PHRAGMÉN'S AND THIELE'S ELECTION METHODS

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Abstract. The election methods introduced in 1894–1895 by Phragmén and Thiele, and their somewhat later versions for ordered (ranked) ballots, are discussed in detail. The paper includes definitions and examples and discussion of whether the methods satisfy some properties, including monotonicity, consistency and various proportionality criteria. The relation with STV is also discussed. The paper also contains historical information on the methods.

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1. Introduction

The purpose of this paper is to give a detailed presentation in English of the election methods by Edvard Phragmén [58; 59; 60; 61] and Thorvald Thiele [78], originally proposed in 1894 and 1895, respectively; we show also some properties of them. (The presentation is to a large extent based on my discussion in Swedish in [42, Chapters 13 and 14].) Both methods were originally proposed for unordered ballots (see below), but ordered versions were later developed, so we shall consider four different methods. We also briefly discuss some other, related, methods suggested by Thiele [78].

Versions of both Phragmén's and Thiele's methods have been used in Swedish parliamentary elections (for distribution of seats within parties), and Phragmén's method is still part of the election law, although in a minor role, while Thiele's method is used for some elections in e.g. city councils, see Appendix D.

Brief biographies of Phragmén and Thiele are given in Appendix A.

1.1. **Background.** The problem that Phragmén's and Thiele's methods try to solve is that of electing a set of a given number s of persons from a larger set of candidates. Phragmén and Thiele discussed this in the context of a parliamentary election in a multi-member constituency; the same problem can, of course, also occur in local elections, but also in many other situations such as electing a board or a committee in an organization.

One simple method that has been used for centuries is the *Block Vote*, where each voter votes for s candidates, and the s candidates with the largest number of votes win. (See further Appendix E.1.1.) In the late 19th century, when (in Sweden) political parties started to became organized, it was evident that the Block Vote tends to give all seats to the largest party, and as a consequence there was much discussion about more proportional election methods that give representation also to minorities.

Many proportional election methods have been constructed. Among them, the ones that dominate today are *list methods*, where each voter votes for a party, and then each party is given a number of seats according to some algorithm, see Appendix E.3. An important example is *D'Hondt's method* (Appendix E.3.1) proposed by D'Hondt [25] in 1882. Phragmén and Thiele were inspired by D'Hondt's method, and Phragmén [58; 59] called his method a generalization of D'Hondt's method, but they did not want a list method. They wanted to keep the voting method of the Block Vote, where each voter chooses a set of persons, arbitrarily chosen from the available candidates, without any formal role for parties. Thus, a voter could select candidates based on their personal merits and views on different questions, and perhaps combine candidates from different parties and independents. (This frequently happened, see Examples 13.1 and 13.2 from the general election 1893.) Then, an algorithm more complicated than the simple Block Vote would give the seats to candidates in a way that, hopefully, would give a proportional representation to minorities. As we shall see in Section 11, both Phragmén's and Thiele's methods achieve this at least in the special case when there are parties with different lists, and every voter votes for one of the party lists; then both methods give the same result as D'Hondt's method. However, the methods were designed to cope also with more complicated cases, when two different voters may vote for partly the same and partly different candidates.

Phragmén's and Thiele's methods, especially the ordered versions, are thus close in spirit to STV (Appendix E.2.1), which also is a proportional election method where parties play no role, see Section 17 for a closer comparison. (Phragmén knew about STV, at least in Andræ's version, and had proposed a version of it before developing his own method, see Section 18.5.)

1.2. **Contents of the paper.** Phragmén's and Thiele's election methods are described in detail in Sections 3–10, in both unordered (Sections 3–7) and ordered (Sections 8–10) versions. Section 11 shows that all the methods reduce to D'Hondt's method in the case of party lists. Section 12 treats the simple special case of a single-member constituency, when only one candidate is elected.

Section 13 contains a number of examples, many of them comparing the methods; some of the examples are constructed to show weak points of some method. (There are also examples in some other sections.) Sections 14–16 discuss further properties of the methods (monotonicity, consistency and proportionality), and Section 17 discusses the relation between Phragmén's method and STV.

Some variants of the methods are described in Section 18.

The purpose of this paper is not to advocate any particular method, but we give a few conclusions in Section 19.

The appendices contain further information, including biographies and the history of the methods. Furthermore, Appendix E gives for the reader's convenience brief descriptions of several other election methods that are related to Phragmén's and Thiele's or occur in the discussions.

2. Assumptions and notation

For the election methods studied here, we assume, as discussed in the introduction, that each voter votes with a ballot containing the names of one or several candidates. (Blank votes, containing no candidates, may also be allowed, but in the methods treated here they are simply ignored.) Parties, if they exist, have no formal role and are completely ignored by the methods.

These election methods are of two different types, with different types of ballots:

- **Unordered ballots:** The order of the names on a ballot does not matter. In other words, each ballot is regarded as a set of names. (Sometimes called *approval ballots*, since the voter can be seen as approving some of the candidates.)
- **Ordered ballots:** The order of the names on a ballot matters. Each ballot is an ordered list of names. (Sometimes called *ranked ballots*, or *preferential voting*.)

Of course, the practical arrangemants may vary; for example, the voter may write the names by hand, or there might be a printed or electronic list of all candidates where the voter marks his choices by a tick (unordered ballots) or by 1, 2, . . . in order of preference (ordered ballots).

In some election methods, there are restrictions on the number of candidates on each ballot (see Appendix E.1–E.2 for examples). We make no such assumptions for Phragmén's and Thiele's methods; each voter can vote for an arbitrary number of candidates. However, the unordered versions can be modified by allowing at most s (the number to be elected) candidates on each ballot (for philosophical reasons or for practical convenience); this was for example done in the version of Thiele's method used in Sweden 1909–1922, see Appendix D.[1] For the ordered versions, there is no point in forbidding (or allowing) more than s names on each ballot, since only the s first names can matter.[2]

It is often convenient to consider the different types of ballots that appear, and count the number of ballots of each type. If α is a type of ballot that appears (or might appear), let v_α denote the number of ballots of that type, i.e., the number of voters choosing exactly this ballot. Furthermore, let $V = \sum_\alpha v_\alpha$ denote the total number of (valid) votes, and let $p_\alpha := v_\alpha/V$ be the proportion of the votes that are cast for α.

Remark 2.1. Many election methods are *homogeneous*, meaning that the result depends only on the proportions p_α. This includes Phragmén's and Thiele's methods.[3]

Remark 2.2. It is possible to let different voters have different weights, which in principle could be any positive real numbers. The only difference is that v_α now is the total weight of all voters choosing α, and that this is a real number, not necessarily an integer.[4] We leave the trivial modifications for this extension to the reader, and continue to talk about numbers of votes.

Remark 2.3. Every election method has to have provisions for the case that a tie occurs between two or more candidates. Usually ties are resolved

[1] I do not know whether Phragmén intended this restriction or not. I cannot find anything stated explicitly about it in Phragmén's papers [58; 59; 61], but all his examples are of this type. Remember also that Phragmén intended his method as an alternative to the then used Block Vote (Appendix E.1.1), where unordered ballots with this restriction were used, and it is possible that he intended the same for his method. Thiele [78], on the contrary, gives several examples with more names on the ballots than the number elected.

[2] This does not hold for all methods using ordered ballots, for example neither for STV (Appendix E.2.1) nor for scoring rules (Appendix E.2.3), as is easily seen be considering the case $s = 1$.

[3] Although homogeneity seems like a natural property, not all election methods used in practice are strictly homogeneous. In quota methods, see e.g. [64], as well as in STV (Appendix E.2.1), a "quota" is calculated, and this is (perhaps by tradition) usually rounded to an integer, see [64, Section 5.8] for several examples, meaning that the election method is not homogeneous. Similarly, in Phragmén's method as used in Sweden, see Appendix C.1, rounding to two decimal places is specified for all intermediary calculations; again this means that the method is not strictly homogeneous. However, in both cases the methods are asymptotically homogeneous, as the number of votes gets large, and for practical purposes they can be regarded as homogeneous, at least for public elections.

[4] This was the case in local elections in Sweden 1909–1918, when a voter had 1–40 votes depending on income, and a modification of Thiele's method was used, see Appendix D.

by lot, although other rules are possible.[5] Phragmén originally proposed a special rule for his method, see Appendix B, but he seems to have dropped this later and we shall do the same. We assume that ties are resolved by lot or by some other rule, and we shall usually not comment on this.

2.1. **Some notation.** We let throughout s be the number of seats, i.e., the number of candidates to be elected; we assume that s is fixed and determined before the election. We use a variable such as i for an unspecified candidate. In examples, candidates are usually denoted by capital letters A, B,...

The *outcome* of the election is the set $\mathcal{E}$ of elected candidates. By assumption, $|\mathcal{E}| = s$, where $|\mathcal{E}|$ denotes the number of elements of the set $\mathcal{E}$. When there are ties, there may be several possible outcomes.

In the discussions below, 'candidate' and 'name' are synonymous. Similarly, we identify a voter and his/her ballot.

In numerical examples, $\doteq$ is used for decimal approximations (correctly rounded).

2.1.1. *Unordered ballots.* In a system with unordered ballots, each ballot can be seen as a set of candidates, so the different types of ballots are subsets of the set of all candidates. We denote such sets by σ. Note that candidate i appears on a ballot of type σ if and only if $i \in \sigma$; hence, the total number of ballots containing i is $\sum_{\sigma \ni i} v_\sigma$.

2.1.2. *Ordered ballots.* In a system with ordered ballots, the different types of ballots are ordered list of some (or all) candidates. We denote such ordered lists by α.

3. Phragmén's unordered method

3.1. **Phragmén's formulation.** Phragmén presented his method in a short note in 1894 [58], followed by further discussions, motivations and explanations in [59; 60; 61], see Appendix B. His definition is as follows (in my words and with my notation). Phragmén assumes that the ballots are of the unordered type in Section 2, i.e., that each ballot contains a set of candidates, without order and without other restrictions. A detailed example is given in Section 3.3; further examples are given in Section 13.

Phragmén's unordered method. *Assume that each ballot has some* voting power t*; this number is the same for all ballots and will be determined later. A candidate needs total voting power* 1 *in order to be elected. The voting power of a ballot may be used by the candidates on that ballot, and it may be divided among several of the candidates on the ballot. During the procedure described below, some of the voting power of a ballot may be already assigned to already elected candidates; the remaining voting power of the ballot is free.*

[5] Non-mathematical rules are occasionally used, for example giving preference to the oldest candidate. (Such rules have not been used in connection with Phragmén's or Thiele's methods as far as I know.)

The seats are distributed one by one.

For each seat, each remaining candidate may use all the free voting power of each ballot that includes the candidate. (I.e., the full voting power t except for the voting power already assigned from that ballot to candidates already elected.) The ballot voting power t that would give the candidate voting power 1 is computed, and the candidate requiring the smallest voting power t is elected. All free (i.e., unassigned) voting power on the ballots that contain the elected candidate is assigned to that candidate, and these assignments remain fixed throughout the election.

The computations are then repeated for the next seat, for the remaining candidates, and so on.

Ties are, as said in Section 2, broken by lot, or by some other supplementary rule. (See Appendix B for Phragmén's original suggestion.)

Note that the required voting power t increases for each seat, except possibly in the case of a tie.

We consider the method defined above in some more detail. We begin with the first seat. Since candidate i appears on $\sum_{\sigma\ni i} v_\sigma$ ballots, the smallest voting power t that makes it possible for i to get the seat, provided it gets all voting power from each available ballot, is thus $t_i = 1/\sum_{\sigma\ni i} v_\sigma$. Phragmén's rule is that the seat is given to the candidate i that requires the smallest voting power t_i. Hence the first seat goes to the candidate appearing on the largest number of ballots.

Suppose that the first seat goes to candidate i, and that this requires voting power $t^{(1)} = t_i$. The ballots containing i have thus all used voting power $t^{(1)}$ for the election of i; this allocation will remain fixed forever. We now increase the voting power t of all ballots, noting that the ballots containing i only have $t - t^{(1)}$ free voting power available for the remaining candidates. We again calculate the smallest t such that some candidate may be given total voting power 1; we give this candidate, say j, the second seat and let $t^{(2)}$ be the required voting power. Furthermore, on the ballots containing j, we allocate all available voting power to the election of j.

We continue in the same way. In general, suppose that $n \geqslant 0$ seats have been allocated so far, and that this requires voting power $t^{(n)}$. Suppose further that on each ballot for the set σ, an amount r_σ of the voting power already is used, with $0 \leqslant r_\sigma \leqslant t^{(n)}$. If the voting power of each ballot is increased to $t \geqslant t^{(n)}$, then each ballot for σ has thus a free voting power $t-r_\sigma$, which can be used by any of the candidates on the ballot. The voting power available for candidate i is thus

$$\sum_{\sigma\ni i} v_\sigma(t - r_\sigma) = t\sum_{\sigma\ni i} v_\sigma - \sum_{\sigma\ni i} v_\sigma r_\sigma \tag{3.1}$$

and for this to be equal to 1 we need t to be

$$t_i = \frac{1 + \sum_{\sigma\ni i} v_\sigma r_\sigma}{\sum_{\sigma\ni i} v_\sigma}. \tag{3.2}$$

The next seat is then given to the candidate with the smallest t_i; if this is candidate i, the required voting power $t^{(n+1)}$ is thus t_i, so r_σ is updated to

$$r'_\sigma := t^{(n+1)} = t_i = \frac{1 + \sum_{\sigma \ni i} v_\sigma r_\sigma}{\sum_{\sigma \ni i} v_\sigma} \tag{3.3}$$

for each σ such that $i \in \sigma$. (r_σ is unchanged for σ with $i \notin \sigma$.)

These formulas give an algorithmic version of Phragmén's method.

Remark 3.1. In [61], Phragmén describes the method in an equivalent way using the term *load* instead of voting power; the idea is that when a candidate is elected, the participating ballots incur a total load of 1 unit, somehow distributed between them. The candidates are elected sequentially. In each round, the loads are distributed and the candidates are chosen such that the maximum load of a ballot is as small as possible. (The same description is used by Cassel [5].) This is also a useful formulation, and it will sometimes be used below.

Remark 3.2. Phragmén [61] illustrates also the method by imagining the different groups of ballots as represented by cylindrical vessels, with base area proportional to the number of ballots in each group. The already elected candidates are represented by a liquid that is fixed in the vessels, and the additional voting power required to elect another candidate is represented by pouring 1 unit of a liquid into the vessels representing a vote for that candidate, distributed among these vessels such that the height of the liquid will be the same in all of them. This is to be tried for each candidate; the candidate that requires the smallest height is elected, and the corresponding amounts of liquid are added to the vessels and fixed there.

Remark 3.3. Sometimes it is convenient to think of the voting power as increasing continuously with time; at time t each ballot has voting power t. The voting power available to each candidate thus also increases with time, and as soon as some candidate reaches voting power 1, this candidate is elected and the free voting power on each participating ballot is permanently assigned to this candidate (which reduces the free voting power to 0 for these ballots, and thus typically reduces the available voting power for other candidates). This is repeated until s candidates have been elected.

3.2. **An equivalent formulation.** The numbers t_i above will in practice be very small, and it is often more convenient to instead use $W_i := 1/t_i$. We also let $q_\sigma := v_\sigma r_\sigma$; this is the total voting power allocated so far from the ballots of type σ, and can be interpreted as the (fractional) number of seats already elected by these ballots; q_σ is called the *place number* of this group of ballots. Note that $\sum_\sigma q_\sigma$ always equals the number of candidates elected so far.

This leads to the following algorithmic formulation. To see that it really is equivalent to the formulation in Section 3.1, it suffices to note that with $W_i = 1/t_i$ and $q_\sigma = v_\sigma r_\sigma$, (3.4) below is the same as (3.2), and the update rule (3.5) is the same as (3.3).

PHRAGMÉN'S UNORDERED METHOD, EQUIVALENT FORMULATION. *Seats are given to candidates sequentially, until the desired number have been elected. During the process, each type of ballot, i.e., each group of identical ballots, is given a* place number*, which is a rational non-negative number that can be interpreted as the fractional number of seats elected so far by these ballots; the sum of the place numbers is always equal to the number of seats already allocated. The place numbers are determined recursively and the seats are allocated by the following rules:*

(i) *Initially all place numbers are* 0*.*

(ii) *Suppose that $n \geqslant 0$ seats have been allocated. Let q_σ denote the place number for the ballots with a set σ of candidates; thus $\sum_\sigma q_\sigma = n$. The total number of votes for candidate i is $\sum_{\sigma \ni i} v_\sigma$, and the total place number of the ballots containing candidate i is $\sum_{\sigma \ni i} q_\sigma$. The reduced vote*[6] *for candidate i is defined as*

$$W_i := \frac{\sum_{\sigma \ni i} v_\sigma}{1 + \sum_{\sigma \ni i} q_\sigma}, \tag{3.4}$$

i.e., the total number of votes for the candidate divided by 1+ *their total place number.*

(iii) *The next seat is given to the candidate i that has the largest W_i.*

(iv) *Furthermore, if candidate i gets the next seat, then the place numbers are updated for all sets σ that participated in the election, i.e., the sets σ such that $i \in \sigma$. For such σ, the new place number is*

$$q'_\sigma := \frac{v_\sigma}{W_i} = \Bigl(1 + \sum_{\sigma \ni i} q_\sigma\Bigr) \frac{v_\sigma}{\sum_{\sigma \ni i} v_\sigma}. \tag{3.5}$$

For σ such that $\sigma \not\ni i$, $q'_\sigma := q_\sigma$.

Steps (ii)–(iv) *are repeated as many times as desired.*

We see that the number W_i may be interpreted as the total number of votes for candidate i, reduced according to the extent to which the ballots containing i already have successfully participated in the election of other candidates. (For that reason, we call W_i the "reduced vote" above.) Cf. D'Hondt's method, see Appendix E.3.1, which as said above, Phragmén tried to generalize.

Remark 3.4. Let $W^{(n)}$ be the winning (i.e., largest) reduced vote when the n-th seat is filled. Then, by (3.5), during the calculations above, the current place number q_σ is $v_\sigma / W^{(\ell)}$, if the last time that some candidate on the ballot (i.e., in σ) was elected was in round ℓ. (Provided any of them has been elected; otherwise $q_\sigma = 0$; in this case we may define $\ell = 0$ and $W^{(0)} := \infty$.)

Remark 3.5. It is in practice convenient to use place numbers q_σ defined for groups of identical ballots as above, but it is sometimes also useful to

[6] The Swedish term is *jämförelsetal* (comparative figure).

consider the place number of an individual ballot; for a ballot of type σ this is $r_\sigma = 1/W^{(\ell)}$, with ℓ as in Remark 3.4.

Remark 3.6. The equivalence with the formulation in Section 3.1 shows that

$$W^{(n)} = 1/t^{(n)} \tag{3.6}$$

with $t^{(n)}$ as in Section 3.1. (And $t^{(0)} := 0$.)

3.3. **An example.** Phragmén [58, 59, 60] illustrates his (unordered) method with the following example (using slightly varying descriptions in the different papers). We present detailed calculations (partly taken from Phragmén) using both formulations above.

Example 3.7 (Phragmén's unordered method)**.**
Unordered ballots. 3 seats. Phragmén's method.

1034 ABC
519 PQR
90 ABQ
47 APQ

The total numbers of votes for each candidate are thus

A 1171
B 1124
C 1034
P 566
Q 656
R 519.

Using the formulation of Phragmén's method in Section 3.1, we see that the smallest voting power that gives some candidate command of voting power 1 is $t^{(1)} = 1/1171 \doteq 0.000854$, which gives A voting power $1171/1171 = 1$. Hence A (which has the largest number of votes) is elected to the first seat.

If now the voting power is increased to $t > t^{(1)}$, then each ballot ABC, ABQ or APQ has free voting power $t - t^{(1)} = t - 1/1171$, while each ballot PQR has free voting power t. Hence, the voting power that each of the remaining candidates can use is

B: $1124(t - t^{(1)})$
C: $1034(t - t^{(1)})$
P: $519t + 47(t - t^{(1)}) = 566t - 47t^{(1)}$
Q: $519t + 137(t - t^{(1)}) = 656t - 137t^{(1)}$
R: $519t$.

In order for these values to be equal to 1, we need for B the voting power t to be, see (3.2),

$$t_{\mathrm{B}} := \frac{1 + 1124t^{(1)}}{1124} = t^{(1)} + \frac{1}{1124} = \frac{2295}{1316204} \doteq 0.001744,$$

and for Q a voting power

$$t_{\mathrm{Q}} := \frac{1+137t^{(1)}}{656} = t^{(1)} + \frac{1-519t^{(1)}}{656} = \frac{327}{192044} \doteq 0.001703,$$

while the remaining candidates obviously require more than either B or Q. Since $t_{\mathrm{Q}} < t_{\mathrm{B}}$, Q is elected to the second seat, and $t^{(2)} = t_{\mathrm{Q}}$.

For the third seat, if the voting power of each ballot is increased further to $t > t^{(2)}$, then each ballot ABC has free voting power $t - t^{(1)}$, and each ballot PQR, ABQ or APQ has free voting power $t - t^{(2)}$. Hence, the voting power that each remaining candidate can use is

B: $1034(t-t^{(1)}) + 90(t-t^{(2)}) = 1124t - 1034t^{(1)} - 90t^{(2)}$
C: $1034(t-t^{(1)})$
P: $566(t-t^{(2)})$
R: $519(t-t^{(2)})$.

In order for this to equal 1, B needs the voting power t to be

$$t_{\mathrm{B}} := \frac{1+1034t^{(1)}+90t^{(2)}}{1124} = \frac{195525}{107928728} \doteq 0.001812$$

while P needs t to be

$$t_{\mathrm{P}} := \frac{1+566t^{(2)}}{566} = t^{(2)} + \frac{1}{566} = \frac{188563}{54348452} \doteq 0.003470$$

while C and R obviously require more than B and Q, respectively. Since $t_{\mathrm{B}} < t_{\mathrm{P}}$, B is elected to the third seat. (Also, $t^{(3)} = t_B$.)

Hence, the elected by Phragmén's method are AQB.

Using "load" instead of "voting power", see Remark 3.1, we do the same calculations; we now say that first A is elected which gives a load $t^{(1)} = 1/1171$ to each ballot ABC, ABQ or APQ (total load $1171t^{(1)} = 1$); then Q is elected which gives a load $t^{(2)} = 327/192044$ to each ballot PQR and an additional load $t^{(2)} - t^{(1)} = 163/192044$ to each ballot ABQ or APQ (total new load $519 \cdot 327/192044 + 137 \cdot 163/192044 = 1$); finally, B is elected which gives an additional load $t^{(3)} - t^{(1)} = 103357/107928728$ to each ballot ABC and $t^{(3)} - t^{(2)} = 11751/107928728$ to each ballot ABQ (total new load $1034 \cdot 103357/107928728 + 90 \cdot 11751/107928728 = 1$). The final loads on the ballots of the four types are $(t^{(3)}, t^{(2)}, t^{(3)}, t^{(2)})$, where $t^{(2)} = 327/192044 \doteq 0.001703$ and $t^{(3)} = 195525/107928728 \doteq 0.001812$.

Using the formulation with place numbers and reduced votes in Section 3.2, we obtain the same result by similar but somewhat different calculations, cf. (3.6). For the first seat, the reduced votes W_i are just the number of votes for each candidate; hence A is elected with $W_{\mathrm{A}} = 1171$. This gives a place number $1/1171$ for each participating ballot, and thus $1034/1171 \doteq 0.8830$ for all ballots ABC together, $90/1171 \doteq 0.0769$ for the ballots ABQ and $47/1171 \doteq 0.0401$ for the ballots APQ.

