approval-based committee voting. *Social Choice and Welfare*, **48** (2017), no. 2, 461–485. Extended abstract: *Proceedings of the 29th AAAI Conference on Artificial Intelligence (AAAI 2015)*, AAAI Press, 2015.

[2] Haris Aziz & Barton Lee: The expanding approvals rule: improving proportional representation and monotonicity. Preprint, 2017. arXiv:1708.07580

[3] Michel L. Balinski & H. Peyton Young: *Fair Representation.* 2nd ed. Brookings Institution Press, Washington, D.C., 2001.

[4] Jan Behrens, Axel Kistner, Andreas Nitsche & Björn Swierczek: *The Principles of LiquidFeedback.* Interaktive Demokratie e. V., Berlin, 2014.

[5] Jean-Charles de Borda: Mémoire sur les élections au scrutin. *Histoire de l'Académie Royale des Sciences. Année MDCCLXXXI*, 657–665 and 31–34, Paris 1784. http://gallica.bnf.fr/ark:/12148/bpt6k35800 (7 October, 2018)

[6] Steven J. Brams & Peter C. Fishburn: *Approval Voting.* Birkhäuser, Boston, 1983.

[7] Steven J. Brams & D. Marc Kilgour: Satisfation approval voting. In *Voting Power and Procedures*, Rudolf Fara, Dennis Leech, Maurice Salles, eds. Springer, Cham, Switzerland, 2014.

[8] Markus Brill, Rupert Freeman, Svante Janson & Martin Lackner: Phragmén's voting methods and justified representation. *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI 2017)*, AAAI Press, Palo Alto, CA, 2017, 406–413.

[9] Andrew McLaren Carstairs: *A Short History of Electoral Systems in Western Europe.* George Allen & Unwin, London, 1980.

[10] Victor D'Hondt: *Système pratique et raisonné de représentation proportionnelle*, Muquardt, Brussels, 1882. http://doi.org/10.3931/e-rara-39876

[11] Victor D'Hondt: Formule du minimum dans la représentation proportionnelle. *La Représentation Proportionnelle, Revue Mensuelle* **2** (1883), no. 6, 117–128.

[12] Charles Lutwidge Dodgson: *The Principles of Parliamentary Representation.* London, 1884.

[13] Henry Richmond Droop: On methods of electing representatives, *Journal of the Statistical Society of London* **44** (1881), no. 2, 141–202.

[14] Michael Dummett: *Voting Procedures.* Clarendon Press, Oxford, 1984.

[15] Edith Elkind, Piotr Faliszewski, Piotr Skowron & Arkadii Slinko: Properties of multiwinner voting rules. *Social Choice and Welfare*, **48** (2017), no. 3, 599–632.

[16] Piotr Faliszewski, Piotr Skowron, Arkadii Slinko & Nimrod Talmon: Multiwinner voting: a new challenge for social choice theory. *Trends in Computational Social Choice*, Ulle Endriss ed., AI Access, 2017.

[17] David M. Farrell: *Electoral Systems.* 2nd ed., Palgrave Macmillan, Basingstoke, 2011.

[18] Philippe Flajolet & Robert Sedgewick: *Analytic Combinatorics.* Cambridge Univ. Press, Cambridge, UK, 2009.

[19] Sixten von Friesen, Gustaf Appelberg, Ivar Bendixson & E. Phragmén: *Betänkande angående ändringar i gällande bestämmelser om den proportionella valmetoden.* 3 December 1913, Stockholm, 1913. [Report of the Royal Commission on the Proportional Election Method]

[20] Günter Hägele & Friedrich Pukelsheim: The electoral systems of Nicholas of Cusa in the Catholic Concordance and beyond. *The Church, the Councils, & Reform – The Legacy of the Fifteenth Century*, 229–249. Catholic University of America Press, Washington, DC, 2008.

[21] Eduard Hagenbach-Bischoff: *Die Frage der Einführung einer Proportionalvertretung statt des absoluten Mehres.* H. Georg, Basel, 1888.

[22] Eduard Hagenbach-Bischoff: Manière de trouver le chiffre répartiteur. *La Représentation Proportionnelle, Revue Mensuelle* **7** (1888), 266–272.
Reprinted from *Bulletin de la Société Suisse pour la Représentation Proportionnelle – Bulletin des Schweizerischen Wahlreform-Vereins für Proportionale Volksvertretung* **5** (1888), 235–243.