Hence, for the second seat, the reduced votes for B and Q are, by (3.4),

$$W_{\mathrm{B}} = \frac{1124}{1 + 1124/1171} = \frac{1316204}{2295} \doteq 573.51$$

(since the 1124 ballots containing B have a combined place number 1124/1171) and

$$W_{\mathrm{Q}} = \frac{656}{1 + 137/1171} = \frac{192044}{327} = 587.29$$

(since the 656 ballots containing Q have a combined place number 137/1171), while the remaining candidates have smaller reduced votes ($W_{\mathrm{C}} \doteq 549.12$, $W_{\mathrm{P}} \doteq 544.16$, $W_{\mathrm{R}} = 519$). Hence Q is elected to the second seat. The place numbers for the four groups of ballots are $\frac{1034}{W_{\mathrm{A}}} = \frac{1034}{1171} \doteq 0.8830$, $\frac{519}{W_{\mathrm{Q}}} = \frac{169713}{192044} \doteq 0.8837$, $\frac{90}{W_{\mathrm{Q}}} = \frac{14715}{96022} \doteq 0.1532$, $\frac{47}{W_{\mathrm{Q}}} = \frac{15369}{192044} \doteq 0.0800$, with sum 2.

For the third seat, we have the reduced votes

$$W_{\mathrm{B}} = \frac{1124}{1 + 1034/1171 + 14715/96022} = \frac{107928728}{195525} \doteq 551.99$$

and

$$W_{\mathrm{P}} = \frac{566}{1 + 169713/192044 + 15369/192044} = \frac{54348452}{188563} \doteq 288.22$$

while the remaining candidates have smaller reduced votes ($W_{\mathrm{C}} \doteq 549.12$ as for the second seat, $W_{\mathrm{R}} \doteq 275.52$). Hence B is elected to the third seat.

4. Thiele's unordered methods

Thiele [78] praised Phragmén's contribution [58], but proposed a different method based on a different idea. Thiele realized that his idea led to a difficult optimization problem that was not practical to solve, so he also proposed two approximations of the method. There are thus three different methods by Thiele for unordered ballots; we may call them *Thiele's optimization method*, *Thiele's addition method* and *Thiele's elimination method*,[7] but since only the addition method has found practical use, we mainly consider this method and we often call the addition method simply *Thiele's method*. (The addition method was also the only of the methods that was considered in the discussions in Sweden in the early 20th century, see Appendix D and e.g. [5], [6].) The three methods are defined below. A detailed example is given in Section 4.4; further examples are given in Section 13.

[7] In Thiele [78] (in Danish), the optimization method has no name, the addition method is called *Tilføjelsesreglen* and the elimination method is called *Udskydelsesreglen*; in his examples, there are also captions in French, with the names *règle d'addition* and *règle de rejet*.

When the addition method was used in Sweden 1909–1922, it was called *Reduktionsregeln* (the Reduction Rule).

4.1. **Thiele's optimization method.** Thiele's idea was that a voter that sees n of the candidates on his[8] ballot elected, will feel a *satisfaction* $f(n)$, for some increasing function $f(n)$, and the result of the election should be the set of s candidates that maximizes the total satisfaction, i.e., the sum of $f(n)$ over all voters.

In formulas, if a set $\mathcal{E}$ of candidates is elected, a voter that has voted for a set σ will feel a satisfaction $f(|\sigma \cap \mathcal{E}|)$.

As Thiele notes, we can without loss of generality assume $f(0) = 0$ and $f(1) = 1$. We also let $w_n := f(n) - f(n-1)$, the added satisfaction when a voter sees the n-th candidate elected; thus

$$f(n) = \sum_{k=1}^{n} w_k. \tag{4.1}$$

Of course, the result of the method depends heavily on the choice of the function $f(n)$. Thiele discusses this, and argues that the function depends on the purpose of the election. He claims that for the election of a government or governing body, each new member is as important as the first, so

$$f(n) = n \tag{4.2}$$

(i.e., $w_n = 1$); he calls this the *strong* method. On the other hand, for the election of a committee for comprehensive investigation of some issue, he sees no point in having several persons of the same meaning, so he sets

$$f(n) = \mathbf{1}\{n \geqslant 1\} := \begin{cases} 0, & n = 0, \\ 1, & n \geqslant 1, \end{cases} \tag{4.3}$$

(i.e., $w_1 = 1$, $w_n = 0$ for $n \geqslant 2$); he calls this the *weak* method. Finally, Thiele claims that there are many cases between these two extremes, in particular when electing representatives for a society, where proportional representation is desired.[9]

Thiele notes that the choice (4.2) gives the result that the s candidates with the largest numbers of votes will be elected; this is thus Approval Voting (Appendix E.1.2). (If each ballot can contain at most s names we obtain Block voting, Appendix E.1.1.) Thiele notes that this yields a method that is far from proportional.

On the other hand, Thiele notes that the choice

$$f(n) = 1 + \frac{1}{2} + \dots + \frac{1}{n}, \tag{4.4}$$

[8]In 1895, only men were allowed to vote, in Denmark as well as in Sweden.

[9]Whether this argument is convincing or not is perhaps for the reader to decide. Tenow [76] argues that Thiele here rather seems to evaluate the methods by the desirability of their outcome, and that his "satisfaction" thus becomes a fiction and is chosen to achieve the desired result. However, while this philosophical question may be relevant for applications of the methods, it is irrelevant for our main purpose, which is to present the methods as algorithms and give some of their mathematical properties.

(i.e., $w_n = 1/n$), yields a proportional method in the case when no name appears on more than one type of ballot, see Section 11. This is therefore Thiele's choice for his *proportional* method.

In the sequel we shall use the choice (4.4), except when we explicitly state otherwise.[10] Thiele's optimization method for proportional elections is thus the following:

THIELE'S OPTIMIZATION METHOD. *For each set $\mathcal{S}$ of s candidates, calculate the "satisfaction"*

$$F(\mathcal{S}) := \sum_{\sigma} v_\sigma f(|\sigma \cap \mathcal{S}|), \tag{4.5}$$

where the function f is given by (4.4)*. Elect the set $\mathcal{S}$ of the given size that maximizes $F(\mathcal{S})$.*

However, Thiele notes that the maximization over a large number of sets $\mathcal{S}$ is impractical. With n candidates to the s seats, there are $\binom{n}{s}$ sets $\mathcal{S}$ that have to be considered, and as an example, Thiele [78] mentions 30 candidates to 10 seats, when there are more than 30 million combinations (30 045 015).[11] Thiele [78] thus for practical use proposes two approximation to his optimization method, where candidates either are selected one by one (the addition method), or eliminated one by one (the elimination method), in both cases maximizing the total satisfication in each step. These methods are described in detail in the following subsections.[12] (Thiele was aware that the methods might give different results, and showed this in some of his examples; see Examples 13.3–13.4 and Remark 5.2.)

Remark 4.1. Thiele's optimization method was reinvented by Simmons in 2001, under the name *Proportional Approval Voting (PAV)*, see [45].

4.2. **Thiele's addition method.** Thiele's addition method is a "greedy" version of his optimization method, where candidates are elected one by one, and for each seat, the candidate is elected that maximizes the increase of the total satisfaction of the voters. For a general satisfaction function f satisfying (4.1), the satisfication of a ballot σ containing a candidate i is increased by w_{k+1} when i is elected, where k is the number of already elected on this ballot. This yields the following simple description, where we use $w_n = 1/n$ to obtain a proportional method. The general version is obtained by replacing $1/(k+1)$ by some (arbitrary) numbers $w_{k+1} \geqslant 0$.

[10]The strong version (4.2) is as said above equivalent to Approval Voting. The weak version (4.3) seems to be more interesting mathematically than for practical applications. See [78, Examples 5 and 6] for an example including all three methods with the weak function (4.3), in this case yielding different results.

[11]Using a concept not existing in Thiele's days, the problem of finding the maximizing set(s) is NP-hard, see [12, Theorem 1] or [72, Theorem 3].

[12]Thiele [78] recommends using the elimination method, for the reason that the addition method selects the elected sequentially, which might give the first elected pretensions to be superior to their colleagues.

Thiele's (addition) method. *Seats are given to candidates sequentially, until the desired number have been elected.*

For each seat, a ballot where $k \geqslant 0$ names already have been elected is counted as $1/(k+1)$ votes for each remaining candidate on the ballot. The candidate with the largest vote count is elected.

As said above, we usually call the method "Thiele's method".

Remark 4.2. Thiele's addition method with the weak satisfaction function (4.3) is studied in Aziz et al. [11] and is there called *Greedy Approval Voting (GAV)*. (In this case, each seat goes to the candidate with most votes among the ballots that do not contain any already elected candidate.)

4.3. **Thiele's elimination method.** Thiele's elimination method works in the opposite direction. Weak candidates are eliminated until only s (the desired number) of them remain. Each time, the candidate is eliminated that minimizes the decrease of the total satisfaction of the voters caused by the elimination. For a general satisfaction function f satisfying (4.1), the satisfication of a ballot σ with k remaining candidates is decreased by w_k when one of them is eliminated. This yields the following description, where again we use $w_n = 1/n$ to obtain a proportional method. The general version is obtained by replacing $1/k$ by some (arbitrary) numbers $w_k \geqslant 0$.

Thiele's elimination method. *Candidates are eliminated one by one, until only s remain. The remaining ones are elected. In each elimination step, a ballot where $k \geqslant 1$ names remain is counted as $1/k$ votes for each remaining candidate on the ballot. The candidate with the smallest vote count is eliminated.*

Note the (superficial?) similarity with the Equal and Even Cumulative Voting in Appendix E.1.5, but note that here the calculation is iterated, with the vote counts changing as candidates are eliminated.

Remark 4.3. Thiele's elimination method has recently been reinvented (under the name *Harmonic Weighting*) as a method for ordering alternatives for display for the electronic voting system *LiquidFeedback* [49].

4.4. **An example.** We illustrate Thiele's unordered methods by the same example as was used to illustrate Phragmén's method in Example 3.7.

Example 4.4 (Thiele's optimization, addition and elimination methods)**.**
Unordered ballots. 3 seats. Thiele's three methods.

1034 ABC
519 PQR
90 ABQ
47 APQ

With Thiele's optimization method (Section 4.1), we note first that in this example, we have dominations A > B > C and Q > P > R, in the sense that, for example, replacing A by B or C always decreases the satisfaction of a set of candidates; hence it suffices to consider the four possible outcomes ABC, ABQ, APQ, PQR instead of all $\binom{6}{3} = 20$ possible sets of three candidates. These sets yield the satisfactions

ABC: $1034 \cdot \frac{11}{6} + 519 \cdot 0 + 90 \cdot \frac{3}{2} + 47 \cdot 1 = 6233/3 \doteq 2077.67$
ABQ: $1034 \cdot \frac{3}{2} + 519 \cdot 1 + 90 \cdot \frac{11}{6} + 47 \cdot \frac{3}{2} = 4611/2 = 2305.5$
APQ: $1034 \cdot 1 + 519 \cdot \frac{3}{2} + 90 \cdot \frac{3}{2} + 47 \cdot \frac{11}{6} = 6101/3 \doteq 2033.67$
PQR: $1034 \cdot 0 + 519 \cdot \frac{11}{6} + 90 \cdot 1 + 47 \cdot \frac{3}{2} = 1112$

Hence the largest satisfaction is given by ABQ, so ABQ are elected.

With Thiele's addition method (Section 4.2), the first seat goes to the candidate with the largest number of votes, i.e., A (1171 votes, see Example 3.7).

For the second seat, all ballots ABC, ABQ and APQ now are worth 1/2 vote each. This gives the vote counts

B $1034/2 + 90/2 = 562$
C $1034/2 = 517$
P $519 + 47/2 = 542.5$
Q $519 + 90/2 + 47/2 = 587.5$
R 519.

Thus Q has the highest vote count and is elected to the second seat.

For the third seat, the ballots ABC and PQR have the value 1/2, and the ballots ABQ and APQ have the value 1/3 each. Hence the vote counts are

B $1034/2 + 90/3 = 547$
C $1034/2 = 517$
P $519/2 + 47/3 \doteq 275.17$
R $519/2 = 259.5$.

Hence B gets the third seat.

With Thiele's elimination method (Section 4.3), in the first round, each ballot is counted as 1/3 (since they all contain 3 names), so R has the smallest vote count (519/3=173) and is eliminated.

This increases the value of the ballots PQR to 1/2 each for P and Q, so the vote counts in the second round are

A $1034/3 + 90/3 + 47/3 = 1171/3 \doteq 390.33$
B $1034/3 + 90/3 = 1124/3 \doteq 374.67$
C $1034/3 \doteq 344.67$
P $519/2 + 47/3 = 1651/6 \doteq 275.17$
Q $519/2 + 90/3 + 47/3 = 1831/6 \doteq 305.17$.

P has the smallest vote count and is eliminated.

In the third round, the ballots are worth $\frac{1}{3}, 1, \frac{1}{3}, \frac{1}{2}$, and thus the vote counts are

A $1034/3 + 90/3 + 47/2 = 2389/6 \doteq 398.17$
B $1034/3 + 90/3 = 1124/3 \doteq 374.67$
C $1034/3 \doteq 344.67$

Q $519 + 90/3 + 47/2 = 1145/2 = 572.5$.

Thus C is eliminated, and the remaining ABQ are elected.

We see that in this example, all three Thiele's methods yield the same result, which also agrees with Phragmén's method (Example 3.7).

5. House monotonicity

The *Alabama paradox* occurs when the house size s (i.e., the number of seats) is increased, with the same set of ballots, and as a result someone loses a seat; see [13] for the historical context. (See also [44].) An election method is *house monotone* if the Alabama paradox cannot occur. More formally, and taking into account the possibility of ties, an election method is house monotone if whenever, for a given set of ballots, $\mathcal{S}_s$ is a possible outcome of the election for s seats, then there exists a possible outcome $\mathcal{S}_{s+1}$ for $s+1$ seats such that $\mathcal{S}_{s+1} \supset \mathcal{S}_s$.

Obviously, any sequential method where seats are assigned one by one is house monotone; this includes Phragmén's method and Thiele's addition method in both the unordered versions above and the ordered versions below. Similarly, Thiele's elimination method is house monotone. However, Thiele's optimization method is not house monotone (as noticed by Thiele [78]), see Examples 13.3 and 13.4. Example 13.4 can be generalized as follows to other satisfaction functions f.

Theorem 5.1. *Let $f(n)$ be a satisfaction function with $f(0) = 0$ and $f(1) = 1$. Then Thiele's optimization method is house monotone if and only if $f(n) = n$.*

Proof. First, if $f(n) = n$, then, as said above, Thiele's optimization method is equivalent to Approval Voting (Appendix E.1.2), which obviously is house monotone.

For the converse, consider an election with 4 votes, for AB, AC, B, C. For $s = 1$, a possible outcome is A, but if $w_2 < 1$, then the only outcome for $s = 2$ is BC. In the opposite direction, in an election with the 2 votes A and BC, a possible outcome for $s = 1$ is A, but if $w_2 > 1$, then the only outcome for $s = 2$ is BC. Hence, if the method is house monotone, then $w_2 = 1$.

More generally, for any $n \geqslant 0$, consider the same two examples but add n further candidates $D_1, \dots, D_n$ to every ballot. Then the only outcome for $s = n$ is $D_1 \cdots D_n$. For $s = n+1$, a possible outcome is $D_1 \cdots D_n$A, but for $s = n+2$, the outcome is $D_1 \cdots D_n$BC in the first example if $w_{n+2} < w_{n+1}$, and in the second example if $w_{n+2} > w_{n+1}$. Hence, if the method is house monotone, then $w_{n+2} = w_{n+1}$ for every $n \geqslant 0$, and thus $w_n = w_1 = 1$ for every $n \geqslant 1$. □

The ties in the examples used in the proof can be avoided by modifying the examples so that the different alternatives have slightly different numbers of votes.

Remark 5.2. Thiele's optimization method and addition method coincide, for a general satisfaction function f, if and only if the optimization method is house monotone. (This is obvious, at least in the absence of ties, from the fact that the addition method is a greedy version of the optimization method. The case with ties is perhaps easiest seen by using Theorem 5.1.) Hence, the two methods coincide only for $f(n) = n$.

Similarly, Thiele's optimization method and elimination method coincide only for $f(n) = n$.

The examples in the proof above also show that the addition and elimination methods do not coincide for any other function f. This can be seen without further calculations by noting that in these examples, for $s = n + 1$, the addition method and the optimization method coincides, while for $s = n+2$ (when only one candidate is eliminated), the elimination method coincides with the optimization method; hence, if the addition method coincides with the elimination method, then the optimization method coincides with the addition method in these examples, which is impossible when it is not house monotone.

As said above, Thiele [78] noted that for the "proportional" satisfaction function (4.4), the three methods may give different result, see Example 13.3. He also gave an example showing the same for the "weak" satisfaction function (4.3) [78, Examples 5 and 6].

6. Unordered ballots, principles

The unordered version of Phragmén's method and Thiele's methods satisfy the following general principles:

Principles for unordered versions of Phragmén's and thiele's methods**.**

(U1) *The ballots are unordered, so that each voter lists a number of candidates, but their order is ignored.*

(U2) *In the allocation of a seat, each ballot is counted fully for every candidate on it (ignoring the ones that already have been elected). The value of a ballot is reduced (in different ways for the different methods) when someone on it is elected, but the effective value of a ballot does not depend on the number of unelected candidates on it.*

As a consequence of (U2), the methods have also the following property, which reduces the need for tactical voting.

(U3) *If any number of candidates are added to a ballot, but none of them is elected, then the result of the election is not affected.*

Note that (U2) and (U3) hold also for Approval Voting (which can be seen as a special case of Thiele's general optimization method, see Section 4.1), but not for some other election methods with unordered ballots, for example Equal and Even Cumulative Voting (Appendix E.1.5), where a vote is split between the names on it, and the value of a ballot for each candidate on it decreases if more names are added.

7. Unordered ballots and decapitation

Unordered ballots have some problems that are more or less independent of the election method used to distribute the seats, and in particular apply to both Phragmén's and Thiele's methods.

If there are organized parties, a party usually fields more candidates than the number that will win seats, for example to be on the safe side since the result cannot be predicted with certainty in advance. If all voters of the party are loyal to the party and vote for the party list, the result will be that all candidates on the party list will tie, and the ones that are elected will be chosen by lot from the party list.

The party can avoid this random selection, and for example arrange so that the party leader gets a seat (if the party gets any seat at all), by organizing a scheme where a small group of loyal members vote for specially selected subsets of the party list, with the result that the party's candidates will get slightly different number of votes, in the order favoured by the party organization. However, this will still be sensitive to a coup from a rather small minority in the party, that might secretly agree not to vote for, say, the party leader. To protect against such coups, the party can give instructions to vote on specific sets of candidates to a larger number of voters, but it is certainly a serious drawback of an election system to depend on complicated systems of tactical voting. (This might also, depending on the system, make it more difficult for the party in the competition with other parties, and the party might risk to get fewer seats.)

Moreover, even if party A is well-disciplined and there is no internal opposition, it is possible that a small group from another party, say B, will cast their votes on some less prominent and perhaps less able candidates from party A, instead of voting for their own party. Of course, that might give party A another seat, but with a small number of tactical votes, this risk is small. On the other hand, there is a reasonable chance that the extra votes will elect the chosen candidates from A instead of the ones preferred by the party. This tactical manoeuvre, called *decapitation*, can thus prevent e.g. the party leader of an otherwise successful party to be elected.

Decapitation does not necessarily occur because of sinister tactics by some groups; it can also occur by mistake, when too many voters believe that their primary candidate is safe and therefore also vote for others. One such situation is discussed in Example 13.17. (There in connection with further complications caused by an election system with additional rules.)

I do not know to what extent such decapitation occurred in practice, but at least the unintentional type in Example 13.17 occurred [6, p. 8], and decapitation was something that was feared and much discussed in the discussions about electoral reform in Sweden around 1900; see for example

[5, p. 25] and [6, pp. 13, 20, 33, 35]. As a result, versions using ordered ballots were developed of both Phragmén's method and Thiele's addition method, as described in the following sections.[13] (We do not know of any ordered versions of Thiele's optimization and elimination methods, except that *Bottoms-up*, see Appendix E.2.2, perhaps might be seen as an ordered version of his elimination method.)

8. Ordered ballots, principles

The ordered versions of Phragmén's method and Thiele's (addition) method differ from the unordered versions in Sections 3 and 4 in two ways:

Principles for ordered versions.

(O1) *The ballots are ordered, so that each voter lists a number of candidates in order.*

(O2) *In the allocation of a seat, each ballot is counted only for one candidate, viz. the first candidate on it that is not already elected.*

This is detailed in the following sections.

Note that (O2) implies the following property, similar to (U3) for the unordered versions.

(O3) *If any number of candidates are added after the existing names on a ballot, then the result of the election is not affected unless all existing candidates are elected.*

Thus, a voter can add names after his or her favourite candidates without risking to hurt their chances.

9. Phragmén's ordered method

Phragmén's ordered method is thus obtained by modifying the version in Section 3 using the principles (O1)–(O2) in Section 8. This method was proposed in 1913 by a Royal Commission on the Proportional Election Method [6] (see Appendix D); Phragmén was one of the members of the commission, so it is natural to guess that he developed also this version.[14] (A similar method had been proposed by Tenow [75] in 1910.)

The method has been used in Swedish elections for the distribution of seats within parties since 1921, although it now plays only a secondary role; see Appendices D and C.1.

[13] Phragmén [59] makes another suggestion to avoid the problem of decapitation (saying that it is just one possibility among many), while keeping unordered ballots. In this proposal, parties could register ordered party lists. The ballots would still be regarded as unordered, but a vote on a party list, say ABCDE, would in addition to the vote ABCDE also be regarded as (for example) $\frac{1}{10}$ extra vote on shorter lists, in this case ABCD, ABC, AB and A, split between them with $\frac{1}{40}$ extra vote each. It seems that everyone ignored this suggestion, including Phragmén himself in later writings, and it seems for good reasons.

[14] Phragmén [57] had already in 1893 used ordered ballots in one example when he discussed a version of STV.

9.1. **First formulation.** The ordered version can thus be defined as follows, cf. Section 3.1. A detailed example is given in Section 9.4; further examples are given in Section 13.

PHRAGMÉN'S ORDERED METHOD, FORMULATION 1. *Assume that each ballot has some* voting power t*; this number is the same for all ballots and will be determined later. A candidate needs total voting power* 1 *in order to be elected. During the procedure described below, some of the voting power of a ballot may be already assigned to already elected candidates; the remaining voting power of the ballot is free, and is used by the ballot's current top candidate, i.e., the first candidate on the ballot that is not already elected.*

The seats are distributed one by one.

For each seat, each ballot is counted for its current top candidate. (If all candidates on a ballot are elected, the ballot is ignored.) The top candidate receives the free voting power of the ballot. (I.e., the full voting power except for the voting power already assigned from that ballot to candidates already elected.) For each candidate, the ballot voting power t that would give the candidate voting power 1 *is computed, and the candidate requiring the smallest voting power t is elected. All free (i.e. unassigned) voting power on the ballots that were counted for the elected candidate is assigned to that candidate, and these assignments remain fixed throughout the election.*

The computations are then repeated for the next seat, with the remaining candidates, and so on.

Remark 9.1. As for the unordered version in Section 3, we can think of the voting power of each ballot increasing continuously with time, see Remark 3.3. Again, the voting power available to each candidate increases with time, and when some candidate reaches voting power 1, that candidate is elected. (A difference from the unordered version is that the available voting powers of the other candidates do not change when someone is elected; in particular, the voting powers of the candidates never decrease.)

9.2. **Second formulation.** The same calculations as in Section 3 show that the formulation above is equivalent to the following more explicit algorithm, cf. Section 3.1:

PHRAGMÉN'S ORDERED METHOD, FORMULATION 2. *Seats are given to candidates sequentially, until the desired number have been elected. During the process, each type of ballot, i.e., each group of identical ballots, is given a* place number*, which is a rational non-negative number that can be interpreted as the (fractional) number of seats elected so far by these ballots; the sum of the place numbers is always equal to the number of seats already allocated. The place numbers are determined recursively and the seats are allocated by the following rules:*

(i) *Initially all place numbers are* 0*.*
(ii) *Suppose that $n \geqslant 0$ seats have been allocated. Let q_α denote the place number for the ballots with a list α of candidates; thus $\sum_\alpha q_\alpha = n$.*

For each candidate i, let A_i be the set of lists α such that i is the first element of α if we ignore candidates already elected. The total number of votes counted for candidate i in this step is $\sum_{\alpha\in A_i} v_\alpha$, and the total place number of the ballots counted for candidate i is $\sum_{\alpha\in A_i} q_\alpha$. The reduced vote for candidate i is defined as

$$W_i := \frac{\sum_{\alpha\in A_i} v_\alpha}{1+\sum_{\alpha\in A_i} q_\alpha}, \tag{9.1}$$

i.e., the total number of votes counted for the candidate divided by 1 + *their total place number.*

(iii) *The next seat is given to the candidate i that has the largest W_i.*

(iv) *Furthermore, if candidate i gets the next seat, then the place numbers are updated for all lists α that participated in the election, i.e., the lists $\alpha \in A_i$. For such α, the new place number is*

$$q'_\alpha := \frac{v_\alpha}{W_i} = \Bigl(1+\sum_{\alpha\in A_i} q_\alpha\Bigr)\frac{v_\alpha}{\sum_{\alpha\in A_i} v_\alpha}. \tag{9.2}$$

For $\alpha \notin A_i$, $q'_\alpha := q_\alpha$.

Steps (ii)–(iv) *are repeated as many times as desired.*

Indeed, the commission report [6] that proposed the method in 1913 introduced it in this form; the committee first discussed the current method and various proposals to improve it, and showed by examples that they all were unsatisfactory, and then presented the method above gradually, through a sequence of examples of increasing complexity, as the correct generalization of D'Hondt's method to ordered lists.[15]

Remark 9.2. As in Section 3, the connection between the two formulations above is that $W_i = 1/t_i$, where t_i is the voting power per ballot that would give i voting power 1 in the current round. In particular, (3.6) still holds for the winning voting power and reduced vote in each round.

9.3. **Third formulation.** In practical applications of the method, it is not necessary to keep track of and calculate place numbers for each individual type of ballot; since each ballot counts only for its current top name, it suffices to group ballots according to their current top name. This simplification was introduced already in the commission report [6] where the method was introduced, and it is used in the Swedish Elections Act, where this method has played a part (within parties) since 1921, see Appendix D, and where the method is defined in the following form (in my words; for the actual text of the Elections Act, see Appendix C.1):

Phragmén's ordered method, formulation 3. *Seats are given to candidates sequentially, until the desired number have been elected. During the*

[15]The commission report also noted that the method was essentially the same as the method for unordered ballots presented in 1894 by one of the members of the commission, Phragmén [58, 59].

process, the ballots are organized in groups; the ballots in each group have the same top candidate, ignoring candidates already elected, and the group is valid for that candidate. Each group is also given a place number*, which is a rational non-negative number that can be interpreted as the (fractional) number of seats elected so far by these ballots; the sum of the place numbers of the different groups is always equal to the number of seats already allocated.*

The groups are created, their place numbers are determined and the seats are allocated by the following rules:

(i) *Initially, the ballots are divided into groups according to their first name. Each group is given place number* 0.

(ii) *Suppose that* $n \geqslant 0$ *seats have been allocated. Denote the current set of groups of ballots by* Γ*, and for each group* $\gamma \in \Gamma$*, let* v_γ *be its number of votes and* q_γ *its place number; furthermore, let* i_γ *be the candidate that the group is valid for. For each candidate* i*, let* $\Gamma_i := \{\gamma \in \Gamma : i_\gamma = i\}$*, the set of groups valid for that candidate, and define the reduced vote by*

$$W_i := \frac{\sum_{\gamma\in\Gamma_i} v_\gamma}{1 + \sum_{\gamma\in\Gamma_i} q_\gamma}, \tag{9.3}$$

i.e., the total number of votes currently valid for the candidate divided by $1 +$ *the total place number of the corresponding groups.*

(iii) *The next seat is given to the candidate* i *that has the largest* W_i.

(iv) *If candidate* i *gets the next seat, then the groups valid for* i *are merged together and then divided into new groups according to the new top candidate on the ballots, i.e., the first candidate after* i *that is not already elected. (Ballots were all candidates are elected are put aside and ignored in the sequel.) Each new group* γ *is given a place number*

$$q_\gamma := \frac{v_\gamma}{W_i}. \tag{9.4}$$

All other groups remain unchanged, and keep their place number.

Steps (ii)–(iv) *are repeated as many times as desired.*

It is easily seen that this version is equivalent to the version in Section 9.2; each group contains all ballots of some set of types (each given by a list α of names), and the number of votes and the place number for the group are just the sums of the number of votes and of the place numbers for these types.

9.4. **An example.** We illustrate Phragmén's ordered method by his unordered example in Example 3.7, but now interpreting the ballots as ordered (with the first candidate first).

Example 9.3 (Phragmén's ordered method)**.**

Ordered ballots. 3 seats. Phragmén's method.

1034 ABC

519 PQR
90 ABQ
47 APQ

We use the formulation of Phragmén's ordered method in Section 9.2.

Since each ballot is counted only for its first name, we have in the first round 1171 votes for A and 519 votes for P, so A is elected.

This gives a place number (= assigned voting power = load) 1/1171 to each ballot ABC, ABQ or APQ, and thus, in the second round, the four groups of ballots have place numbers $\frac{1034}{1171}, 0, \frac{90}{1171}, \frac{47}{1171}$, see (9.2). (If we use the formulation in Section 9.3, we group the ballots ABC and ABQ together; this group has 1124 votes and place number $\frac{1124}{1171}$.)

In the second round, the top names are B and P. They have 1124 and 566 votes with total place numbers $\frac{1124}{1171} \doteq 0.9599$ and $\frac{47}{1171} \doteq 0.0401$ (with sum 1), respectively, and thus reduced votes

$$W_{\mathrm{B}} = \frac{1124}{1+\frac{1124}{1171}} = \frac{1316204}{2295} \doteq 573.51$$

and

$$W_{\mathrm{P}} = \frac{566}{1+\frac{47}{1171}} = \frac{331393}{609} \doteq 544.16$$

Thus B is elected to the second seat.

The place numbers for the four groups of ballots are now $\frac{1034}{W_{\mathrm{B}}} = \frac{1186515}{658102} \doteq 1.8029$, 0, $\frac{90}{W_{\mathrm{B}}} = \frac{103275}{658102} \doteq 0.1569$, $\frac{47}{1171} \doteq 0.0401$ (with sum 2).

In the third round, the top names on the four types of ballots are C, P, Q, P, and their reduced votes are

$$W_{\mathrm{C}} = \frac{1034}{1+\frac{1186515}{658102}} = \frac{680477468}{1844617} \doteq 368.90$$
$$W_{\mathrm{P}} = \frac{566}{1+\frac{47}{1171}} = \frac{331393}{609} \doteq 544.16$$
$$W_{\mathrm{Q}} = \frac{90}{1+\frac{103275}{658102}} = \frac{59229180}{761377} \doteq 77.79.$$

Hence, P is elected to the third seat.

Elected: ABP.

10. Thiele's ordered method

Thiele's ordered method is obtained by modifying his unordered addition method in Section 4.2 using the principles in Section 8.

Thiele's ordered method. *Seats are given to candidates sequentially, until the desired number have been elected.*

For each seat, a ballot is counted for the ballot's current top candidate, i.e., the first candidate on the ballot that has not already been elected; if this is the k-th name on the ballot (so the preceding $k-1$ have been elected), then *the ballot is counted as* $1/k$ *vote. (Ballots where all candidates have been elected are ignored.) The candidate with the largest vote count is elected.*

A detailed example is given in Section 10.1; further examples are given in Section 13.

Remark 10.1. This is based on Thiele's proportional satisfaction function (4.4). Of course, just as in Section 4, other functions f may be used. We leave this to the reader to explore.

The ordered version of Thiele's method was proposed in the Swedish parliament in 1912 (by Nilson in Örebro), as an improvement of the then used unordered Thiele's method for distribution of seats within parties (see Appendix D). It was not adopted, but it was later adopted, still within parties, for elections inside city and county councils, and it is still used for that purpose, see Appendix C.2.

10.1. **An example.** We illustrate also Thiele's ordered method with the same example as in Examples 3.3, 4.4 and 9.4.

Example 10.2 (Thiele's ordered method)**.**

Ordered ballots. 3 seats. Thiele's method.

1034 ABC
519 PQR
90 ABQ
47 APQ

Since each ballot is counted only for its first name, we have in the first round (as in Example 9.3) 1171 votes for A and 519 votes for P, so A is elected.

For the second seat, all ballots ABC, ABQ and APQ now have a value 1/2 each. This gives the vote counts

B $1034/2 + 90/2 = 562$
P $519 + 47/2 = 542.5$

Thus B has the highest vote count and is elected to the second seat.

For the third seat, the four types of ballots have the values $\frac{1}{3}, 1, \frac{1}{3}, \frac{1}{2}$ each. Hence the vote counts are

C $1034/3 \doteq 344.67$
P $519 + 47/2 = 542.5$
Q $90/3 = 30.$

Hence P gets the third seat.

Elected: ABP. (As with Phragmén's method in Example 9.3.)

11. Phragmén's and Thiele's methods generalize D'Hondt's

The election methods considered in this paper are blind to the possible existence of parties. Nevertheless, it is important to see how they perform in the presence of parties. We therefore consider the following special case:

PARTY LIST CASE. *There is a number of parties such that:*

- *Each party has a single list of candidates. (Ordered or unordered, depending on the election method.)*
- *The lists are disjoint. (I.e., no person runs for more than one party.)*
- *Every voter votes for one of the party lists.*

In this case, Phragmén's and Thiele's methods reduce to D'Hondt's method (described in Appendix E.3.1). This was explicitly stated (and shown) by Phragmén [59, pp. 38–40] for his method; Thiele [78, p. 425] states explicitly only the special case when the fraction of votes for each party is an integer times $1/s$, so that a perfectly proportional representation is possible, but his proof really shows the general result.

Theorem 11.1 (Phragmén, Thiele)**.** *In the party list case just described, Phragmén's unordered and ordered methods and Thiele's three unordered methods and his ordered method (assuming the proportional satisfaction function* (4.4)*) all yield the same distribution of seats between parties as D'Hondt's method.*

Proof. More or less obvious, and left to the reader. Note that for the unordered methods, all candidates from the same party will tie, and that for Phragmén's methods, the place number of a list equals the number of seats already given to that list (since lists are disjoint). □

Remark 11.2. For the unordered methods, there is a minor difference with D'Hondt's method in the case of a tie, if that is resolved by lot. In D'Hondt's method, a tie will be resolved by choosing one of the tying parties at random, while in Phragmén's and Thiele's methods, one of the tying candidates is chosen at random, and thus a party with many candidates will have a larger chance of getting the seat.

Remark 11.3. For the unordered methods, by the assumptions of the party list case, all candidates from the same party will tie, so which of them that will be elected to the seats won by the party will typically be determined by lot. (See also Section 7.) For the ordered methods, the seats of a party will be assigned in the order the candidates appear on the list.

Remark 11.4. The theorem extends to Thiele's methods with a different satisfation function $f(n) = \sum_{k=1}^{n} w_k$; in the party list case they all give the same result as the divisor method with divisors $1/w_n$ (see Appendix E.3). See [20], where also further related results are given.

12. Electing a single person

Another simple special case is when $s = 1$, so that only one person is to be elected. (A single-member constituency.) Of course, in this case, there are simpler methods, but it is of interest to see how the methods above perform.

Theorem 12.1. *In the case $s = 1$, the following holds.*

(i) *For unordered ballots, Phragmén's method and Thiele's optimization and addition method elect the candidate with the largest number of votes. I.e., they are equivalent to Approval Voting (Appendix E.1.2).*

(ii) *For ordered ballots, Phragmén's and Thiele's methods elect the candidate with the largest number of first votes. I.e., they are equivalent to Single-Member Plurality (First-Past-The-Post, see Appendix E.1.1). (Names after the first name on a ballot are thus ignored.)*

Proof. Obvious. □

Remark 12.2. Note that Thiele's elimination method may give a different result, see Example 13.4.

13. Examples

We give several examples to illustrate and compare the different methods. (Many of them are constructed by various authors to illustrate weaknesses of some method.) See also Examples 3.7, 4.4, 9.3, and 10.2.

It is seen that, except in very simple cases, Phragmén's methods lead to rational numbers with large denominators and that they thus are not suitable for hand calculations; this is not a problem with computer assistance.[16 17]

In the examples below, ABC denotes a ballot with the names A, B and C; in the case of ordered ballots, the names are given in order, with A the first name.

In several of the examples, there are obvious ties because of symmetries. We then usually give only one, typical, outcome, sometimes without comment.

Example 13.1 (Phragmén's unordered method)**.** Phragmén [59, pp. 19, 28, 44] discusses the following example, taken from the parliamentary election 1893 (Stockholm's second constituency). There were actually 2624 valid ballots, but for simplicity Phragmén ignores 73 ballots of various types with less than 10 ballots each.[18]

Unordered ballots. 5 seats. Phragmén's method.[19 20]

1233 TVFPE
585 TLNBQ
124 TVFLN

[16]In the use of the method in Sweden (Appendix C.1), all numbers are calculated to two decimal places (rounded down), so exact rational arithmetic is not used.

[17]For computational aspects, see also [19].

[18]The five remaining types, shown below, were all promoted by various political organizations; note that there are considerable overlaps.

[19]T = Themptander, V = von Friesen, F = Fredholm, P = Palme, E = Eklund, L = Lovén, N = Nordenskiöld, B = Backman, Q = Pettersson, J = Johansson, W = Wikman.

[20]The actual election used the Block Vote, and the elected were, counting also the ballots ignored here, TVFPE with 1995, 1470, 1458, 1316 and 1313 votes, respectively; the next candidate had 1301 votes [9, p. 29].

547 LNBQJ
62 VFPEW

Phragmén uses this example to discuss several conceivable methods, and gives detailed calculations for his method which results in the election of, in order (after resolving ties arbitrarily) TLVNF.

Example 13.2 (Phragmén's and Thiele's unordered methods)**.** Phragmén [59, p. 49] gives also another example from the parliamentary election 1893 (Stockholm's fifth constituency). There were actually 1520 valid votes, of which 57 (of types with less than 10 ballots each) are ignored in the example.

Unordered ballots. 5 seats. Phragmén's and Thiele's methods.[21] [22]

680 HEOBG
341 HEXBG
322 HYPQG
49 HPXQG
47 ZYPQL
14 HEOBX
10 HPOQG

Phragmén shows that his method would elect HGEBQ (or HGEBP; the last seat is tied, as is the order between E and B, but both are elected in any case).

Thiele [78, Examples 1 and 2] uses the same example (omitting Z and L to show that all ballots do not have to have the same number of names, and perhaps also to shorten the calculations) and shows that his addition and elimination methods both also elect HGEBQ (or HGEBP).

Thiele's optimization method yields also the same result. (Thiele [78] did not calculate this, presumably because even with his omission of two candidates, there are $\binom{9}{5} = 126$ sets to consider.)

Example 13.3 (Thiele's three unordered methods)**.** The following example by Thiele [78, Examples 3 and 4], here slightly modified to avoid ties, shows that the three methods by Thiele (see Section 4) can give different results.

Unordered ballots. 2 seats.

960 ACD
3000 BCD
520 BC
1620 AB
1081 AD
1240 AC
360 BD

[21] H = Höjer, G = G. Eriksson, E = P.J.M. Eriksson, B = Bergström, O = Olsson, X = Branting, Q = Östberg, P = Palm, Y = Hellgren, Z = Billmansson, L = Lindvall.

[22] The actually elected, using the Block Vote and counting also the ballots ignored here, were HGEBO, with 1454, 1421, 1055, 1053, 715 votes, respectively, with 469 votes for the next candidate [9, p. 29].

360 D
120 C
60 B

Thiele's optimization method elects AB, his addition method elects CA and his elimination method elects BD.

This example also shows that the optimization method is not house monotone, see Section 5; if only 1 is elected, the optimization method elects C (as the addition method does; the elimination method elects D).

Phragmén's method elects CA, and thus gives the same result as Thiele's addition method.

(With 3 seats, Thiele's optimization method and elimination method elect BCD, while the addition method and Phragmén's method elect CAD.)

Example 13.4 (Thiele's three unordered methods)**.** Tenow [76] gave a simpler example showing that Thiele's three unordered methods are different (and thus also that Thiele's optimization method is not house monotone).

Unordered ballots.

12 AB
12 AC
10 B
10 C

For 1 seat, Thiele's optimization method elects A, but for 2 seats, the method elects BC.

Thiele's addition method elects A for 1 seat, and AB or AC for 2 seats.

Thiele's elimination method eliminates first A; hence it elects BC for 2 seats, and B or C for 1 seat.

In the examples above, Phragmén's method yields the same result as Thiele's (addition) method. The following examples are constructed to show differences between the two methods (and to show weaknesses of Thiele's methods). The first three examples are rather similar, and have two parties (or factions) that agree on one (or several) common candidates.

Example 13.5 (Phragmén's and Thiele's unordered or ordered methods)**.** Phragmén [61] compares his method and Thiele's (addition) method for unordered ballots (and argues in favour of his own) using the following example (here slightly modified to avoid ties). The example works also for ordered ballots, with the given order.

Unordered or ordered ballots.

2001 $AB_1B_2B_3\dots$
1000 $AC_1C_2C_3\dots$

This could be two parties B and C, and a highly respected independent candidate A, or two factions that both accept the same leader.

In an example like this, with two types of ballots that have one common candidate, which in the ordered case comes first, it is obvious that both Phragmén's and Thiele's methods first elect the common candidate A.

After that, as is easily seen from the formulation in Sections 3.1 and 9.1, Phragmén's method treats the remaining parts of the two lists as two disjoint party lists, and thus (see Theorem 11.1) the remaining seats are distributed as by D'Hondt's method. Hence, Phragmén's method will elect candidates in the order $AB_1B_2C_1B_3B_4C_2B_5B_6C_3\dots$.[23]

Thiele's method, on the other hand, reduces the votes of both lists by the same factor after the election of A. As Tenow [76] comments for a related example, Thiele's method here treats A as two different persons; if the two lists had had two different persons A_1 and A_2, then, when both were elected, Thiele's method would have reduced the votes in the same way. (In that case, we would have had disjoint party lists and by Theorem 11.1 the elected would have been $A_1B_1A_2B_2B_3C_1B_4B_5C_2B_6B_7C_3\dots$.) Consequently, the candidates are elected in order $AB_1B_2B_3C_1B_4B_5C_2B_6B_7C_3\dots$

We thus see that Thiele's method favours the larger party. In particular, with 4 seats, Phragmén's method elects $AB_1B_2C_1$ and Thiele's method elects $AB_1B_2B_3$.

Example 13.6 (Phragmén's and Thiele's unordered or ordered methods)**.** Phragmén [61] considers also (with unordered ballots) the following modification of the previous example.

Unordered or ordered ballots.

2000 $AB_1B_2B_3\dots$
1000 $AC_1C_2C_3\dots$
550 $C_1C_2C_3\dots$

A calculation shows that Phragmén's method will elect candidates in the order $AB_1C_1B_2C_2B_3C_3B_4C_4B_5B_6\dots$, while Thiele's method will elect in the order $AC_1B_1B_2C_2B_3C_3B_4C_4B_5B_6\dots$. (The next seat that differs is no. 25, which is a tie for Thiele's method.)

It is not clear that one of these results is "better" or "more fair" than the other, but Phragmén points out that both methods generally give more seats to B than to C (as they should, because B got more votes), but if there are 2 seats, then Thiele's method will give these to AC; Phragmén concludes that at least it is not an advantage of Thiele's method that for 2 seats it puts C ahead of B, while for larger numbers of seats it puts B ahead of C.[24]

In this simple example, it is possible to analyse exactly what happens also for a large number of seats (and candidates, cf. Section 18.4 and [55]). Let $b = b(c)$ be the number elected from the B party before c candidates are elected from the C party, for $c \geqslant 1$.

With Phragmén's method, the election of A gives a load 2/3 to the first group of ballots, and 1/3 to the second. Thus, when C_c is elected, the load on each ballot in the first group is $(b(c) + \frac{2}{3})/2000$ and the load on each

[23] In the unordered case, all B's are tied, as are all C's, so the order among them is arbitrary.

[24] I do not know whether the same can happen with Phragmén's method in another situation.

ballot in one of the two last groups is $(c+\frac{1}{3})/1550$. Consequently, $b = b(c)$ is the largest integer such that

$$\frac{b+\frac{2}{3}}{2000} \leqslant \frac{c+\frac{1}{3}}{1550}, \tag{13.1}$$

i.e.

$$b(c) = \left\lfloor \frac{2000}{1550}\Big(c+\frac{1}{3}\Big) - \frac{2}{3} \right\rfloor = \left\lfloor \frac{40}{31}c - \frac{22}{93} \right\rfloor. \tag{13.2}$$

With Thiele's method, C_c gets elected with $1000/(c+1) + 550/c$ votes, while B_b was elected with $2000/(b+1)$ votes. Thus $b = b(c)$ is the largest integer such that

$$\frac{2000}{b+1} \geqslant \frac{1000}{c+1} + \frac{550}{c}, \tag{13.3}$$

and thus

$$b(c) = \left\lfloor \frac{2000}{\frac{1000}{c+1} + \frac{550}{c}} - 1 \right\rfloor = \left\lfloor \frac{40}{31}c - \frac{161}{961} - \frac{8800}{961(31c+11)} \right\rfloor. \tag{13.4}$$

The claims above are easily verified from (13.2) and (13.4). Note that $\frac{22}{93} \doteq 0.2366 > \frac{161}{961} \doteq 0.1675$. Furthermore, since the last fraction in (13.4) tends to 0 as $c \to \infty$, it follows that except for the first seats, both methods elect B and C in sequences that are periodic with period 71, with 40 B and 31 C. Moreover, it is easily verified that for $c > 1$, (13.2) and (13.4) differ if and only if $c \equiv 11$ or $18 \pmod{31}$, and hence the two methods differ if and only if $s = 2$ or $s \equiv 25$ or $41 \pmod{71}$; when they differ, except for $s = 2$, Phragmén's method favours C and Thiele's method favours B (except that for $s = 25$, as said, Thiele's method gives a tie for the last seat).

Example 13.7 (Phragmén's and Thiele's unordered or ordered methods)**.** Cassel [5] gave an example similar to Example 13.5 (using unordered ballots).

Unordered or ordered ballots. 9 seats.

4200 ABCDEFGHI
1710 ABCUVWXYZ

With both Phragmén's method and Thiele's method, the 3 common candidates ABC are obviously elected first (in arbitrary order in the case of unordered ballots).