[23] *Inter-Parliamentary Union*: PARLINE database on national parliaments. `http://www.ipu.org/parline/` (7 October, 2018)

[24] Svante Janson: Proportionella valmetoder. Preprint, 2012–2018. `http://www2.math.uu.se/~svante/papers/sjV6.pdf` (7 October, 2018)

[25] Svante Janson: Asymptotic bias of some election methods. *Annals of Operations Research*, **215** (2014), no. 1, 89–136.

[26] Svante Janson: Phragmén's and Thiele's election methods. Preprint, 2016. `arXiv:1611.08826`

[27] Michael A. Jones & Jennifer M. Wilson: Evaluation of thresholds for power mean-based and other divisor methods of apportionment. *Math. Social Sci.* **59** (2010), no. 3, 343–348.

[28] D. Marc Kilgour: Approval balloting for multi-winner elections. *Handbook on Approval Voting*, J.-F. Laslier and M. R. Sanver eds., Chapter 6. Springer-Verlag, Berlin, 2010

[29] Klaus Kopfermann: *Mathematische Aspekte der Wahlverfahren.* Wissenschaftsverlag, Mannheim, 1991.

[30] Arend Lijphart & Robert W. Gibberd: Thresholds and payoffs in list systems of proportional representation. *European Journal of Political Research*, **5** (1977), no. 3, 219–244.

[31] Iain McLean: The Borda and Condorcet principles: three medieval applications. *Social Choice and Welfare* **7** (1990), no. 2, 99–108.

[32] Antonio Palomares & Victoriano Ramírez: Thresholds of the divisor methods. *Numer. Algorithms* **34** (2003), no. 2-4, 405–415.

[33] Edvard Phragmén: Sur une méthode nouvelle pour réaliser, dans les élections, la représentation proportionelle des partis. *Öfversigt av Kongl. Vetenskaps-Akademiens Förhandlingar 1894*, N:o 3, Stockholm, 133–137.

[34] Edvard Phragmén: *Proportionella val. En valteknisk studie.* Svenska spörsmål 25, Lars Hökersbergs förlag, Stockholm, 1895.

[35] Edvard Phragmén: Sur la théorie des élections multiples, *Öfversigt av Kongl. Vetenskaps-Akademiens Förhandlingar 1896*, N:o 3, Stockholm, 181–191.

[36] Edvard Phragmén: Till frågan om en proportionell valmetod. *Statsvetenskaplig Tidskrift* **2** (1899), nr 2, 297–305.

[37] *The Politics of Electoral Systems.* Eds. Michael Gallagher & Paul Mitchell. Oxford Univ. Press, Oxford, 2005.

[38] Friedrich Pukelsheim: *Proportional Representation. Apportionment Methods and Their Applications.* 2nd ed., Springer, Cham, Switzerland, 2017.

[39] Douglas W. Rae: *The Political Consequences of Electoral Laws.* Rev. ed. Yale Univ. Press, New Haven, 1971.

[40] Douglas Rae, Victor Hanby & John Loosemore: Thresholds of representation and thresholds of exclusion. An analytic note on electoral systems. *Comparative Political Studies* **3** (1971), no. 4, 479–488.

[41] André Sainte-Laguë: La représentation proportionnelle et la méthode des moindres carrés. *Ann. Sci. École Norm. Sup.* (3) **27** (1910), 529–542.
Summary: *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, **151** (1910), 377–378.

[42] Luis Sánchez-Fernández, Norberto Fernández, Jesús A. Fisteus & Pablo Bastanta Val: Some notes on justified representation. *Proceedings of the 10th Multidisciplinary Workshop on Advances in Preference Handling (MPREF)*, 2016.
`http://www.mpref-2016.preflib.org/program/` (7 October, 2018)

[43] Luis Sánchez-Fernández, Edith Elkind, Martin Lackner, Norberto Fernández, Jesús A. Fisteus, Pablo Bastanta Val & Piotr Skowron: Proportional justified representation. *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI 2017)*, AAAI Press, Palo Alto, CA, 2017, 670–676.
Full version (with appendices): `arXiv:1611.09928`

[44] Nore B. Tenow: Felaktigheter i de Thieleska valmetoderna. *Statsvetenskaplig Tidskrift* 1912, 145–165.

[45] Thorvald N. Thiele: Om Flerfoldsvalg. *Oversigt over det Kongelige Danske Videnskabernes Selskabs Forhandlinger* 1895, København, 1895–1896, 415–441.

[46] Nicolaus Tideman: The Single Transferable Vote, *Journal of Economic Perspectives*, **9** (1995), no. 1, 27–38.
[47] Robert J Weber: Approval voting. *J. Economic Perspectives* **9** (1995), no. 1, 39–49.
[48] Douglas R. Woodall: Properties of preferential election rules. *Voting matters* **3** (1994), 8–15.

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