With Phragmén's method, the 6 seats after the 3 first are distributed as for two disjoint parties (again easily seen from the formulation in Sections 3.1 and 9.1), i.e., as with D'Hondt's method (see Section 11). The result is 4 of these seats to the larger "party" and 2 to the smaller (since $4200/4 > 1710/2 > 4200/5$); i.e., ABCDEFGUV.

With Thiele's method, however, the last 6 seats go to DEFGHI; thus the largest party (faction) gets all 9 seats, again as by D'Hondt's method regarding ABC as different persons on the two ballots. (The last elected, I, is elected with $4200/9 \doteq 466.67$ votes against $1710/4 = 427.5$ for U.)

Example 13.8 (Phragmén's and Thiele's unordered or ordered methods)**.** An example from the commission report [6] (there with unordered ballots):

Unordered or ordered ballots. 2 seats.

21 AB
20 AC
12 D

Again there are two parties, with the first split into two factions, both recognizing the same leader. Note that since the first party gets more than three times as many votes as the second, an election without splitting the vote between two lists would give both seats (and also a third, if there were one) to the first party by any of the methods considered here (since they would reduce to D'Hondt's method by Theorem 11.1).

Phragmén's method gives the first seat to A (41 votes), and then the second to B (with C next in line); the reduced votes are $W_{\mathrm{B}} = 21/(1+\frac{21}{41}) = \frac{861}{62} \doteq 13.89$, $W_{\mathrm{C}} = 20/(1+\frac{20}{41}) = \frac{820}{61} \doteq 13.44$ and $W_{\mathrm{D}} = 12$.

Thiele's unordered or ordered method gives the first seat to A, which reduces the votes for the first two types of ballots so much that the second seat goes to D, with 12 votes against 10.5 for B and 10 for C. Hence, AD are elected, and the larger party gets only one seat.

Example 13.9 (Phragmén's and Thiele's unordered or ordered methods)**.** Cassel [5, p. 53] gave a similar example (with unordered ballots).

Unordered or ordered ballots.

90 $\mathrm{A}_1\mathrm{A}_2\mathrm{A}_3\dots$
90 $\mathrm{B}_1\mathrm{B}_2\mathrm{B}_3\dots$
90 $\mathrm{B}_1\mathrm{C}_1\mathrm{C}_2\mathrm{C}_3\dots$

There are three different parties (factions) of equal size, but the C party supports B_1.

With Thiele's method, evidently B_1 is elected first. However, as Cassel notes, once B_1 is elected, the B party has no advantage at all of the partial support from the C party, and parties A and B will get equally many seats (possibly tying for the last one), while the C party pays for the support of B_1 as much as if the party had elected one of its own, and will get one seat less than the others (up to ties for the last place). Hence, the candidates are elected in order $\mathrm{B}_1\mathrm{A}_1(\mathrm{A}_2\mathrm{B}_2\mathrm{C}_1)(\mathrm{A}_3\mathrm{B}_3\mathrm{C}_2)\dots$, where the parentheses indicate ties. For example, with 5 seats, the elected are 2 A, 2 B and 1 C, and with 6 seats, the additional seat is a tie between all three parties.

With Phragmén's method, B_1 is still elected first, but this gives only a place number $\frac{1}{2}$ to each of the last two lists. It follows, e.g. by considering loads as in Remark 3.1, that the candidates are elected in order $\mathrm{B}_1\mathrm{A}_1(\mathrm{B}_2\mathrm{C}_1)\mathrm{A}_2(\mathrm{B}_3\mathrm{C}_2)\mathrm{A}_3\dots$. For example, with 5 seats, the elected are 2 A, 2 B and 1 C, as by Thiele's method, but with 6 seats, the additional seat is a tie between B and C only; thus B gets some advantage from the partial support from C, and C does not pay quite as much for it as with Thiele's method.

Example 13.10 (Phragmén's and Thiele's unordered or ordered methods)**.** Tenow [76] gave the following example (with unordered ballots):

Unordered or ordered ballots. 8 seats.

21 ABCDH
21 ABCEI
21 ABCFJ
21 ABCGK
12 OPQRS

In this example, there are two disjoint parties, but one party is split into four different lists. The second party has $12/96 = 1/8$ of the votes, so a proportional representation between the parties would give 7 seats to the first party and 1 to the second, electing for example ABCDEFGO (with many equivalent choices, by symmetry, in the unordered case).

This is also the result of Phragmén's method.

However, Thiele's optimization method, addition method and elimination method in the unordered case and Thiele's ordered method in the ordered case all elect ABCDEFOP, with two seats to the second party.[25]

It seems that Thiele's methods reduce the votes for lists with common names too much in the examples above. On the other hand, the opposite seems to happen with Thiele's ordered method when two list have a common name in a lower position; in this case the method does not reduce the votes enough.

Example 13.11 (Phragmén's and Thiele's ordered methods)**.** An example from the commission report [6]:

Ordered ballots. 3 seats.

34 AC
34 BC
32 D

With Thiele's ordered method, the three seats go to ABC (34 votes each, since for the third seat, C has $34/2 + 34/2$ votes), while D is not elected (32 votes).

Phragmén's method elects instead ABD; for the third seat, the two first lists have place number 1 each, so C has reduced vote $(34 + 34)/3 \doteq 22.67$, while D has 32. This seems to be a more proportional result.

Example 13.12 (Phragmén's and Thiele's ordered methods)**.** Another example from the commission report [6]:

Ordered ballots. 4 seats.

33 AB
32 AC

[25] It is natural that parties that split their votes on several lists may be hurt by this, and that happens also with Phragmén's method in other examples, see e.g. Example 16.7, so one should perhaps be careful with drawing conclusions from this example.

18 DF
17 EF

This example combines the features of Examples 13.8 and 13.11.

With Thiele's ordered method, A is elected first (65 votes), and then D, E, F are elected to the remaining seats (with 18, 17 and 17.5 votes, against 16.5 for B and 16 for C). Thus the ABC party, with 65% of the votes, gets only 1 seat, while the smaller DEF party gets 3.

Phragmén's method also elects A first (65 votes), but then B and C are elected with reduced votes $W_{\mathrm{B}} = 33/(1+\frac{33}{65}) = \frac{2145}{98} \doteq 21.89$ and $W_{\mathrm{C}} = 32/(1+\frac{32}{65}) = \frac{2080}{97} \doteq 21.44$ against $W_{\mathrm{D}} = 18$ and $W_{\mathrm{E}} = 17$; finally, D takes the last seat. Elected: ABCD.

Furthermore, in the following examples, Thiele's methods seem to reduce the votes too little for candidates that appear both on a list where someone else already has been elected, and on a list without any elected. Moreover, and perhaps more seriously, this yields possibilities for tactical voting, where a party may gain seats by carefully splitting its votes on different lists.

Example 13.13 (Phragmén's and Thiele's unordered methods)**.** An example by Tenow [76]:

Unordered ballots. 3 seats.

37 ABC
13 KLM

In this case, there are two disjoint party lists, so both Phragmén's and Thiele's methods reduce to D'Hondt's method (see Theorem 11.1) and, say, ABK are elected. Thus the larger party gets 2 seats and the smaller 1.

However, with Thiele's method, the larger party may cunningly split their votes on five different lists as follows:

1 A
9 AB
9 AC
9 B
9 C
13 KLM

Then A gets the first seat (19 votes), and the next two go to B and C (in some order) with 13.5 votes each, beating KLM with 13. Thus the large party gets all seats. The reason is obviously that only some of the ballots contain A, and thus get their votes reduced for the following seats.

Nevertheless, if news of this scheme is leaked to the small party, they may thwart it by letting two persons vote BKLM. The resulting election is

1 A
9 AB
9 AC
9 B
9 C

2 BKLM
11 KLM

Now, still with Thiele's method, B gets the first seat (20 votes); then C gets the second (18 votes); finally, K, say, gets the third seat with 12 votes, beating A (10 votes). Thus, KLM have gained a seat by voting for the enemy! (Cf. Section 14.)

I do not know whether there are in this example even more cunning schemes that can not be thwarted, and what the best strategies of the two parties are if they do not know in advance how the others vote. (Possibly the best strategies are mixed, i.e. random.) Thiele's method thus leads to interesting mathematical problems in combinatorics and game theory, but for its practical use as a voting method, this example of sensitivity to tactical voting is hardly an asset.

Phragmén's method seems to be much more robust in this respect. In the example above, KLM can always get one seat by voting KLM, regardless of how the voters in the other party vote, see Theorem 16.1.

Example 13.14 (Phragmén's and Thiele's ordered methods)**.** The opportunities for tactical voting shown by Thiele's unordered method in Example 13.13 exist for Thiele's ordered method as well, in a somewhat simpler and perhaps more obvious form. The following example is (partly) from the commission report [6].

Ordered ballots. 2 seats.

61 AB
39 CD

Again, there are two disjoint party lists, so both Phragmén's and Thiele's methods reduce to D'Hondt's method (see Theorem 11.1) and AC are elected. Thus the parties get 1 seat each. (Since both have more than a third of the votes.)

However, with Thiele's method, the larger party may split their votes as follows:

41 AB
20 B
39 CD

With Thiele's method, the first seat goes to A (41 votes, against 20 for B and 39 for C). For the second seat, B has $41/2 + 20 = 40.5$ votes, and beats C. Elected: AB.

In this example, C thus is not elected, in spite of being supported by 39% of the voters. (See further Section 16.) Unlike Example 13.13 with the unordered method, in this case there is nothing that the CD party can do to prevent AB from being elected; even if they help B to be elected to the first seat, A will still beat them to the second seat.

Phragmén's method is more robust (as for the unordered methods); using it, CD will always get (at least) one seat by voting CD, regardless of how the others vote, see Theorem 16.1. In the example above, Phragmén's method elects A to the first seat. This gives place number 1 to the first list, and for the second seat, B has an reduced vote $(41 + 20)/2 = 30.5$ by (9.1). Hence C gets the second seat. Elected: AC.

Example 13.15 (Thiele's ordered method)**.** Consider the following variation of the second election in Example 13.14.

Ordered ballots. 3 seats. Thiele's method.

30 AB
15 B
55 CD

C gets the first seat, but then D has only $55/2 = 27.5$ votes and the second and third seats go to A and B with 30 votes each. Elected: ABC.

Thus the CD party gets only 1 seat in spite of a majority of the votes.

Also Phragmén's method can behave strangely, as seen in the following example. Further examples of non-intuitive behaviour are given in Sections 14 and 15.

Example 13.16 (Phragmén's ordered method)**.** This example was constructed by Lanke [47].

Ordered ballots. 2 seats. Phragmén's method.

15 AB
12 BX
14 CY
3 ZW

Phragmén's method gives the first seat to A (reduced vote 15) and the second to C (reduced vote 14, against $27/2 = 13.5$ for B and 3 for Z). Elected: AC.

Now suppose that the three ZW voters change their minds and vote AC:

15 AB
12 BX
14 CY
3 AC

Then the first seat goes to A (reduced vote 18). For the second seat, the four groups of ballots have place numbers $\frac{5}{6}, 0, 0, \frac{1}{6}$, giving B and C the reduced votes

$$W_{\mathrm{B}} = \frac{15+12}{1+5/6} = \frac{162}{11} \doteq 14.73 \quad \text{and} \quad W_{\mathrm{C}} = \frac{14+3}{1+1/6} = \frac{102}{7} \doteq 14.57;$$

thus the second seat goes to B.

Thus C lost the seat because of the new votes for AC! (Cf. Section 14, and note that this example does not contradict Theorem 14.3, since the change here involves also A and not only C.)

The explanation of this seems to be that Phragmén's method implicitly regards A and B (and X) as a party. The new votes AC help C, but they are also extra votes for A that help the AB "party", and thus (since A already is elected) B. (In both cases above, the first seat goes to A. The extra votes for A in the second case, i.e., after the change, means that the place numbers get smaller so each vote is reduced less, and more votes are counted for B in the second round.) The calculations above show that, perhaps surprisingly, the indirect gain for B is larger than the direct gain for C.

Nevertheless, it seems strange that more votes on the winning combination AC will make someone else elected, and Lanke [47] regards this example is an argument against Phragmén's ordered method. (On the other hand, as discussed in Section 14, it seems impossible for any election method with ordered ballots to avoid similar examples.)

The final example involves the combination of methods used in Sweden 1909–1921.

Example 13.17 (Ranking Rule + Thiele's unordered method)**.** An example from the commission report [6].

Ordered and unordered ballots.

570 ABC
290 BC
20 C

With ordered ballots, the candidates are obviously elected in order A, B, C by any method considered here, and with unordered ballots, they are elected in the opposite order C, B, A. Both results are obviously fair by all standards, but a problem arises with the combination of ordered and unordered rules used in Sweden 1909–1921, see Appendix D. With those rules, A would get the first seat (because of the ordered Ranking Rule in Appendix D), but then the unordered Thiele's method would be used, and the remaining candidates would be elected in order C, B. Thus, if 2 were elected, it would be AC, which hardly seems fair by any interpretation of the ballots.

This example illustrates a situation that actually occurred in some cases [6]. In fact, it could easily happen if the party organization recommends a list ABC, in order of priority, and most of the voters like this and vote as the party says, but a sizeable minority prefers B and therefore omits A, but still keeps C after B on the ballot (perhaps not fully aware of the implications of keeping C).

Note also that if 20 of the BC voters had voted ABC instead, then the Ranking Rule (Appendix D) would have applied to both A and B, and thus with 2 seats, AB would have been elected. Hence, the voters favouring B have thus in reality eliminated their candidate by voting BC instead of ABC!

In the opposite direction, suppose that there are also 300 votes on other lists with the same party name but other candidates (other factions, or another party in electoral alliance, see Appendix D); then the Ranking Rule would not be applicable at all, and if the lists above still would get 2 seats, they would go to C and B, while the party's top candidate A would not be elected in spite of a majority of these voters preferring A. And if the same list only appoints 1 candidate, it would be C and not A. This can be seen as a form of decapitation, see Section 7. A real example of this type from the general election in 1917 (Skaraborg County south) is discussed in [77].

14. Monotonicity

It is well-known that no perfect election method can exist. For example, the Gibbard–Satterthwaite theorem [36; 71] says briefly that every deterministic election method for ordered ballots is susceptible to tactical voting, i.e., there exist situations where a voter can obtain a result that she finds better by not voting according to her true preferences.[26] A simple and well-known (also in practice) example of this is when a single candidate is elected by simple plurality (First-Past-The-Post, Appendix E.1.1); recall that this includes both Phragmén's and Thiele's ordered methods when $s = 1$, see Theorem 12.1. If candidates B and C have equally strong support, and candidate A a much weaker, then a voter that prefers them in order ABC may do better by voting BAC, since this might help getting B elected instead of C, while the voter's first choice A is without a chance in any case.

Nevertheless, some election methods are more imperfect than others. Particularly disturbing are cases of non-monotonicity when a candidate can become elected if some votes are changed to less favourable for the candidate; equivalently, changing some votes in favour of an elected candidate A might result in A losing the election. Unfortunately, this can happen in Thiele's ordered method.[27]

Example 14.1 (Non-monotonicity for Thiele's ordered method)**.**
Ordered ballots. 2 seats. Thiele's method.

5 ABC
2 ACB
9 BCA
8 CAB

With Thiele's method, first B is elected (9 votes), and then C is elected to the second seat (with 12.5 votes against 7 for A), so the result is BC. However, suppose that the two ACB voters move A to the second place (although they really favour A):

[26] Assuming only that there are at least three candidates that can win, and that the election method is not a dictatorship where the result is determined by a single voter. See also the related Arrow impossibility theorem [10].

[27] This type of non-monotonicity is also a well-known problem with STV (Appendix E.2.1), where it may occur in particular because of the eliminations. This can happen even in a single-seat election, when STV reduces to Alternative Vote (regardless of the version used for transfering surplus, since there is no surplus when $s = 1$). A simple example is 6 ABC, 6 BCA, 5 CAB, where C is eliminated and A is elected; however, if A attracts more support from two voters that change from BCA to ABC, so that the votes are 8 ABC, 4 BCA, 5 CAB, then B is eliminated and C is elected instead of A, see further [83]. See also the impossibility theorem by Woodall [81], but note that Phragmén's and Thiele's methods do not satisfy property (4) there.

5 ABC
9 BCA
10 CAB

Then C will be elected first (10 votes), followed by A (10 votes, against 9 for B), so the result is AC. Hence, these voters have got A elected by moving A to a lower place.

The type of non-monotonicity in Example 14.1 cannot happen with Phragmén's ordered method, or by any of Phragmén's and Thiele's unordered methods. For the unordered methods, this was shown by Phragmén [60] for a class of methods that includes his method, see Theorem 18.4; he did not consider Thiele's methods, but the same proof applies to them too, and yields the following result for the unordered methods considered so far. (The proof applies also to Thiele's methods with other satisfaction functions, except that a minor complication may occur with ties if some $w_n = 0$ as for the weak method.) However, note that there may be other types of non-monotonicity; for Thiele's unordered method, one example is given in Example 13.13.

Theorem 14.2 (partly Phragmén [60]). *Phragmén's unordered method and Thiele's three unordered methods satisfy the following: Consider an election and let A be one of the candidates. Suppose that the votes are changed such that A gets more votes, either from new voters that vote only for A, or from voters that add A to their ballots (thus changing the vote from some set σ to $\sigma \cup \{A\}$), but no other changes are made (thus all other candidates receive exactly the same votes as before). Then this cannot hurt A; if A would have been elected before the change, then A will be elected also after the change.*[28] [29]

Proof. For Thiele's optimization method, note that the total satisfaction is unchanged for every set $\mathcal{S}$ of candidates that does not include A, but it is increased for every set of candidates that includes A. Hence, if one set including A maximized the total satisfaction before the change, then the same holds after the change.[30] Thus A will still be elected.

For Phragmén's unordered method and Thiele's addition method, we note that as long as A is not elected, then all other candidates have exactly the same voting power or vote count before and after the change, while this has increased for A. Hence, if A would have been elected in round ℓ before the

[28] There is no problems with ties. The proof shows that there are strict improvements, so if A was tied before the change, then A will certainly win after the change. (This is called *positive responsiveness*, see e.g. [51].)

[29] Note that the elected set is not necessarily the same. Adding A to some ballots can affect, for example, whether B is elected or not together with A. This applies to all the methods. For example, if we modify Examples 14.4 and 14.5 below by adding only a vote on A_1, then this results in replacing A_2 by C or B.

[30] The maximum set is not necessarily the same.

change, then after the change either A has already been elected, or A will be elected in the same round ℓ.

Finally, for Thiele's elimination method, note that if there are N candidates, the elimination method can be seen as repeatedly using the optimization method to select $s = N-1, N-2, \dots$ candidates, until the desired number remains, after each round eliminating the non-selected candidate from all further consideration. The result just proved for the optimization method shows that if A survives all rounds before the change, then A survives all rounds also after the change. □

For Phragmén's ordered method we have the following corresponding result. Note that, as shown by Example 14.1, this theorem fails for Thiele's ordered method.

Theorem 14.3. *Phragmén's ordered method satisfies the following: Consider an election and let A be one of the candidates. Suppose that some votes are changed in favour of A, either by adding a new ballot containing only A, by adding A to an existing ballot at an arbitrary place (if A did not already exist on the ballot), or by moving A to a higher position (if A already existed on the ballot); in all cases without changing the other candidates and their order. Then the change cannot hurt A; if A would have been elected before the change, then the same is true after the change.*

Proof. We use the formulation in Section 9.1 and Remark 9.1 with increasing time; thus, at time t, each ballot has voting power t. Let $f_i(t)$ and $\hat{f}_i(t)$ be the total voting power for candidate i at time t for the original set of ballots and for the changed ballots, respectively. Note that $0 \leqslant f_i(t) \leqslant 1$ and that i has been elected (with the original ballots) at time t or earlier if and only if $f_i(t) = 1$, and similarly for $\hat{f}_i(t)$. We claim the following:

$$\text{If } \hat{f}_A(t) < 1, \text{ then } f_A(t) \leqslant \hat{f}_A(t) \text{ and } f_i(t) \geqslant \hat{f}_i(t) \text{ for } i \neq A. \tag{14.1}$$

To prove this, we let $\{t_\ell\}_{\ell=1}^L$ be the set of times when some candidate is elected for either the original or the changed ballots, ordered in increasing order. We let $\hat{t}_{\mathrm{A}}$ be the time that A is elected for the changed ballots, so the assumption $\hat{f}_{\mathrm{A}}(t) < 1$, meaning that A is not yet elected, is equivalent to $t < \hat{t}_{\mathrm{A}}$. We also let $t_0 = 0$. We prove the claim (14.1) by induction, assuming that it is true for $0 \leqslant t \leqslant t_\ell$, for some $\ell \geqslant 0$ with $t_\ell < \hat{t}_{\mathrm{A}}$, and show that (14.1) holds also for $t \in I_\ell := [t_\ell, t_{\ell+1}]$.

To show this induction step, note that by the inductive assumption, for every candidate $i \neq$ A that has not yet been elected at t_ℓ for the original ballots, we have $1 > f_i(t_\ell) \geqslant \hat{f}_i(t_\ell)$, and thus i is also not elected for the changed ballots. Furthermore, $t_\ell < \hat{t}_{\mathrm{A}}$, so A has not been elected at t_ℓ for the changed ballots. Consequently, every candidate that at t_ℓ has been elected for the changed ballots has also been elected for the original ballots.

Consider now a candidate $i \neq$ A. If i has been elected for the original ballots at t_ℓ, then $f_i(t_\ell) = 1$ and thus $f_i(t) = 1 \geqslant \hat{f}_i(t)$ for all $t \geqslant t_\ell$. On the other hand, suppose that i has not been elected yet for the original ballots at t_ℓ, so $f_i(t_\ell) < 1$. Consider a ballot that, after the change, is valid for i immediately after t_ℓ; this means that i is on the ballot and any candidate j before i on the ballot has been elected, i.e., $\hat{f}_j(t_\ell) = 1$. In particular, $j \neq \mathrm{A}$. Hence, before the change, this ballot had the same names before i, and we have seen that they have been elected for the original ballots too at t_ℓ; thus the ballot is valid for i also in the original set of ballots. If we increase time (i.e., voting power) from t_ℓ to $t_{\ell+1}$, then, because there are no elections inside this interval, the voting power of i increases linearly, with a derivative equal to the number of ballots valid for i. Hence we have shown that $f_i'(t) \geqslant \hat{f}_i'(t)$ for $t \in (t_\ell, t_{\ell+1})$, and we see that $f_i(t) \geqslant \hat{f}_i(t)$ for $t \in [t_\ell, t_{\ell+1}]$ because, by the induction hypothesis, this holds for $t = t_\ell$.

We have verified the last part of the claim (14.1). Now, the total voting power assigned or available from a ballot b at time t equals $\min(t, t_b)$, where t_b is the time the last candidate on the ballot is elected. We have seen that if the last candidate on the ballot is elected at some time $\hat{t}_b$ for the set of changed ballots, and $\hat{t}_b \leqslant t_\ell < \hat{t}_\mathrm{A}$, then at time $\hat{t}_b$ all candidates on the ballot have been elected also for the original set of ballots, so $t_b \leqslant \hat{t}_b$. Consequently, for each ballot, and each time $t < t_{\ell+1}$, the total voting power that is used from a ballot is at least as large after the change as before. Summing over all ballots (and noting that some ballots may have been added, but none removed), we find that the total voting power used for all candidates is at least as large after the change as before, i.e., $\sum_i \hat{f}_i(t) \geqslant \sum f_i(t)$, $t < t_{\ell+1}$. For $t \in [t_\ell, t_{\ell+1})$, we have also shown that $\hat{f}_i(t) \leqslant f_i(t)$ for every $i \neq A$, and it follows that $\hat{f}_\mathrm{A}(t) \geqslant f_\mathrm{A}(t)$. By continuity, this holds for $t = t_{\ell+1}$ too, which completes the inductive step verifying (14.1).

Finally, (14.1) says that the conclusion of (14.1) holds when $\hat{f}_\mathrm{A}(t) < 1$, i.e., when $t < \hat{t}_\mathrm{A}$. By continuity, the conclusion holds also for $t = \hat{t}_\mathrm{A}$. Now suppose that A is not elected with the changed ballots; then a set $\widehat{\mathcal{E}}$ of s other candidates have been elected at time $\hat{t}_\mathrm{A}$. This together with the claim (14.1) implies that for each $i \in \widehat{\mathcal{E}}$, $f_i(\hat{t}_\mathrm{A}) \geqslant \hat{f}_i(\hat{t}_\mathrm{A}) = 1$, and thus i has at time $\hat{t}_\mathrm{A}$ been elected also for the original ballots. On the other hand, for any $t < \hat{t}_\mathrm{A}$, by (14.1), $f_\mathrm{A}(t) \leqslant \hat{f}_\mathrm{A}(t) < 1$, so A has not been elected earlier. Consequently, A is not in the set $\mathcal{E}$ of the s first elected with the original ballots, which completes the proof. (In the case of A tying the last place both for the original and changed ballots, we have to assume that the tie is resolved in the same way.) ☐

However, also Phragmén's methods are liable to non-monotonicity of other types. One such case for Phragmén's ordered method is shown by Example 13.16. We give two examples for Phragmén's unordered method, taken from Mora and Oliver [55, Section 7.5], which show that Theorem 14.2 does not extend to sets of more than one candidate, even if they always appear together (and thus are tied).

Example 14.4 (Non-monotonicity for Phragmén's unordered method)**.** An example from Mora and Oliver [55, (43)].

Unordered ballots. 3 seats. Phragmén's method.

10 A_1A_2
3 B
12 C
21 A_1A_2B
6 BC

A calculation shows that Phragmén's method elects A_1, C, A_2. (For the second seat, C has reduced vote 18 against $30/(1+\frac{21}{31}) = 465/26 \doteq 17.88$ for B and $31/2 = 15.5$ for A_2. Then A_2 is elected with reduced vote $31/2 = 15.5$ against $2790/187 \doteq 14.92$ for C.)

Now add another vote for A_1A_2:

11 A_1A_2
3 B
12 C
21 A_1A_2B
6 BC

The first seat still goes to A_1, but its load on each ballot A_1A_2B is changed from 1/31 to 1/32, and as a consequence, for the second seat, B's reduced vote is increased to $30/(1+\frac{21}{32}) = 960/53 \doteq 18.11$, and thus B gets the second seat. This, in turn, increases the load on each ballot A_1A_2B to 53/960, and thus for the third seat, the reduced vote of A_2 is only $10240/801 \doteq 12.78$ against $960/71 \doteq 13.52$ for C, so C gets the third seat. Thus the elected are A_1, B, C.

Consequently, the extra vote on A_1A_2 means that one of them loses a seat.

Example 14.5 (Non-monotonicity for Phragmén's unordered method)**.** An example from Mora and Oliver [55, (42)].

Unordered ballots. 3 seats. Phragmén's method.

4 A_1A_2
7 B
1 A_1A_2B
16 A_1A_2C
4 BC

A calculation shows that Phragmén's method elects A_1, B, A_2.

Now suppose that one of the voters for B adds A_1 and A_2 to the ballots. Thus the ballots are:

4 A_1A_2
6 B
2 A_1A_2B
16 A_1A_2C
4 BC

A_1 still gets the first seat, but now the second seat goes to C, and then the third seat goes to B; the elected are thus A_1, C, B.

Consequently, also in this example, the extra vote on $\mathrm{A}_1\mathrm{A}_2$ means that one of them loses a seat.

15. Consistency

Consistency [85] of an election method is the property that if there are two disjoint sets $\mathcal{V}_1$ and $\mathcal{V}_2$ of voters voting such that the votes from only $\mathcal{V}_1$ would elect a set $\mathcal{E}$ of candidates, and the votes from only $\mathcal{V}_2$ would elect the same set $\mathcal{E}$, then the combined set of votes from $\mathcal{V}_1 \cup \mathcal{V}_2$ elects the same set $\mathcal{E}$. To be precise (to allow for ties), we require that if $\mathcal{E}$ is a possible outcome for the votes from only $\mathcal{V}_1$, and also a possible outcome for the votes from only $\mathcal{V}_2$, then $\mathcal{E}$ is a possible outcome for the combined set of votes from $\mathcal{V}_1 \cup \mathcal{V}_2$.[31]

Remark 15.1. Consistency implies, in particular, that multiplying the number of ballots of each type by some integer will not affect the outcome of the election; in other words, the election method is homogeneous, see Remark 2.1. We assume that all voters are equal, so that only the number of ballots of each type matters.[32] Then the input to the election method, for a given set $\mathcal{C}$ of candidates, can be seen as a vector of non-negative integers, indexed by subsets of $\mathcal{C}$ in the unordered case and by ordered subsets of $\mathcal{C}$ in the ordered case. Homogeneity means that we can extend the method (uniquely) to vectors of non-negative rational numbers, such that the set $C_{\mathcal{E}}$ of such vectors that (may) elect a set $\mathcal{E}$ is a cone (or empty), for any set $\mathcal{E}$ of s candidates. Furthermore, as is easily seen, consistency is equivalent to the cone $C_{\mathcal{E}}$ being convex, for every possible outcome $\mathcal{E}$. (Hence the property is also called *convexity* [82].)

Consistency is a very natural property. Unfortunately, it is not compatible with some other natural properties, and it is not satisfied by most election methods. In fact, Young [85] showed that (assuming some technical conditions) the only consistent election methods for ordered ballots are the scoring rules (Borda methods) (Appendix E.2.3). Similarly, Lackner and Skowron [46] show that (again assuming some technical conditions) every consistent method for unordered ballots equals Thiele's optimization method for some satisfaction function. We record the easy part of this as a theorem.

Theorem 15.2. *Thiele's optimization method is consistent (for any satisfaction function).*

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[31] In the case of ties, [85] also requires that, conversely, if there exists a set $\mathcal{E}$ that is a possible outcome for both $\mathcal{V}_1$ and $\mathcal{V}_2$, then any other possible outcome $\mathcal{E}'$ of $\mathcal{V}_1 \cup \mathcal{V}_2$ (so $\mathcal{E}'$ ties with $\mathcal{E}$) is a possible outcome for both $\mathcal{V}_1$ and $\mathcal{V}_2$. We will not consider this, more technical, condition.

[32] A property called *anonymity*.

Proof. Obvious from the definition of the method in Section 4.1. □

By the theorems just mentioned, Phragmén's unordered and ordered methods and Thiele's unordered (addition) and ordered methods are not consistent. We give some explicit counterexamples.

Remark 15.3. In the case $s = 1$, Theorem 12.1 says that Phragmén's method and Thiele's optimization and addition methods for unordered ballots reduce to Approval Voting, while Phragmén's and Thiele's methods for ordered ballots reduce to Single-Member-Plurality; Approval Voting and Single-Member-Plurality are both obviously consistent, so all of Phragmén's and Thiele's methods (in the latter case for any satisfaction function), except possibly Thiele's elimination method, are consistent for $s = 1$.

Example 15.4 (Phragmén's and Thiele's unordered methods are not consistent)**.**

Unordered ballots. 2 seats. Phragmén's or Thiele's method.

First set of voters:

12 AB
1 B
9 C

For both Phragmén's and Thiele's (addition) methods, first B is elected and then C. Elected: BC.

Second set of voters:

12 AC
9 B
1 C

This is the same with B and C interchanged, so the elected are again BC. (They are elected in different order, but that is irrelevant for the outcome of the election.)

Combined election:

12 AB
12 AC
10 B
10 C

This is the same as Example 13.4. Phragmén's and Thiele's method both elect first A, and then B or C (a tie, by symmetry). Elected: AB or AC.

Thus neither of the methods is consistent.

Example 15.5 (Thiele's ordered method is not consistent)**.**

Ordered ballots. 2 seats. Thiele's method.

First set of voters:

12 A
10 BC
9 C

Thiele's (ordered) method elects A and B.

Second set of voters:

11 B
10 AC
9 C

Thiele's method elects B and A.

Combined election:

12 A
10 AC
11 B
10 BC
18 C

Thiele's method elect A first (22 votes), but then C gets the second seat (23 votes, against 21 for B). Elected: AC.

Thus Thiele's ordered method is not consistent.

In these examples, the same candidates were elected by the two sets of voters $\mathcal{V}_1$ and $\mathcal{V}_2$, but in different order. It is also possible to find counterexamples where the candidates are elected in the same order.

Example 15.6 (Phragmén's ordered method is not consistent)**.** We modify Example 13.16.

Ordered ballots. 2 seats. Phragmén's method.

First set of voters:

15 AB
12 BX
14 CY

Phragmén's method gives, just as in Example 13.16, the first seat to A and the second to C. Elected (trivially): AC.

Second set of voters:

3 AC

Elected: AC.

Combined election:

15 AB
12 BX
14 CY
3 AC

As seen in Example 13.16, now the elected are A and B.

Hence, Phragmén's ordered method is not consistent.

15.1. **Ballots with all candidates.** Consider unordered ballots and the case when the voters in a subset $\mathcal{V}_1$ of all voters $\mathcal{V}$ vote for all candidates. Will their votes affect the outcome at all, or will the result of the election be the same as if only $\mathcal{V}_2 := \mathcal{V} \setminus \mathcal{V}_1$ had voted? We call a ballot containing all candidates a *full ballot*, and we say that an election method *ignores full* *ballots* if it always gives the same result (including ties) if we add or remove full ballots.[33]

This is a special case of consistency; if only $\mathcal{V}_1$ had voted, with full ballots, then the result would have been a tie between any set of s candidates. Hence any consistent method will ignore full ballots. In particular, Thiele's optimization method ignores full ballots by Theorem 15.2. Moreover, the following holds.

Theorem 15.7. *Thiele's unordered methods all ignore full ballots (for any satisfaction function).*

Proof. Obvious from the definitions in Section 4, since a ballot with all candidates gives the same contribution to each candidate at every step of the calculations. □

On the other hand, Phragmén's method does *not* ignore full ballots. This follows from the study of party versions by Mora and Oliver [55, Section 7] (discussed in Section 18.4 below); we give two explicit examples below. (See [55] for further strange behaviour of Phragmén's method in situations where some voters vote for all candidates.)

Example 15.8 (Phragmén's method does not ignore full ballots)**.** This example is based on general results in Mora and Oliver [55, Section 7.7].

Unordered ballots. 4 seats. Phragmén's method.

5 $A_1A_2A_3A_4$
3 $B_1B_2B_3B_4$

This is a case with party lists, so by Theorem 11.1, the candidates are elected as by D'Hondt's method, which gives $A_1B_1A_2A_3$. (For example, the last seat goes to A_3 because $5/3 > 3/2$.

Now add a large number of votes for all candidates:

5 $A_1A_2A_3A_4$
3 $B_1B_2B_3B_4$
10 $A_1A_2A_3A_4B_1B_2B_3B_4$

A calculation shows that the elected will be $A_1B_1A_2B_2$. (For example, for the fourth seat, the three types of ballots have already loads $\frac{34}{195}, \frac{25}{195}, \frac{34}{195}$ and thus the three groups of ballots have place numbers $\frac{34}{39}, \frac{15}{39}, \frac{68}{39}$, and the reduced votes of the A's and B's (which of course tie among themselves) are $\frac{195}{47} \doteq 4.1489$ and $\frac{507}{122} \doteq 4.1557$, so the last seat goes to B.)

Example 15.9 (Phragmén's method does not ignore full ballots)**.** Another example, showing that already the second seat can differ.

Unordered ballots. 2 seats. Phragmén's method.

[33]Whether this is a natural, and perhaps important, property or not is a philosphical question rather than a mathematical, and is left to the reader; however, it seems natural to argue that full ballots with all candidates do not express any real preference, and therefore ought to be ignored just as blank ballots with no names.

10 A
3 AB
2 C

The first seat goes to A (13 votes). The second seat goes to B (reduced vote $3/(1+\frac{3}{13}) = 39/16 = 2.4375$ against 2 for C). Elected: AB.

Now add a large number of votes for all candidates:

10 A
3 AB
2 C
10 ABC

A still gets the first seat. A calculation shows that for the second seat, B has reduced vote $13/(1+\frac{13}{23}) = 299/36 \doteq 8.31$ while C has reduced vote $12/(1+\frac{10}{23}) = 92/11 \doteq 8.36$; thus C gets the second seat. Elected: AC.

Since the result of Phragmén's method thus can be affected by adding full ballots, it is (mathematically) interesting to ask for the result if we add a very large number N of full ballots.

Theorem 15.10. *Consider an election with Phragmén's unordered method. If we add N full ballots, then for all sufficiently large N, the outcome of the new election will be the same as for the original ballots with the following election method, at least providing that there are no ties for the latter:*

A LIMIT OF PHRAGMÉN'S UNORDERED METHOD. *Seats are given to candidates sequentially, until the desired number have been elected. In round k (i.e., for electing to the k-th seat), a ballot β is counted as $1-\ell_\beta/k$ votes for each candidate on the ballot (except the ones already elected), where ℓ_β is the last round where some candidate from the ballot was elected, with $\ell_\beta = 0$ if no candidate on the ballot has been elected yet. The candidate with the largest number of votes is elected.*

Remark 15.11. Equivalently, we may count ballot β as $k-\ell_\beta$ votes. We can thus also describe the method as follows:

Elect the candidate with the largest number of votes. Discard all ballots with the elected candidate; then add a new copy of each original ballot. Repeat s times.

Proof. We consider an election with s seats and V 'real' votes, to which we add N full ballots. We use the formulation in Section 3.1; in particular, $t^{(n)}$ is the woting power required to elect n candidates. Since an additional voting power $1/N$ gives each candidate voting power 1 just from the full ballots, which is enough to elect any of them, $t^{(n+1)} \leqslant t^{(n)} + 1/N$. In particular, for every $n \leqslant s$, $t^{(n)} \leqslant n/N \leqslant s/N$. This implies that at time $t^{(n)}$, the elected candidate gets voting power at most Vs/N from the real votes, and thus at least $1 - Vs/N$ from the full votes. Hence, $1/N - Vs/N^2 \leqslant t^{(n)} - t^{(n-1)} \leqslant 1/N$, and thus

$$t^{(n)} - t^{(n-1)} = N^{-1} + O\big(N^{-2}\big). \tag{15.1}$$

When the k-th candidate is elected, a ballot where the last round that someone was elected was ℓ thus has a free voting power $t^{(k)} - t^{(\ell)} = (k - \ell)N^{-1} + O\big(N^{-2}\big)$. The total voting power available for candidate i is thus

$$T_i = \sum_{\beta \ni i} (k - \ell_\beta) N^{-1} + O\big(N^{-2}\big) + N(t^{(k)} - t^{(k-1)}), \tag{15.2}$$

summing over all real ballots containing i. Since the total voting power T_i is 1 for the elected candidate, and at most 1 for every other, we see that if i_k is the elected in round k, then for every i, $T_i \leqslant T_{i_k}$, and thus by (15.2)

$$\sum_{\beta \ni i} (k - \ell_\beta) N^{-1} \leqslant \sum_{\beta \ni i_k} (k - \ell_\beta) N^{-1} + O\big(N^{-2}\big) \tag{15.3}$$

for every i, and thus

$$\sum_{\beta \ni i} (k - \ell_\beta) \leqslant \sum_{\beta \ni i_k} (k - \ell_\beta) + O\big(N^{-1}\big). \tag{15.4}$$

It follows by induction on k, that for all large N, i_k is the winner of the k-th seat by the method defined above, provided there are no ties in that method. □

This limit method seems more interesting theoretically than for practical use. In particular, the following example shows that in the party list case, the limit method does *not* reduce to D'Hondt's method.

Example 15.12 (The limit method in a party list case)**.** Consider the simplest case with two parties.

Unordered ballots. The limit of Phragmén's method defined above.

a $A_1A_2A_3\dots$
b $B_1B_2B_3\dots$

Suppose that $a > b$, and let $k := \lfloor a/b \rfloor$. Furthermore, to avoid ties, suppose that a/b is not an integer, so $kb < a < (k+1)b$. It is easy to see that the limit method above will elect in the periodic order $A_1,\dots,A_k,B_1, A_{k+1},\dots,A_{2k},B_2,\dots$ with k A's followed by a single B, repeatedly.

In particular, if $b < a < 2b$, and the number of seats is even, then the two parties get an equal number of seats, in spite of the fact that one party has more votes than the other. For example, with $a = 5$, $b = 3$ and $s = 4$ as in Example 15.8, the elected are $A_1B_1A_2B_2$, while D'Hondt's method would give $A_1B_1A_2A_3$.

16. Proportionality

There is no precise definition of proportionality for an election method, and whether a method is proportional or not is partly a matter of degree (and perhaps taste). The general idea of proportionality is that a sufficiently large group of voters gets a "fair" share of the elected. When there are no formal parties, as in the election methods considered here, we should here consider arbitrary groups of voters, possibly with some restriction on how they vote. This can be made precise in many different ways. We study in this section some such properties relevant for Phragmén's and Thiele's methods.

Phragmén's methods satisfy the following proportionality criteria.

Theorem 16.1. *Phragmén's methods satisfy the following properties, where V is the number of votes, s the number of seats and ℓ is an arbitrary integer with $1 \leqslant \ell \leqslant s$.*

(i) *For Phragmén's unordered method: If a set of more than $\frac{\ell}{s+1}V$ voters vote for the same list containing at least ℓ candidates, then at least ℓ candidates from this list are elected.*

(ii) *For Phragmén's ordered method: If a set of more than $\frac{\ell}{s+1}V$ voters vote with the same ℓ candidates first on their ballots, but not necessarily in the same order, then these ℓ candidates are elected. In particular, if at least $\frac{\ell}{s+1}V$ voters vote for the same list (containing at least ℓ candidates), then the ℓ first candidates on this list will be elected.*

Proof. Let $\mathcal{A}$ be such a set of $|\mathcal{A}| > \frac{\ell}{s+1}V$ voters, and let $\mathcal{C}$ in case (i) be the set of candidates on the list voted for by $\mathcal{A}$ (thus $|\mathcal{C}| \geqslant \ell$), and in case (ii) the set of the ℓ candidates that are first on the ballots from $\mathcal{A}$ (thus $|\mathcal{C}| = \ell$). Suppose, to obtain a contradiction, that only $k < \ell$ of the candidates in $\mathcal{C}$ are elected.

We use the formulation with voting power in Sections 3.1 and 9.1, and let $t = t^{(s)}$ be the final voting power of each ballot.

In the unordered case (i), the k elected candidates from $\mathcal{C}$ are together assigned voting power k, of which some part may come from voters not in $\mathcal{A}$. The total free (unassigned) voting power of the ballots in $\mathcal{A}$ is at most 1, since otherwise another candidate from $\mathcal{C}$ would have been elected. (This free voting power may equal 1 if there was a tie for the last seat.) Hence, the total voting power of the ballots in $\mathcal{A}$ is at most $k + 1 \leqslant \ell$,

In the ordered case (ii), since at least one candidate in $\mathcal{C}$ is not elected, each ballot in $\mathcal{C}$ has when the election finishes a current top candidate that belongs to $\mathcal{C}$. Hence, if we regard the free voting power of a ballot as assigned to its current top candidate, then the voting power of each ballot in $\mathcal{A}$ is fully assigned to one or several candidates in $\mathcal{C}$ (elected or not). Since each candidate is assigned a total voting power at most 1, and $|\mathcal{C}| = \ell$, the total voting power of the ballots in $\mathcal{A}$ is at most ℓ in this case too.

Since there are $|\mathcal{A}|$ ballots in $\mathcal{A}$ and each has voting power t, this shows that in both cases

$$\ell \geqslant |\mathcal{A}|t > \frac{\ell}{s+1}Vt \tag{16.1}$$

and thus

$$Vt < s + 1. \tag{16.2}$$

On the other hand, $s - k$ candidates not in $\mathcal{C}$ have been elected, and thus a total voting power $s - k \geqslant s + 1 - \ell$ has been assigned to them. Since the ballots in $\mathcal{A}$ do not contribute to this, all this voting power comes from the $V-|\mathcal{A}|$ other ballots, and thus

$$s+1-\ell \leqslant (V-|\mathcal{A}|)t < \Bigl(V-\frac{\ell}{s+1}V\Bigr)t = \frac{s+1-\ell}{s+1}Vt. \tag{16.3}$$

However, this contradicts (16.2). □

Corollary 16.2. *Phragmén's unordered and ordered methods satisfy: If more than half of the voters vote for the same list containing at least $s/2$ candidates, then at least $s/2$ of these are elected.*

Proof. By Theorem 16.1 with $\ell = \lceil s/2 \rceil \leqslant (s+1)/2$. □

In particular, if s is odd, a majority of the voters will, if they vote coherently, get a majority of the seats.

Note that Theorem 16.1 does *not* hold for Thiele's unordered and ordered methods, as is seen by Example 13.13 (the second election, where KLM has $13 > V/(s+1) = 50/4$ votes but does not get any seat) and Example 13.14 (the second election, where CD has $39 > V/(s+1) = 100/3$ votes but does not get any seat). Also Corollary 16.2 does not hold for Thiele's unordered and ordered methods, as is seen by the Example 13.15 and the following example:

Example 16.3. Consider the following election:

Unordered ballots. 5 seats. Thiele's method.

9 $A_1A_2A_3$
2 X_1X_2
2 X_1X_3
2 X_2
2 X_3

The first two seats go to A_1 and A_2 (say). Then X_1, X_2 and X_3 tie for the third seat (4 votes each), and if X_1 is elected, then A_3, X_2 and X_3 tie for the remaining 2 seats (3 votes each). Hence, depending on tie-breaks, X_1, X_2 and X_3 may be elected to the last 3 seats, and thus the A party, which has a majority of the votes (9 of 17) may end up with only 2 seats. (In this and similar examples, it is easy to modify the example to avoid ties by multiplying all votes by some large number and then adding a few votes on suitable ballots, giving an arbitrarily small change of the proportions.)

Remark 16.4. The proportion of (more than) $\frac{\ell}{s+1}$ of the voters is exactly the proportion required to get ℓ seats of s in an election with two parties by D'Hondt's method, as is easily seen. Hence, by Theorem 11.1, this proportion in Theorem 16.1 is the best possible.

Remark 16.5. The proportion $\frac{\ell}{s+1}$ is less that $\frac{\ell}{s}$, which often is regarded as a proportion that ought to guarantee at least ℓ seats. The proportion $\frac{\ell}{s}$ might be more intuitive, but as pointed out already by Droop [29], in an election of s representatives using simple plurality[34], if some candidate A has more than $\frac{1}{s+1}$ of the votes, then there cannot be s other candidates that have at least as many votes, and thus A is elected. The proportion $\frac{1}{s+1}$ of the votes (called the *Droop quota*, see Appendix E.3) is thus a reasonable requirement for having the "right" to be elected. And a party with more than $\frac{\ell}{s+1}V$ votes could in principle split them on ℓ candidates that would be elected. Note also that in the case $\ell = s = 1$, $\frac{1}{s+1} = \frac{1}{2}$, so the criterion, and Theorem 16.1, reduces to saying that a candidate supported by a majority is elected.

Remark 16.6. STV (Appendix E.2.1) satisfies a property similar to Theorem 16.1, called the *Droop Proportionality Criterion* by Woodall [82]. The version in [82] assumes that the Droop quota $Q_D := V/(s+1)$ is used without rounding,[35] and then says:

If $m \geqslant \ell$ and a set of more than $\ell V/(s+1)$ voters put the same m names first on their ballots, but not necessarily in the same order, then at least ℓ of these will be elected.

Note that Theorem 16.1(ii) is this property in the special case $m = \ell$. (So the Droop Proportionality Criterion is stronger.) The fact that the property holds for STV also for arbitrary $m > \ell$ depends on the eliminations of weak candidates, which will concentrate the votes of this set of voters to smaller sets of candidates until at least ℓ of them are elected. For Phragmén's ordered method there is no similar mechanism, and splitting the votes on more than ℓ candidates can lead to less than ℓ of them being elected; one example is given in Example 16.7 below. This is in contrast to the unordered method, see Theorem 16.1(i), where splitting the vote on several candidates does not decrease the support of any of them (except when one of them is elected).

Example 16.7 (Phragmén's ordered method)**.**

Ordered ballots. 3 seats. Phragmén's method.

9 ABC
9 ACB
9 BAC
9 BCA
9 CAB
9 CBA
46 XYZ

[34]E.g. by SNTV (Appendix E.1.4); Droop really discussed Cumulative Voting (Appendix E.1.5), but SNTV can be seen as a special case, and the difference between the methods does not matter here.

[35]For STV with a quota $Q > Q_D$ (recall that $Q \geqslant Q_D$ almost always holds in practice), the property still holds in the case $\ell = 1$, but not in general; however, it holds for sets of at least ℓQ voters. See e.g. [42, Section 12.5.1].

With Phragmén's ordered method, the first two seats go to X and Y (reduced votes 46 and 23), while the third to any of A, B or C (reduced vote 18). Elected: AXY.

The ABC party has together 54 of the 100 votes, and if they all voted, say, ABC, then they would be guaranteed two seats by Theorem 16.1 (with $\ell = 2$ and $s = 3$) or by Corollary 16.2. (In fact, it suffices that they all vote with ballots beginning with AB or BA.) However, by splitting their votes, they lose one seat, and thus the majority.

For election methods with unordered ballots, Aziz et al. [11] defined two properties *JR (justified representation)* and (stronger) *EJR (extended justified representation)*; Sánchez-Fernández et al. [70] then defined a related property *PJR (proportional justified representation)* such that *EJR* $\implies$ *PJR* $\implies$ *JR*. These are defined as follows, where V is the total number of votes, and $\mathcal{A}$ is a set of voters.

PJR: Let $1 \leqslant \ell \leqslant s$ *and suppose that* $|\mathcal{A}| \geqslant \frac{\ell}{s}V$ *and that* $\left|\bigcap_{\sigma\in\mathcal{A}}\sigma\right| \geqslant \ell$, *i.e., the voters in* $\mathcal{A}$ *all vote for a common set* $\mathcal{C}$ *of at least* ℓ *candidates, but may also vote for other candidates. Then* ℓ *candidates that some voter in* $\mathcal{A}$ *has voted for are elected, i.e.* $\left|\mathcal{E}\cap\bigcup_{\sigma\in\mathcal{A}}\sigma\right| \geqslant \ell$.

EJR: Let $1 \leqslant \ell \leqslant s$ *and suppose that* $|\mathcal{A}| \geqslant \frac{\ell}{s}V$ *and that* $\left|\bigcap_{\sigma\in\mathcal{A}}\sigma\right| \geqslant \ell$, *i.e., the voters in* $\mathcal{A}$ *all vote for a common set* $\mathcal{C}$ *of at least* ℓ *candidates, but may also vote for other candidates. Then some voter in* $\mathcal{A}$ *has voted for* ℓ *candidates that are elected, i.e.* $\left|\mathcal{E}\cap\sigma\right| \geqslant \ell$ *for some* $\sigma \in \mathcal{A}$.

JR: The condition above for PJR (or EJR) holds for the special case $\ell = 1$. (Note that the difference between *PJR* and *EJR* disappears for $\ell = 1$.)

We state some results from the papers above (and [19]) without proofs.

Theorem 16.8 ([11])**.** *Thiele's optimization method satisfies EJR.*

However, with any other satisfaction function instead of the proportional (4.4)*, EJR fails.*

Thiele's optimization method with a general satisfaction function (4.1) *with* $w_1 = 1$ *satisfies JR if and only if* $w_j \leqslant 1/j$ *for every* $j > 1$.

Theorem 16.9 ([70])**.** *Thiele's addition method satisfies JR for* $s \leqslant 5$*, but not for* $s \geqslant 6$.

Theorem 16.10 ([11])**.** *Thiele's addition method with a general satisfaction function* $f(n)$ *satisfies JR if and only if* $f(n)$ *is the weak satisfaction function* (4.3).

Theorem 16.11 ([19])**.** *Phragmén's unordered method satisfies PJR, but not EJR.*

17. Phragmén's ordered method and STV

Phragmén's and Thiele's ordered methods use ordered ballots just as the more well-known election method STV (Appendix E.2.1). Both methods are clearly different from STV, which is seen already in the case $s = 1$ of electing a single person, when they reduce to simple plurality (Theorem 12.1), while STV reduces to Alternative Vote. Nevertheless, there are strong connections between Phragmén's method and STV; both methods can be regarded as giving each ballot a certain voting power that can be used to elect the candidates on the ballot (one at a time, in order), and when the voting power of a candidate is more than enough to elect the candidate, the surplus is transfered to the next candidate. Thiele's method seems to be founded on different principles and will not be considered further in this section.

To see the connection in more detail, consider an election of s seats by Phragmén's ordered method, using the formulation in Section 9.1 (and the notation in Section 3.1). We thus give each ballot voting power $t^{(s)}$ (found at the end of the calculations). We obtain an equivalent formulation of the method where each ballot has voting power 1 if we define $Q := 1/t^{(s)}$ and multiply every voting power by Q. This means that every ballot has voting power 1, and that a candidate needs voting power Q to be elected, just as in STV if the quota is Q. Note that by (3.6), $Q = W^{(s)}$ is the reduced vote for the last elected candidate if we use the formulation in Sections 9.2–9.3.

Let us use the algorithm in Section 9.2, and let i_j be the candidate elected in round j, $j = 1, \dots, s$. Consider a ballot β in some round $k \geqslant 1$, and let $\ell < k$ be the latest round before k when the ballot participated in the election of a candidate (meaning that the elected candidate i_ℓ was the current top name on the ballot that round); if the ballot has not yet participated in the election of any candidate let $\ell = 0$. Then the place number of the ballot in round k, which is the voting power assigned to the already elected candidates, is $t^{(\ell)}$, with $t^{(0)} = 0$ (see Remarks 3.5–3.6). In the new scale, the ballot has used $Qt^{(\ell)}$ of its value to the already elected, and its remaining value is $1 - Qt^{(\ell)}$. (Note that $Qt^{(\ell)} = t^{(\ell)}/t^{(s)} \leqslant 1$.)

In round k, i_k is elected. Suppose that then there are n_k ballots currently valid for i_k (i.e., with i_k as the current top name), and that of these $n_{k\ell}$ have participated in the election of i_ℓ, but not in the election in any later round, $\ell = 1, \dots, k-1$, while n_{k0} of them have not participated in the election of any candidate before i_k. Thus, the number of ballots valid for i_k is

$$n_k = \sum_{\ell=0}^{k-1} n_{k\ell}, \tag{17.1}$$

and their total place number is

$$P_k := \sum_{\ell=1}^{k-1} n_{k\ell} t^{(\ell)} = \sum_{\ell=0}^{k-1} n_{k\ell} t^{(\ell)}. \tag{17.2}$$

By the formulation of Phragmén's method in Section 9.1, if each ballot has voting power $t^{(k)}$, then the elected candidate i_k will have voting power 1 from the n_k ballots valid for i_k, i.e., subtracting the voting power assigned to previously elected candidates,

$$1 = n_k t^{(k)} - P_k. \tag{17.3}$$

On the other hand, if we consider the values of the ballots in the new scale defined above, then the total remaining value of the n_k ballots valid for i_k is, using (17.1) and (17.2),

$$\sum_{\ell=0}^{k-1} n_{k\ell}(1 - Qt^{(\ell)}) = \sum_{\ell=0}^{k-1} n_{k\ell} - Q \sum_{\ell=0}^{k-1} n_{k\ell} t^{(\ell)} = n_k - QP_k. \tag{17.4}$$

Let us, as in STV, subtract Q (which is used to elect i_k); then there remains a surplus, using (17.3),

$$n_k - QP_k - Q = n_k - Q(P_k + 1) = n_k - Qn_k t^{(k)} = n_k(1 - Qt^{(k)}). \tag{17.5}$$

(Since $Qt^{(k)} = t^{(k)}/t^{(s)} \leqslant 1$, this surplus is $\geqslant 0$, which shows that the total value (17.4) is at least Q.) After the election of i_k, these ballots have, as we have seen, the value $1 - Qt^{(k)}$ each, which equals the surplus (17.5) divided equally between the n_k ballots that participated in the election of i_k. This is exactly as in the inclusive Gregory method for STV (Appendix E.2.1(iv)).

We summarize in a theorem.

Theorem 17.1. *Consider an election with ordered ballots. and let $Q := W^{(s)}$, the reduced vote for the last elected using Phragmén's method. Then Phragmén's method yields the same result as STV with the inclusive Gregory method and this quota Q, provided that the surpluses are transfered in the order the candidates are elected by Phragmén's method.*

We see also that every elected will reach the quota Q, so that there will be no eliminations; in fact, the final surplus $n_s(1 - Qt^{(s)}) = 0$, and thus Q is the largest quota that allows s candidates to be elected without eliminations (still provided the surpluses are transfered in the prescribed order).

Consequently, Phragmén's ordered method can be seen as a variant of STV without eliminations, and with a quota Q that is calculated dynamically instead of as a fixed proportion of the number of votes. (The latter difference is the same as the difference between divisor and quota methods for list election methods, see Footnote 76 in Appendix E.3. Thus Phragmén's method can be seen as a variant of STV related to divisor methods, which is not surprising since Phragmén's method was conceived as a generalization of D'Hondt's method.)

However, this correspondence with STV is not perfect, since STV is sensitive to the order the surpluses are transfered, and standard versions of STV may do this in a different order than Phragmén's method yields.

Example 17.2 (Phragmén's method and STV)**.**

Ordered ballots. 4 seats.

22 ABCD
11 ABE

11 CE

Using Phragmén's method, candidates are elected in the following order:

(1) A is elected with reduced vote $W_{\text{A}} = 33$.

(2) B is elected with reduced vote $W_{\text{B}} = \frac{33}{2}$.

(3) The three groups of ballots now have the place numbers $\frac{4}{3}$, $\frac{2}{3}$, 0 (with sum 2), which give C and E the reduced votes

$$W_{\text{C}} = \frac{22+11}{1+4/3} = \frac{99}{7} \qquad \text{and} \qquad W_{\text{E}} = \frac{11}{1+2/3} = \frac{33}{5}.$$

Thus C is elected.

(4) The three groups of ballots now have the place numbers

$$\frac{22}{W_{\text{C}}} = \frac{22}{99/7} = \frac{14}{9}, \qquad \frac{11}{W_{\text{B}}} = \frac{11}{33/2} = \frac{2}{3} = \frac{6}{9}, \qquad \frac{11}{W_{\text{C}}} = \frac{11}{99/7} = \frac{7}{9}$$

(with the sum 3), which gives D and E the reduced votes

$$W_{\text{D}} = \frac{22}{1+14/9} = \frac{22\cdot 9}{23} \qquad \text{and} \qquad W_{\text{E}} = \frac{22}{1+13/9} = \frac{22\cdot 9}{22} = 9.$$

Thus E is elected.

Elected: ABCE.

Consider now instead STV with the inclusive Gregory method and quota $Q = W_{\text{E}} = 9$ as in Theorem 17.1. (Q happens to be the Droop quota rounded upwards in this example. We could also obtain the Droop quota without rounding by adding a single vote on F. In general, there is no direct relation between Q in Theorem 17.1 and the Droop quota.) By Theorem 17.1, we will obtain the same elected ABCE if the surpluses are transfered in this order. This is also easy to see directly: A has surplus 24, which is transfered to B. B has surplus 15 which is transfered with 10 to C and 5 to E. C has surplus 12 of which is 8 is transfered to D and 4 to E. E now has exactly $Q = 9$ votes and gets the fourth seat, while D only has 8 votes.[36]

However, in standard versions of STV, such as the inclusive Gregory method used for the Australian Senate, the surpluses are transfered in a different order: A and C reach the quota on their first-preference votes and are elected. Their surpluses 24 and 2 are transfered to B and E, respectively. This makes B reach the quota, so B is elected to the third seat, with a surplus 15 that is transfered with 10 to D and 5 to E. Consequently, D gets the final seat with 10 votes against 7 for E. Elected: ABCD.

Remark 17.3. Theorem 17.1 yields also the following (rather complicated) description of Phragmén's method purely in terms of STV: With s seats, do s distributions of seats by STV with the inclusive Gregory method, the first time for 1 seat, the second for 2 seats, and so on. In the k-th distribution,

[36] In this example, and assuming that the surpluses are transfered in this order, the result will be the same for every quota $Q \leqslant 99/7 = 14\frac{1}{7}$, although if $Q > 9$ then E will not reach the quota and if $Q \leqslant 198/23 \doteq 8.61$ then both D and E will reach the quota, although E wins.

the surpluses of the $k-1$ already elected shall be transfered in the order that these have been elected, and the quota Q_k shall be chosen such that these $k-1$ will reach the quota, and that (after all transfers), there will be another candidate that reaches exactly Q_k, so this candidate is elected to seat k.

18. Some variants of Phragmén's and Thiele's methods

18.1. **Unordered ballots with two groups.** As said in Section 7, a problem with unordered ballots is the risk of decapitation, which led to the introduction of the ordered versions of Phragmén's and Thiele's methods.

Another modification to avoid decapitation is to let the names on each ballot be separated into two groups (instead of a complete ranking), with preference to the first group but no ordering inside each groups. (This modification too applies to both Phragmén's and Thiele's methods, as well as to some other similar methods with unordered ballots.)

Unordered ballots with two groups. *Each ballot has two groups of names. The orders inside the two groups do not matter. A ballot is regarded as a vote for only the names in the first group as long as at least one of these names remains unelected; when all names in the first group are elected, the ballot is regarded as a vote for all remaining names. (The second group might be empty; equivalently, ballots with a single group are also accepted.)*

The names in the second group can thus be seen as reserves, that are not used until required.

This modification was described for Phragmén's method (and attributed to Phragmén) by Cassel [5] in 1903, and was proposed for Thiele's method in parliament in 1906 (by Petersson in Påboda) [6, p. 20].

In practice, the groups might be shown on the ballot by, for example drawing a line between the two groups of names [5], or by underlining the names in the first group [6].

18.2. **Weak ordering.** To have two groups on a ballot is an intermediary between unordered and ordered voting. More generally, a ballot could be allowed to have a set of candidates with an arbitrary *weak ordering* (= *total preorder*), i.e., a list of groups of candidates, with the different groups in order of preference but ignoring the order inside each group.[37]

Weakly ordered ballots. *Each ballot contains a weakly ordered set of names. In practice, this could be a list of names separated into groups by one or several lines (or a single group without a line); the orders inside the groups are ignored. A ballot is regarded as a vote for only the names in the first group where there is someone that is not yet elected.*

[37] This version was, as far as I know, not proposed by Phragmén, Thiele or any of their contemporaries.

Note that this version includes (as extreme cases) both the unordered version (every ballot contains only a single group) and the ordered version (every ballot contains a total ordering), as well as the version with two groups in Section 18.1. The commission report [6, p. 20] briefly mentions (and dismisses as impractical) also the possibility of having some names on a ballot ranked, and the remaining names coming after these but unordered among themselves.

Ballots with weakly ordered sets of candidates have been discussed in the context of STV (Appendix E.2.1), see e.g. Meek [52] and Hill [39], and such ballots have been used in STV elections in some organizations [39]. However, the method discussed in [52] and [39] to handle weak orderings differs from Phragmén's: they regard a weak ordering as a vote split equally between all total orderings compatible with the weak ordering. On the other hand, Phragmén's version in Section 18.1, extended to weak orderings as above, means that each ballot is counted as a full vote for each candidate in its first group, as long as none of them is elected, cf. the principle in (U2). For example, a ballot beginning with the group (AB) is, as long as neither A nor B is elected, regarded as $\frac{1}{2}$ vote for A and $\frac{1}{2}$ for B by [52] and [39], but as 1 vote for A and 1 vote for B by Phragmén. (If A or B is elected, then the value of the ballot is reduced for the remaining candidate in both systems, by different mechanisms.)

It would be interesting to compare the two ways to handle weakly ordered list. It seems that Phragmén's principle can be applied to STV elections too, and it might have some advantages over splitting the vote between total orderings as described above. (Cf. Section 18.5 for the case of unordered ballots.)

18.3. **Phragmén's method recursively for alliances and factions.** In Swedish elections 1924–1950, ballots could (but did not have to) contain not only a party name but also an alliance name and a faction name (in addition to the names of the candidates), see Appendix D; a ballot could thus be labelled in up to three levels. (Only the party name was compulsory.) When ballots with alliance and faction names were used, Phragmén's method was used recursively in up to three steps: (D'Hondt's method was used to distribute seats between alliances and parties outside alliances.)

PHRAGMÉN'S METHOD WITH ALLIANCES AND FACTIONS. *Ballots are ordered, and contain a party name and possibly an optional alliance name and an optional faction name.*

First, Phragmén's method is applied to each faction separately. Thus, for each faction name, the method is used to determine an ordering of the candidates on the ballots with that faction name; call this ordering the faction list. In the sequel, all ballots with this faction name are regarded as containing the faction list instead of their original lists of names.

Secondly, Phragmén's method is applied to each party. For each party name, the method is used to determine an ordering of the candidates on the *ballots with that party name; call this ordering the party list. In the sequel, all ballots with this party name are regarded as containing this party list instead of their original lists of names (or the faction list).*

Thirdly, Phragmén's method is applied to each alliance. For each alliance name, the method is used to determine an ordering of the candidates on the ballots with that alliance name: call this ordering the alliance list.

Seats given to an alliance or a separate party are assigned to candidates according to the alliance list or party list.

This means that if no candidate appears on the lists for more than one party or faction, then (by Theorem 11.1) the seats given to an alliance are distributed between the participating parties according to D'Hondt's method, and similarly the seats given to a party are distributed between the factions according to D'Hondt's method; finally Phragmén's method is applied for each faction (if there are different ballots within the faction). (Since D'Hondt's method also was used for the distribution of seats between alliances and separate parties, this gives a nice consistency.) However, the method above in an elegant way handles also case where the same name appears on ballots from different factions, or even different parties. Moreover, the method above handles cases where some but not all ballots have a faction name.

18.4. **Party versions.** Mora and Oliver [55] discuss a variant of Phragmén's unordered method where each ballot contains a set of *parties* instead of candidates. The seats are distributed to the parties one by one as in Phragmén's method, with the difference that a party can receive more than one seat, and thus parties that have received a seat are not ignored in the sequel.

PHRAGMÉN'S METHOD FOR PARTIES. *Each ballot contains a set of parties. The seats are distributed as in Section 3.1 or 3.2, but parties that have received seats continue to participate.*

It is easily seen that this extension of Phragmén's method also can be seen as a special case of it: if each party has a set of (at least) s candidates, with these sets disjoint, and we on each ballot replace each party by its set of candidates, then an election by Phragmén's (unordered) method will give the same number of seats to each party as the party version above for the original ballots with parties. (All candidate within the same party would obviously tie, so the choice of elected within each party would be uniformly random.) Hence the party version is equivalent to assuming that each party has a list of candidates (with at least s names), and that each voter votes for some union of party lists, i.e., if the voter votes for one candidate from some party, he or she also votes for all other from the same party. (We ignore here that in the case of a tie, the probabilities for the different possible outcomes may be different.)

A party version of Thiele's method can be defined in the same way.

The party version of Phragmén's method has interesting and surprising mathematical properties, see Mora and Oliver [55]; these will be further studied elsewhere. (The corresponding party version of Thiele's method is much better behaved.)

18.5. **Phragmén's first method (Eneström's method) – STV with unordered ballots.** Cassel [5, pp. 47–50] describes what he calls "Phragmén's first method", which is a version of STV (Appendix E.2.1), but much resembles Phragmén's later method described above (which is called "Phragmén's second method" in [5]). The same method was earlier described by Eneström [31] in 1896.[38] Note that the method, unlike all other versions of STV that I know of, older and newer, uses unordered ballots; it follows the principles (U1)–(U2) in Section 6.

PHRAGMÉN'S FIRST METHOD (ENESTRÖM'S METHOD). *Unordered ballots. Each ballot has initially voting power* 1*. Each ballot is counted fully, with its*

[38]The history is murky. Phragmén seems to have invented this method c. 1893; it is similar to his discussion and examples in the newspaper article [57] from 1893, which clearly describes the weighted inclusive Gregory method, but of his two examples in [57] one uses unordered ballots and the other ordered ballots (without comment), and at least one uses the Droop quota, so the method was presumably not completely developed yet. As far as I know, Phragmén never published anything about his "first method". On the other hand, the method is described in detail (including examples) by Eneström [31] in 1896, together with Phragmén's and Thiele's methods; Eneström calls it "my method", and does not mention Phragmén in connection with this method. He had also a few months earlier briefly described the method in a letter to a newspaper [30], as a simpler alternative to Phragmén's method. Flodström [35, pp. 29–31] calls it "Eneström's method". Nevertheless, in 1903, Cassel [5] attributes the method to Phragmén, without mentioning or giving a reference to Eneström. It seems improbable that this is a mistake by Cassel, since [5] is an appendix to the commission report [4], and Phragmén was one of the members of the commission. Moreover, it seems obvious that Phragmén and Cassel must have seen [31], which was published in the proceedings of the Royal Academy of Science in the same volume as [60]. It is perhaps possible that Cassel regarded Phragmén as having invented the method first and deliberately ignored Eneström [31]. Note also that Eneström [31] and Cassel [5] illustrate the method by the same example (Example 18.1 below), earlier used by Phragmén [58, 59, 61] for his method; Eneström [31] and Cassel [5] even round the numbers in the calculations in the same way, but they use different labels for the candidates.

Maybe the method ought to be called *Eneström's method*? At least Eneström [31] seems to be the first publication of it.

Gustaf Eneström (1852–1923) was a Swedish mathematician. He did (as both Phragmén and Thiele, see Appendix A) work in Actuarial Science, but his main contributions are to the history of mathematics, where he published *Bibliotheca mathematica*, initially an appendix to Mittag-Leffler's *Acta Mathematica*, but 1888–1913 his own independent journal. (Eneström was assistant to Mittag-Leffler, helping with *Acta Mathematica*, but there occured a break between them in 1888, and Eneström was replaced by Phragmén, who became a coeditor.) Eneström did not have a university position and worked as a librarian. I do not know anything about his personal relations with Phragmén, but they were possibly strained. Eneström was an outsider in Swedish mathematics and was disappointed that Mittag-Leffler and other established mathematicians looked down upon his work on the history of mathematics. [28], [73, Gustaf Hjalmar Eneström].

present voting power, for each unelected candidate on the ballot. Let Q be the Hare quota (see Appendix E.3). For each seat, the candidate is elected that has the largest sum of voting powers (from all ballots that contain the candidate's name). If this total voting power is v, and $v > Q$, then each of these ballots has its voting power multiplied by $(v - Q)/v$. If $v \leqslant Q$, then these ballots all get voting power 0 *(and are thus ignored in the sequel). This is repeated until the desired number of candidates are elected.*

Note that the total voting power of all ballots is decreased by $(Q/v) \cdot v = Q$ each time, as long as someone reaches the quota.

The method is presented in [5] as Phragmén's improvement of *Andræ's method* (Appendix E.2.1), resolving two major problems with the latter:

First, since Andræ's method uses ordered ballots, and only counts the first unelected candidate on each (except possibly at the end), as in (O1)–(O2) above, a party that gets many votes (maybe several times the quota) but with its candidates in different orders may not get any seat at all. Phragmén thus solves this by using unordered ballots. (In modern versions of STV, with ordered ballots, this problem is solved by eliminations, see Appendix E.2.1.)[39]

Secondly, Andræ's method uses effectively a random selection of the ballots that are transferred from a successful candidate, so the outcome may be random. (This is still true for some versions of STV, see Appendix E.2.1(i)–(ii).) Phragmén resolves this by transferring all ballots but reducing their voting power proportionally, by what is now known as the *weighted inclusive Gregory method*, see Appendix E.2.1(v), which thus was invented by Phragmén (and then forgotten for until reinvented almost a century later).

Phragmén seems to have invented this method c. 1893, see Footnote 38, but then he instead developed the ideas further to the method described in Section 3 by eliminating the fixed quota Q and instead using a variable quota (or, equivalently, as in the description in Section 3, a variable voting power); note also that there are other modifications, and that the weighted inclusive Gregory method is gone, and replaced by something similar to the unweighted inclusive Gregory method in Appendix E.2.1(iv), although the mechanism is different, see Section 17.

Phragmén seems to have returned to new versions of the quota-based method (with a simplified method for reducing the votes) in 1906, according to some unpublished notes and drafts, see [55, Appendix B.1].

Example 18.1. Phragmén's first method (Eneström's method) was illustrated by both Eneström [31] and Cassel [5, p. 49] by the example used by Phragmén [58, 59, 60] for his method described in Section 3, and used

[39] There is no point in adding eliminations to Phragmén's version with unordered ballots, since every candidate on a ballot gets its full remaining voting power, regardless of whether there are other remaining candidates on the ballot or not; hence eliminations would not change the sum of voting powers for the remaining candidates.

above in Examples 3.7, 4.4, 9.3, 10.2. We follow [31] and [5], and present the calculations using decimal approximations.

Unordered ballots. 3 seats. Phragmén's first method (Eneström's method).

1034 ABC
519 PQR
90 ABQ
47 APQ

1690 votes. The Hare quota $Q = 1690/3 \doteq 563.3$.

The total numbers of votes for each candidate are

A 1171
B 1124
C 1034
P 566
Q 656
R 519.

The first seat goes to A, who has the largest number of votes. Since A has 1171 votes, which exceed the quota Q, each ballot containing A has its voting power (value) reduced from 1 to $1 - \frac{Q}{1171} = 1 - \frac{563.3}{1171} \doteq 0.519$.

The total voting power of each group of ballots is now

ABC $1034 \cdot 0.519 \doteq 536.6$
PQR 519
ABQ $90 \cdot 0.519 \doteq 46.7$
APQ $47 \cdot 0.519 \doteq 24.4$

(The sum is $V - Q = 2Q \doteq 1126.7$.)

By summing these values for the ballots containing a given candidate, the voting power that each candidate can collect is: B 583.3; C 536.6; P 543.4; Q 590.1; R 519. Since Q has the highest voting power, Q is elected to the second seat.

The voting power 590.1 of Q exceeds the quota Q=563.3, and thus the ballots PQR, ABQ and APQ get their voting power multiplied by $1 - \frac{563.3}{590.1} \doteq 0.045$. The total voting power of each group of ballots is now

ABC 536.6
PQR $519 \cdot 0.045 \doteq 23.5$
ABQ $46.7 \cdot 0.045 \doteq 2.1$
APQ $24.4 \cdot 0.045 \doteq 1.1$

(The sum is $V - 2Q = Q \doteq 563.3$.)

The voting powers available to the remaining candidates are thus: B 538.7; C 536.6; P 24.6; R 23.5. Thus B is elected to the third seat.

Elected: ABQ.

This is the same result as produced by both Phragmén's and Thiele's methods, see Examples 3.7 and 4.4.

See further examples in [31].

18.6. **Versions of Phragmén's method based on optimization criteria.** Phragmén [60] discusses election methods (for unordered ballots) from the general point of view that each voter should, as far as possible, obtain the same representation as everyone else.[40] More precisely, each elected candidate is counted as one unit, which is divided between the voters that have voted for that candidate. This gives each voter a measure of the voter's representation, and the goal is to keep these as equal as possible.

This "amount of representation" is the same as *load* in Remark 3.1 and for convenience we use the latter term here. (Recall that load also equals *voting power* in Section 3.) The methods obtained by this approach can thus be described as: *For each set of s candidates, compute the loads for each voter. Elect the set of candidates that minimizes the "inequality" of the load distribution.*

In the paper, Phragmén discusses several alternatives, as follows. We use the notation that there are V voters, and that the load of voter k is ξ_k; thus $\sum_{k=1}^{V} \xi_k = s$, the number of elected candidates, and we define $\bar{\xi} := \sum_{k=1}^{V} \xi_k / V = s/V$, the average load per voter; note that $\bar{\xi}$ is the same for all sets of elected (with s fixed) and for all distributions of their loads.

- (a) The total load 1 of each candidate can be:
  - (a1) divided equally between all voters voting for that candidate,
  - (a2) divided arbitrarily between the voters voting for that candidate (in a way that minimizes the final inequality).
- (b) The "measure of inequality" (that is to be minimized) for a set $\{\xi_k\}$ of loads can be taken as:
  - (b1) The sum of squares $\sum_{k,l}(\xi_k - \xi_l)^2$. (This choice is thus an instance of the general method of *least squares*.) Equivalently, as is well-known (and noted by Phragmén [60]), we can minimize the variance $V^{-1}\sum_k(\xi_k - \bar{\xi})^2$ (or just $\sum_k(\xi_k - \bar{\xi})^2$), or simply (because $\bar{\xi}$ is fixed) the sum of squares $\sum_k \xi_k^2$.
  - (b2) The maximum difference $\max_k(\xi_k - \bar{\xi})$. This is equivalent to minimizing the maximum load $\max_k \xi_k$.
- (c) Furthermore, Phragmén discussed two versions for the optimization (the same as two of the three versions discussed by Thiele [78], see Section 4):
  - (c1) The set of s candidates that minimizes the inequality is elected. We call these methods *optimization* methods.
  - (c2) The candidates are elected sequentially. In each round, the existing loads for the previously elected are kept fixed and for each remaining candidate, the loads for that candidate, if elected, are added to the previous loads. The candidate that minimizes the

[40] It seems likely that this was a kind of response to Thiele [78], which also tries to derive an election method from general principles by some optimization criterion. Phragmén notes that his classification according to "inequality" is different from from Thiele's based on "satisfaction", but does not compare the two approaches.

inequality of the resulting set of loads is elected. (This is thus a greedy version of the optimization method.) We call these methods *sequential*.

This gives $2 \cdot 2 \cdot 2 = 8$ possible combinations (we shall see below that they all yield different methods), but Phragmén does not discuss all of them.

Phragmén first says that it is natural to divide the load of a candidate equally between the voters (i.e., (a1) above), but that this cannot be regarded as the definitive answer; he gives the following example.

Example 18.2. From Phragmén [60].

Unordered ballots. 2 seats.

100 AB
100 AC

An optimization method (c1) is assumed. BC will give a uniform distribution of the load, while with (a1), this is impossible if A is elected. Thus the outcome is BC (for any inequality measure), in spite of the fact that everyone has voted for A.

We may also note that (a2) would enable all three choices AB, AC and BC to have uniform load distributions, so although this would enable A to be elected, the result would be a tie which hardly is satisfactory in this case, see also Example 18.5 below. Finally, note that any sequential method (c2) would elect A first.

Phragmén nevertheless continues to study (a1), and says that it is natural to measure the inequality of the loads by the sum of the squared differences, i.e. (b1) above. He shows that this is equivalent to minimizing the sum $\sum_k \xi_k^2$, and that if there are v_i votes for candidate i, of which v_{ij} also are for j, then, if we elect a set $\mathcal{S}$,

$$\sum_k \xi_k^2 = \sum_{i\in\mathcal{S}} \frac{1}{v_i} + \sum_{i,j\in\mathcal{S},\, i<j} \frac{2v_{ij}}{v_i v_j}. \tag{18.1}$$

Phragmén then continues to study the sequential version (a1)(b1)(c2), which avoids the problem in Example 18.2, but shows by another example that this too suffers from undesirable non-monotonicity, see Example 18.3 below.

We may note (although Phragmén did not do so) that the election method (a1)(b1)(c2) can be given an algorithmic description similar to the one in Section 3.2. The proof follows from (18.1) by noting that the increase of the sum in (18.1) if candidate i is added the elected set $\mathcal{S}$ equals $1/W_i$ in (18.2).

Sequential version with equipartitioned loads and least squares criterion. *Seats are given to candidates sequentially. Let v_i be the number of votes for candidate i. If a set $\mathcal{S}$ of candidates already has been elected, then each ballot with a set σ of candidates is given a place number $q'_\sigma := \sum_{i\in\sigma\cap\mathcal{S}} 1/v_i$, i.e. each elected candidate i contributes $1/v_i$ to the place number on each ballot containing i. Let further v_σ be the number of* *ballots with the set σ, and let $q_\sigma := v_\sigma q'_\sigma$, their total place number. The reduced vote for candidate i is then defined as*

$$W_i := \frac{v_i}{1 + 2\sum_{\sigma \ni i} q_\sigma} = \frac{\sum_{\sigma \ni i} v_\sigma}{1 + 2\sum_{\sigma \ni i} q_\sigma}. \tag{18.2}$$

The candidate with the largest reduced vote is elected.

As said above, (a1)(b1)(c2) suffers from non-monotonicity, as shown by the following example. (See Example 14.1 for a similar example for Thiele's method.)

Example 18.3. This example is in principle from Phragmén [60], but Phragmén's numerical example is incorrect; the following corrected version is due to Xavier Mora (personal communication).

Unordered ballots. 2 seats.

1145	A
885	B
900	C
55	AB
100	AC

Using (a1)(b1)(c2), the first elected is A (who has most votes), and a calculation, e.g. using (18.2), shows that the second place is tied between B and C. However, if some of the voters for A change their mind and add B, so that the numbers of votes instead are

$1145 - x$	A
885	B
900	C
$55 + x$	AB
100	AC

for some $x > 0$ (with $x < 360$ to ensure that A still is elected first), then C will be elected; conversely, with a change in the opposite direction ($x < 0$), B is elected. The election of B can thus be prevented by more voters voting for B!

Phragmén draws the conclusion that (a1) has to be abandoned, and replaced by (a2), which does not have this kind of non-monotonicity as shown by the following result. (See also Section 14 and in particular the overlapping Theorem 14.2.)

Theorem 18.4 (Phragmén [60])**.** *Every election method based on* (a2)*, for any measure of inequality, satisfies the following: Consider an election and let A be one of the candidates. Suppose that the votes are changed such that A gets more votes, either from new voters that vote only for A, or from voters that add A to their ballots (thus changing the vote from some set σ to $\sigma \cup \{A\}$), but no other changes are made (thus all other candidates receive exactly the same votes as before). Then this cannot hurt A; if A would have been elected before the change, then A will be elected also after the change.*

Proof. Consider first an optimization method. Each set of candidates not containing A has the same possible distributions of loads, and thus the same minimum inequality as before the change. On the other hand, a set containing A still has all possible distributions of loads that existed before the change, and possibly some new ones; hence the minimum inequality for such a set is at most the same as before. If A would be elected before the change, then a set of the latter type had smaller (or possibly equal) minimum inequality than every set of the former type, and then the same holds after the change, so A will still be elected.

Consider now a sequential method.[41] Consider the round when A would have been elected, if there was no change. After the change, either A already has been elected before this round, and we are done, or the preceding rounds have elected the same candidates as before the change, and then A will be elected in this round, by the same argument as in the optimization case. □

Having thus argued for (a2), Phragmén says that the election method is determined by the measure of inequality.[42] He notes that this measure can be chosen in different ways, and continues that in order to obtain as simple calculations as possible, it is recommended to use the difference between maximum and average load (b2) and the sequential version (c2).

Phragmén thus has finally come to the election method (a2)(b2)(c2), and notes that this is the method that he earlier has proposed in [58; 59], which is described in Section 3; it is easily seen that the optimal distribution of loads is the same as the voting powers assigned in the formulation in Section 3.1.[43]

Although Phragmén thus favoured the method (a2)(b2)(c2), one might consider also other combinations than the ones he studied. In particular, the other three versions with (a2) also seem interesting, at least from a mathematical point of view; they are (a2)(b2)(c1), (a2)(b1)(c1) and (a2)(b1)(c2) above:

Optimization version of Phragmén's method. *Elect the set of s candidates such that, with loads distributed optimally, the maximum load is minimal.*

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[41]Phragmén did not mention this case explicitly.

[42]Phragmén did not explicitly discuss the difference between optimization and sequential methods, or the computational problems with the former.

[43]This was obviously the intention of the paper [60]. It seems that one of the main objections to Phragmén's method (from its proposal in 1894 to the adoption of the ordered version 27 years later) was that, regardless of whether it had mathematical advantages or not, it was too complicated to be understood and to be used in practice. Phragmén tried varying formulations and motivations in different papers, and in [60] he thus tries to present the method as simple by giving even more complicated alternatives. (He does not explicitly say that the method is the simplest possible among the acceptable alternatives, but he possibly wanted to give that impression.) However, in retrospect, this attempt to present the method as simple does not seem succesful, since Phragmén later did not use this argument again.

Optimization least squares version of Phragmén's method. *Elect the set of s candidates such that, with loads distributed optimally, the sum of the squares of the loads is minimal.*

Sequential least squares version of Phragmén's method. *Seats are given to candidates sequentially. When a candidate is elected, each ballot with this candidate is given a load for that candidate, in addition to any load that might exist from previously elected candidates; the additional loads are chosen such that their sum is* 1 *and the sum of the squares of the total loads of the ballots is minimal. In each round, the candidate is elected such that the resulting sum of squares of loads after electing the candidate is smallest.*

The two optimization methods are studied by Mora and Oliver [55, Section 8] (together with their party versions as in Section 18.4) and in [19] (under the names *max-Phragmén* and *var-Phragmén*). The sequential least squares version is studied by Mora [54].

The optimization methods, although optimal in the mathematical sense of optimizing some function, do not always yield results that seem optimal from other points of view. This was seen in Example 18.2, and we can modify the example to make it more striking.

Example 18.5.

Unordered ballots. 2 seats.

100 AB
100 AC
1 B
1 C

For any optimization method (c1), using either (a1) or (a2), and either (b1) or (b2) or any other method of inequality, BC will be elected, since this gives a perfectly uniform load distribution, while any other combination leaves one voter with load 0.

Since every voter but two votes for A, this result seems questionable. Indeed, any sequential method (of the types considered here) begins by electing the candidate with most votes, so A is elected (and the second seat is a tie between B and C).

This example also shows that the optimization methods (c1) considered in this subsection differ from the sequential ones (c2).

Cf. the similar Example 13.4, showing the difference between the optimization and the sequential versions of Thiele's method. That example would work here too, but not conversely. (In this type of example, with the four ballot types above and symmetric in B and C, Thiele's optimization and addition methods yield the same result unless at least a third (and at most half) of the votes are on B or C only, while for the methods studied here, the fraction of such votes can be arbitrarily small.)

Mora [54] found a technical problem with the sequential least squares method (a2)(b1)(c2), shown in the following example.

Example 18.6 (Mora [54])**.** The sequential least squares method (a2)(b1)(c2). Unordered ballots. 3 seats.

9 AB
1 ABC
3 CD

The first seat goes to A, and the loads on the three types of ballots are $(0.1, 0.1, 0)$. The next seat goes to C, and then the loads are $(0.1, 0.275, 0.275)$. The third seat goes to B. However, if we would distribute the total loads of the ballots containing B (including one unit for the election of B) uniformly, we would get the load distribution $(0.2175, 0.2175, 0.2750)$, where the load on the ballot has decreased. This is not allowed by the formulation above, and instead we have to distribute the load of B on the 9 ballots AB only, giving the loads $(0.2111, 0.275, 0.275)$. So either we have to accept this behaviour, which complicates the implementation and reduces the usual advantage of least squares methods, or the method has to be modified by allowing the load of a ballot to decrease.[44]

Consider now the party list case (see Section 11). In this case, there is no difference between (a1) and (a2), since an equidistribution of loads is optimal; if a party has v votes and m candidates elected, then each of its v ballots thus has load m/v. The problem of optimizing these quotients for party lists by the method of least squares was considered in 1910 (thus after Phragmén's paper, but presumably independent of it) by Sainte-Laguë [67], who showed that this leads to the method now named after him, see Appendix E.3.2. In particular, since Sainte-Laguë's method is sequential, the optimization problem can in this case be solved sequentially (greedily).

Sainte-Laguë [67] also more briefly, again for party lists, considered minimizing the maximum load, i.e., (b2) above, and showed that this leads to D'Hondt's method (Appendix E.3.1); this had also earlier been shown by Rouyer [66] and Equer [32], see Mora [53].[45]

We thus can conclude the following, cf. Theorem 11.1.

Theorem 18.7. *In the party list case, the following holds:*

(i) *The least squares methods using* (b1)*, with either* (a1) *or* (a2) *and either optimizational* (c1) *or sequential* (c2)*, all yield the same result as Sainte-Laguë's method.*

[44] An analogous phenomenon occurs in the version of STV that uses the inclusive Gregory method (Appendix E.2.1(iv)), where the voting value of a ballot can increase when the surplus is transferred, see [34].

[45] As said in Section 11, Phragmén showed this for his sequential method (c2); it is easy to see that in the party list case, there is no difference between the optimization and sequential versions, so both yield D'Hondt's method, but I do not know whether Phragmén did observe this; he did not consider the optimization version at all except implicitly in [60] as discussed above.

(ii) *The maximum load methods using* (b2)*, with either* (a1) *or* (a2) *and either optimizational* (c1) *or sequential* (c2)*, all yield the same result as D'Hondt's method.*

Example 18.8.

Unordered ballots. 2 seats.

5 AB
2 CD

This is a party list case, and by Theorem 18.7, every least squares method (b1) (of the types considered above) will elect (e.g.) AC, while every maximum load method (b2) will elect AB.

This example shows that the least squares methods differ from the maximum load methods.

Remark 18.9. Sainte-Laguë [67] also considered (for party lists) maximizing the minimum load (we might call this (b3)); this yields *Adams' method* (see [13]), which is equivalent to giving every party 1 seat and distributing the rest by D'Hondt's method. In the context of general unordered ballots (instead of party lists), in typical cases there will always be some ballots that do not vote for any of the elected candidates, so their load is 0 and the minimum load is 0, which seems to make (b3) less interesting.

Sainte-Laguë [67] also considered optimizing the distribution of votes per elected candidate, again using the least squares method, showing that (at least assuming that each party gets at least one seat), this yields a method that is now known as *Huntington's method* (proposed by Huntington in 1921), which since 1941 is used for the allocation of seats in the US House of Representatives among the states, see [13]. This version of optimization could perhaps be extended to general unordered ballots, using a dual of (a2) where each voter has 1 vote which is divided between the names on the ballots in an arbitrary way, and the goal is to make the resulting total votes on the elected candidates high and close to equal. However, it is not clear how to treat candidates that are not elected, and how to define the quantity that should be optimized.

Examples 18.5 and 18.8 thus show that of the 8 election methods obtained by combining the alternatives above, the only ones that possibly could be equal (in the sense of always giving the same outcome) would be two that differ only in using (a1) or (a2). For the combination (b1)(c2) (sequential least squares), Example 18.3 and Theorem 18.4 show that the two methods differ. For completeness, we give another example to show that also in the other cases, (a1) and (a2) give different methods, and thus all 8 methods are different.

Example 18.10.

Unordered ballots. 2 seats.

1 AB
1 A

1 B
c C

Here c is a positive rational number with $c < 2$; we can obtain integer numbers of votes by multiplying all numbers by the denominator of c.

By symmetry, there will be ties; we may suppose that these are resolved lexicographically, with A before B. In the sequential versions (c2), then A will be elected first (because $c < 2$); thus for the second seat we consider AB and AC. In the optimization versions (c1), by symmetry we also consider AB and AC.

For AC, the loads on the four types of ballots will in all cases ((a1) or (a2), (b1) or (b2), (c1) or (c2)) be $\left(\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{c}\right)$, with maximum $\frac{1}{c}$ and sum of squares (recalling that we have c ballots of the last type) $\frac{1}{2} + \frac{1}{c}$.

For AB, we obtain using (a1), in both the optimization and sequential versions, the loads $\left(1, \frac{1}{2}, \frac{1}{2}, 0\right)$ with maximum 1 and sum of squares $\frac{3}{2}$; thus, for all four methods using (a1), C will get a seat if $c > 1$.

Using (a2), we instead find that in the optimization version (c1), the loads for AB will be $\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 0\right)$, with maximum $\frac{2}{3}$ and sum of squares $\frac{4}{3}$. Hence, C wins a seat if $c > \frac{3}{2}$ for method (a2)(b2)(c1), and if $c > \frac{6}{5}$ for method (a2)(b1)(c1).

Using (a2) and the sequential version (c2), the loads for AB will be $\left(\frac{3}{4}, \frac{1}{2}, \frac{3}{4}, 0\right)$, with maximum $\frac{3}{4}$ and sum of squares $\frac{11}{8}$. Hence, C wins a seat if $c > \frac{4}{3}$ for method (a2)(b2)(c2), and if $c > \frac{8}{7}$ for method (a2)(b1)(c2).

Together with the examples above, this shows that all 8 combinations discussed above yield different methods.

18.7. **Recent STV-like versions of Phragmén's method.** Olli Salmi [68; 69] has proposed a modification of Phragmén's method to something similar to STV by introducing the Droop quota as a criterium for election and eliminations when no-one reaches the quota. This has been further developed by Woodall [84] and, in several versions, Hyman [40].

However, it seems doubtful whether it is possible to modify Phragmén's method in some way without losing some of its advantages.

19. Some conclusions

As said in the introduction, our purpose of is not to advocate any particular method. Nevertheless, we draw some conclusions for practical applications.

It seems that Thiele's unordered (addition) and ordered methods both have serious problems in some situations, shown by several of the examples in Section 13 and by further results in Section 16, and that these methods therefore are not satisfactory in general. (This was also the conclusion of Cassel [5], Tenow [76] and the commission reports [6] and [7, pp. 213–220].) On the other hand, these methods are simple and may be useful in some situations, and Thiele's ordered method has been used for a long time inside local councils in Sweden, see Appendix D.1, as far as I know without any problems.

Thiele's optimization method is computationally difficult, but might be used in some (small) situations. It is perhaps not sufficiently investigated, but note the property in Theorem 16.8. On the other hand, this method is likely to have at least some of the same problems with tactical voting as Thiele's other methods.

Phragmén's unordered and ordered methods seem quite robust in many situations, and they have good proportionality properties (see Section 16), but they have the disadvantage of leading to rather complicated calculations that only in simple cases can be made by hand.

Phragmén's unordered method does not ignore full ballots, but that is perhaps more a curiosity than a real problem in practice.

Note also the general problems with unordered methods discussed in Section 7. Nevertheless, unordered ballots seem to work well in practice in many situations without organized parties, for example in elections in non-political associations and organizations.

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Appendix A. Biographies

A.1. **Edvard Phragmén.** Lars Edvard Phragmén (1863–1937) was a Swedish mathematician, actuary and insurance executive. He was born in Örebro 2 October 1863, and died in Djursholm (outside Stockholm) 13 March 1937.

Edvard Phragmén began his university studies in Uppsala in 1882, but transferred in 1883 to Stockholm, where Gösta Mittag-Leffler had become professor in 1881. Phragmén obtained a Licentiate degree in 1889, and was awarded a Ph.D. h.c. by Uppsala University in 1907.

In 1888, Edvard Phragmén was appointed coeditor of Mittag-Leffler's journal *Acta Mathematica*, where he immediately made an important contribution by finding an error in a paper by Henri Poincaré on the three-body problem. The paper had been awarded a prize in a competition that Mittag-Leffler had persuaded King Oscar II to arrange, but Phragmén found a serious mistake when the journal already had been printed; the copies that had been released were recalled and a new corrected version was printed. Phragmén continued to be an editor of Acta Mathematica until his death in 1937.

In 1892, Edvard Phragmén became professor of Mathematics at Stockholm University (at that time Stockholm University College). In 1897, he became also actuary in the private insurance company Allmänna Lifförsäkringsbolaget. His interest in Actuarial Science and insurance companies seems to have grown, and in 1904 he left his professorship to become the first head of the Swedish Insurance Supervisory Authority. He left that post in 1908 to become director of Allmänna Lifförsäkringsbolaget, a post that he held until 1933.

Edvard Phragmén became a member of the Royal Swedish Academy of Sciences in 1901. He was President of the Swedish Society of Actuaries 1909–1934.

His best known mathematical work is the Phragmén-Lindelöf principle in complex analysis, a joint work with the Finnish mathematician Ernst Lindelöf which was published in 1908.

His interest in election methods is witnessed by his publications [57; 58; 59; 60; 61]. Moreover, he was a member of the Royal Commission on a Proportional Election Method 1902–1903 [4], and of a new Royal Commission on the Proportional Election Method 1912–1913 [6]. He was also a local politician and chairman of the city council of Djursholm 1907–1918.

See further [73, L Edward Phragmén], [28] and [56, Lars Edvard Phragmén].

A.2. **Thorvald Thiele.** Thorvald Nicolai Thiele (1838–1910) was a Danish astronomer and mathematician. He was born in Copenhagen 24 December 1838, and died there 26 September 1910.

Thorvald Thiele studied Astronomy at the University of Copenhagen, where he obtained his Master's degree in 1860 and his Ph.D. in 1866, with a thesis on a double star. He was Professor of Astronomy at the University of Copenhagen from 1875 until his retirement in 1907. He became a member of the Royal Danish Academy of Sciences and Letters in 1879.

Thiele had an increasingly poor eye-sight (severe astigmatism) and turned to theoretical work and mathematics instead of observational astronomy; he published papers in both Astronomy and Mathematics. Among his mathematical contributions are "Thiele's interpolation formula" for finding a rational function taking given values at given points, published in 1909. He made also contributions to Statistics, where he introduced the half-invariants (later known as cumulants) as well as many other fundamental concepts in his book from 1889. Moreover, he was interested in Actuarial Science, where he both did theoretical work and was a founder of the insurance company Hafnia where he was Mathematical Director from 1872 until 1901; from 1903 he was the chairman of the board of the company. Thiele was an original thinker, and his ideas were often ahead of his time.

Thorvald Thiele was one of the founders of the Danish Mathematical Society in 1873, and one of the founders of the Danish Society of Actuaries in 1901; he was the president of the latter until his death. In 1901, he also became a foreign member of the Institute of Actuaries in London.

See further [48], [23, Thiele, Thorvald Nicolai] and [56, Thorvald Nicolai Thiele].

Appendix B. Phragmén's original formulation

Phragmén first presented his method in a short note [58], dated 14 March 1894, in the Proceedings of the Royal Swedish Academy of Sciences. The note is written in French, and the method is defined as follows, using the term *force électrice*:

Désignons par k une quantité variable, et

1:o) *commençons par donner à la force électrice de tous les bulletins cette valeur k.*

2:o) *Avant de proclamer l'élection d'un premier représentant, nous établirons entre les candidats un certain ordre de préférence, en calculant pour chacun d'eux la valeur de k qui donne la valeur un à la somme de force électrice de tous les bulletins qui contiennent son nom, et en donnant toujours la préférence au candidat pour qui cette valeur est moins grande.*

3:o) *Nous proclamerons élu pour représentant le candidat qui se trouve en tête de cette liste, et nous réduirons en même temps la force électrice des bulletins portant son nom, en y soustrayant la valeur de cette force électrice qui correspond à l'élection.*

4:o) *Nous répèterons les opérations 2:o et 3:o alternativement jusqu'à ce que le nombre prescrit de représentants soient proclamés élus.*

5:o) *S'il arrive, en faisant l'opération 2:o, qu'il y a deux ou plusieurs candidats pour lesquels la valeur de k devient égale, on déterminera leur ordre relatif d'après la liste analogue obtenue à l'opération précédente. Si, même à la première opération, ils ont la même valeur de k, leur place relative est déterminée par la sort (ou par tout autre moyen qu'on y préfèrerait).*

Phragmén gave his method an expanded treatment in the book [59] the following year; he also gave a different motivation for the method in 1896 [60] (see Section 18.6) and made further comments, including comparisons with Thiele's methods in 1899 [61]. Rule 5:o) above, on ties, seems to have been dropped; otherwise there are only minor variations (without mathematical significance) in the formulations; for example, in [61] he tries to make the method more easily understood by talking about the "load" a ballot receives by the election of a candidate, instead of "voting power" (see Remark 3.1).

Appendix C. Phragmén's and Thiele's methods as formulated in current Swedish law

Both Phragmén's ordered method and Thiele's ordered method are used officially in Sweden for some purposes, see Appendix D. We give here official formulations (in Swedish) from current laws.

The methods are not called "Phragmén's" and "Thiele's" anywhere in the laws; Phragmén's method has no name in the Elections Act, but in the Parliament Act (Riksdagsordningen) [3, 12 kap 8 §, 12.8.5] it is called "heltalsmetoden" (the whole number method), which otherwise is the Swedish name for D'Hondt's method. This is really a misnomer; although the method clearly is related to D'Hondt's method, and Phragmén argued that his method was a generalization of D'Hondt's, it is in several important ways different from it. Moreover, from a mathematical point of view, an important feature is the use of non-integer place numbers (generalizing the integer ones in D'Hondt's method), which makes the name "heltalsmetoden" a bit bizarre.

C.1. **Phragmén's ordered method in the Swedish Elections Act.** Phragmén's ordered method has been used in the Swedish Elections Act since 1921 for distribution of seats within each party, although since 1998 only as a secondary method that rarely is used, see Appendix D.

The official formulation in the current (2016) Elections Act [1, 14 kap. 10 §] is as follows. (The formulations have been essentially identical since 1921.)

> *Vid första uträkningen gäller en valsedel för den kandidat som står först på sedeln varvid bortses från kandidater som redan tagit plats. Valsedlar med samma första kandidat bildar en grupp. Varje grupps röstetal räknas fram. Röstetalet är lika med det antal valsedlar som ingår i gruppen. Samma tal är också jämförelsetal för den kandidat som står först på gruppens valsedlar. Den kandidat vars jämförelsetal är störst får den första platsen i ordningen.*
>
> *Vid varje följande uträkning gäller en valsedel för den kandidat som står först på sedeln, men man bortser från kandidater som redan fått plats i ordningen. Den eller de grupper, vilkas valsedlar vid närmast föregående uträkning gällde för den kandidat som fick plats i ordningen, upplöses och ordnas i nya grupper, så att valsedlar som vid den pågående uträkningen gäller för en och samma kandidat bildar en grupp. Övriga befintliga grupper behålls däremot oförändrade. För varje nybildad grupp räknas röstetalet fram. Röstetalet är lika med det antal valsedlar som ingår i gruppen. För samtliga kandidater som deltar i uträkningen beräknas röstetal och jämförelsetal.*
>
> *Röstetalet för en kandidat är lika med röstetalet för den grupp eller det sammanlagda röstetalet för de grupper vilkas valsedlar gäller för kandidaten. Jämförelsetalet för en kandidat är lika med kandidatens röstetal, om inte den grupp av valsedlar som gäller för kandidaten deltagit i besättandet av en förut utdelad plats. Om detta är fallet, får man kandidatens jämförelsetal genom att kandidatens röstetal delas med det tal som motsvarar den del gruppen tagit i besättandet av plats eller platser som utdelats (gruppens platstal), ökat med 1, eller, om flera grupper av valsedlar som gäller för kandidaten deltagit i besättandet av förut*

utdelad plats, med dessa gruppers sammanlagda platstal, ökat med 1. Platstalet för en grupp beräknas genom att gruppens röstetal delas med det största jämförelsetalet vid uträkningen närmast före gruppens bildande. För kandidat som redan stod först på någon valsedel beräknas nytt platstal endast för nytillkomna valsedlar. Bråktal som uppkommer vid delning beräknas med 2 decimaler. Den sista decimalsiffran får inte höjas.

Den kandidat vars jämförelsetal är störst får nästa plats i ordningen.

The Elections Act specifies that calculations should be done to two decimal places, rounded downwards.[46] Otherwise, the formulation is equivalent to the one in Section 9.3.

C.2. **Thiele's ordered method in current Swedish law.** Thiele's ordered method is used in Sweden for the distribution of seats within parties at elections in city and county councils, for example in the election of the city executive board and other boards, see Appendix D.1. The method is formulated as follows in the Act on Proportional Elections (Lagen om proportionella val) [2, 15–18 §]:

15 § *Ordningen mellan namnen inom varje valsedelsgrupp skall bestämmas genom särskilda sammanräkningar, i den utsträckning sådana behövs.*

16 § *Efter varje sammanräkning skall det namn som enligt 18 § har fått det högsta röstetalet föras upp på en lista för valsedelsgruppen, det ena under det andra. Namnen gäller i den ordningsföljd som de har blivit uppförda på listan.*

17 § *Vid varje sammanräkning gäller en valsedel bara för ett namn. Valsedeln gäller för det namn som står först på valsedeln, så länge detta namn inte har förts upp på listan. Därefter gäller valsedeln för det namn som står näst efter det namn som redan har förts upp på listan.*

18 § *En valsedel som gäller för sitt första namn räknas som en röst. När den gäller för sitt andra namn räknas den som en halv röst. En valsedel som gäller för sitt tredje namn räknas som en tredjedels röst, och så vidare efter samma grund.*

---

[46]This was obviously of practical importance in 1921. Today, with computers, it would seem better to use exact calculations with rational numbers.

Already the commission [6] that suggested the method in 1913 proposed that the calculations should be done with decimal numbers, and that two decimal places would be enough for practical purposes; they also for simplicity recommended consistently either rounding up or down, but favoured rounding up for reasons not further explained. The law introduced in 1921 chose rounding down, but otherwise followed these recommendations.

Appendix D. History and use of Phragmén's and Thiele's methods in Sweden[47]

As said above, Phragmén [58] proposed his method in 1894 (see Appendix B), and Thiele [78] as a response proposed his method in 1895.

This was a period when electoral reform was much discussed in Sweden, both the question of universal suffrage and the election method, see e.g. [21]. The two questions were linked; the conservatives expected to become a minority when universal suffrage was introduced, and therefore many of them wanted a proportional election system; conversely, many liberals and socialdemocrats wanted a plurality system with single-member constituencies.

Sweden had 1867–1970 a parliament with two chambers, where the Second Chamber was elected in general elections, while the First Chamber was elected indirectly, by the county councils. (Until 1909, the county councils were also elected indirectly, by electors chosen by city councils and rural municipality councils; these councils were elected in local elections,[48] and thus the First Chamber was elected indirectly in three steps.) Until 1909, the Second Chamber was elected by the *Block Vote* (Appendix E.1.1), mainly in single-member constituencies but in larger cities in constituencies with several members; only men with a certain income or real estate of a certain value were allowed to vote. There were no formal parties, and no registration of candidates before the election; every eligible man was a possible candidate, and the voters could write any names they wanted on the ballots, although parties and other political organizations recommended certain lists and printed ballots that were used by most voters. (Even these organized lists often overlapped, see Examples 13.1 and 13.2.)

In 1896, the government proposed a minor extension of the suffrage (lower limits for income and real estate), together with introduction of a proportional method in the constituencies that elected several members of parliament (*Andræ's method*, a form of STV, see Appendix E.2.1); however, the parliament voted against.

Nevertheless, the questions continued to be discussed. Several bills on universal suffrage for men were introduced in parliament and defeated, mainly because of disagreement on whether the election system should be proportional or not. A Royal Commission on a Proportional Election Method was appointed in 1902, with Phragmén as one of its members. (Nevertheless, the commission proposed the cumulative method, see Appendix E.1.5, and not Phragmén's own.)

[47]As far as I know, no version of Phragmén's or Thiele's method has ever been used outside Sweden. (Thiele was Danish. I do not know whether his, or Phragmén's, method was discussed in Denmark.)

[48]Until 1918, in local elections, each voter had a number of votes proportional to his (or in exceptional cases her) tax. There were some limitations, in particular were the number of votes limited to at most 100 per voter in cities and, from 1900, at most 5000 in rural municipalities; in 1909–1918 the number of votes per voter were 1–40.

In 1909, finally, universal suffrage for men was accepted together with a proportional election system, based on *D'Hondt's method* (Appendix E.3.1). Thus parties were formally introduced; there was still no registration before the election, but the writer was supposed to write a party name on the ballot, followed by a list of candidates as before.[49 50] Each party was then, by D'Hondt's method, given a number of seats according to the party names on the ballots.

It seems that parties had now become better organized and dominated the political scene, and that it was natural to base the election system on them; to use a method like Phragmén's or Thiele's to distribute the seats without party labels was no longer realistic (as it was when the methods were proposed some 15 years earlier). However, the parties were still rather loosely organized and consisted of various more or less clearly defined factions, and since Sweden kept the old system of open lists, where the voter could list any candidates on the ballot, a system was needed for the distribution of seats within the parties. The method adopted 1909 was *Thiele's unordered method*, but combined with a special rule, called *rangordningsregeln (The Ranking Rule)*, to prevent decapitation (see Section 7):[51]

THE RANKING RULE. *The ballots are ordered. If more than half of the ballots for a given party have the same first name, then this candidate gets the party's first seat. If further more than* $2/3$ *of the ballots have the same first and second names, then the second name gets the second seat, and so on. I.e., if more than* $k/(k+1)$ *of the ballots have the same* k *names first, in the same order, then these get the first* k *seats (for any* $k \geqslant 1$*).*

The combination chosen in 1909 for distribution of seats within each party was thus:

Ballots are ordered. The Ranking Rule is used for as many seats as possible. The remaining seats, if any, are distributed by Thiele's unordered method, thus ignoring the order of the names on the ballot.

---

[49] It was until 1921 also possible to omit the party name and just list candidates, as previously. These ballots ("the free group") were treated as a separate party, with some special rules. Such ballots were few, and they had only rarely (in some local elections) any effect on the outcome. In the elections to parliament 1911, only 210 votes of 607 487 were without a party name. In the county council elections 1910, the free group dominated in a few constituencies, and in total 7 county councillors (of 1217 in the whole country) were elected from the free group.

[50] In more recent years, party names and candidates are usually registered before the election, and ballots are printed by the Election Authority. However, registration is compulsory only from the general election in 2018; until now, it has been possible for voters to use blank ballots and fill in any party name and any candidates.

[51] It is easily verified that both the later introduced ordered version of Phragmén's method and the ordered version of Thiele's method obey the Ranking Rule, in the sense that any seat allocated by the Ranking Rule will be allocated to the same candidate by these nethods too.

In practice, it turned out that most seats were distributed by the Ranking Rule. For example, in the elections to the parliament in 1911 were only 21 of the 150 members of the First Chamber, and 18 of the 230 in the Second Chamber, elected by Thiele's method.[52]

However, the combination was not successful, and there were several examples where the outcome did not seem fair; all of them coming from the application of Thiele's method. One important reason seems to have been that the combination of two rules, where one depended on the order of the names on the ballot but the other did not, was confusing, and that votes could have an effect not intended by the voter. And even when the voter fully understood the method, it was often difficult to predict how many from the party that would be elected by the Ranking Rule, and therefore whether the addition or deletion of a name would help or hinder another (more favoured) candidate. Moreover, the problem was compounded by the fact that two parties often agreed to an electoral alliance in a constituency, where they would appear under a common party name in the election, but otherwise promote their own candidates and their own policies; this was possible by the open nature of the election system, and it was encouraged by D'Hondt's method, which works in a superadditive way, so that small parties are somewhat disadvantaged (see e.g. [43] for detailed calculations) and two parties that form an electoral alliance can together never lose a seat by that.[53] Since the different names and lists within a "party" thus in reality often represented different parties, it was imperative that the distribution inside each party was fair. One typical problem is described in Example 13.17.

As a result of dissatisfaction with the results of the new method, a Royal Commission on the Proportional Election Method was appointed in 1912 to suggest improvements; one of the four memebers was Edvard Phragmén.[54] The commission report in 1913 [6] (with many constructed examples) discusses several types of errors that could appear, both because of the combination of the Ranking Rule and Thiele's method, and because of deficiencies in the latter itself. They discuss several possible alternatives[55], and come to the conclusion that the only satisfactory solution is to use ordered ballots. Moreover, they discuss first the ordered version of Thiele's method (which

[52]Note that the same election method always was used for the general elections to the Second Chamber and for the elections to the First Chamber made by the county councils.

[53]It also happened that a party participated in the election using another party's name, without any agreement or consent. (For example because the first party did not expect to get enough votes to get a seat on their own.)

[54]The three other members were Sixten von Friesen, a leading liberal politician; Ivar Bendixon, who was Phragmén's successor as professor of Mathematics at Stockholm University, and also Vice-Chancellor of the University and a local politician in the city council of Stockholm; Gustaf Appelberg, a civil servant from the department of Justice. Two of the four members were thus mathematicians.

[55]Including two proposed in the Swedish parliament, a scoring rule (Appendix E.2.3), and several methods that seem to have been invented and discussed within the commission.

had been proposed in parliament in 1912 by Nilson in Örebro) but find serious problems with this method too. Finally, the commission presents and proposes the ordered version of *Phragmén's method*.

Nothing came out of this proposal immediately, but a new commission repeated the proposal in 1921, in connection with the reform introducing universal suffrage for both men and women. This time the method was adopted, and *Phragmén's method* has been used for the distribution of seats within parties since 1921, at least to some extent (see below).[56]

In 1924, the well-established practice of forming electoral alliances was formalized and alleviated by allowing ballots to contain, besides the party name also a alliance name and a faction name; the ballot could thus be labelled in three levels. (Only the party name was compulsory.) These possibilities disappeared in 1952, when D'Hondt's method was replaced by the modified Sainte-Laguë's method, and thus the incentive to form alliances disappeared. When ballots with alliance and faction names were used, Phragmén's method was used recursively in up to three steps, as described in Section 18.3.

Phragmén's method is still used in the distribution of seats within parties (see Appendix C.1), but it was demoted to a secondary role in 1998, when a system of preference votes was introduced. Each party has one or several ballots, with lists of candidates decided by the party, and the voter in addition to choosing a ballot, also can (but does not have to) vote for one of the candidates on the ballot. The seats given to the party are primarily given to its candidates in order of these preference votes (i.e., by the *SNTV method*, Appendix E.1.4), but only to candidates that obtain at least 5% of the total number of votes for the party. Only when there are not enough candidates that have reached this threshold, the remaining seats are distributed according to Phragmén's method.[57]

In most cases, the party chooses to have just one list in a constituency, and then Phragmén's method only means that any seats not assigned by preference votes are assigned in the order the party has put them on the ballot. And if there are several lists but they have disjoint sets of names, Phragmén's method is just the same as D'Hondts. Thus, Phragmén's method really plays a role only in the few cases where the party chooses to

[56]The seats were still distributed between parties by D'Hondt's method. In 1952, this was replaced by the *modified Sainte-Laguë's method* (Appendix E.3.3). In 1971, the two chambers were replaced by a parliament with a single chamber; furthermore, adjustment seats were introduced for the distribution between parties. None of this affected the use of Phragmén's method within parties.

[57]Ignoring the candidates that have got seats because of their preference votes. This seems to be a mistake in the combination of the two methods. Suppose that a party has two lists ABCD and EFGH in a constituency, and get 3 seats. Suppose further that of the ballots for the party, 55% are ABCD and 45% EFGH, and also that A and D, but no-one else, each gets preference votes on more than 5% of the ballots. Then A and D are elected by their preference votes, and the remaining seat goes to the largest list, i.e. to B. Hence all three elected come from the same list. It seems that it would be better to calculate place numbers taking into account also those elected by preference votes.

have more than one list, with some of the names on more than one list, and moreover the voters choose to give not too many preference votes.

Local elections have followed essentially the same rules as parliamentary elections; in particular, they too have for the distribution of seats inside parties used Thiele's unordered method together with the Ranking Rule in 1909–1921, and after 1921 Phragmén's ordered method, but since 1998 only for seats not assigned by preference votes.

D.1. **Elections within political assemblies.** *Phragmén's method* is in Sweden, besides its use in general elections described above, also used for elections of committees in the parliament [3, 12 kap. 8 §]. Also in this case, the method has in practice a very small role; the members of parliament can be expected to be loyal to their parties and vote on a party list, and thus the seats a party gets are assigned according to this list. Moreover, parties usually form electoral alliances for these elections;[58] each party in the electoral alliance can be expected to vote on its own party list, and then the result of Phragmén's method will be that the seats are distributed within the electoral alliance by D'Hondt's method.[59]

In local councils (cities and other municipalities, and counties), elections to various committees are instead done by *Thiele's ordered method* [2].[60] Again, the method has in practice a very minor role, for the same reasons as in the parliament.

Appendix E. Some other election methods

We give here brief (and incomplete) descriptions of some other election methods, as a background to the methods by Phragmén and Thiele and for comparisons with them. The selection of methods is, by necessity, far

[58] These elections are performed using D'Hondt's method (Appendix E.3.1), which has the important property that a party or electoral alliance that has a majority in the parliament will get a majority in each committee. This method is superadditive and encourages electoral alliances, as said above.

[59] In practice the elections are usually done by consent by voice vote, often unanimous, to a proposal from the nominating committee, so Phragmén's method is not even formally used. However, the nominating committee has of course calculated what the result would be of an election, so the election method still plays the same role, although hidden.

[60] I do not know the reason why different methods are used in parliament and in local councils. One possibility could be that it has been thought that the parliament is more important and that it is justified to use the more complicated but usually better Phragmén's method there, while local councils can do with the simpler Thiele's method. However, this is only a guess. In any case, the choice of method very rarely matters, since as said above, the votes can be expected to be on party lists, and then the result for both methods will be the same as by D'Hondt's method.

Note that a political assembly, where everyone can predict how everyone else will vote, is an ideal setting for tactical voting as in Example 13.13 when Thiele's method is used. However, this only works inside an electoral alliance, and I do not know that anyone has ever tried it – it would hurt your partners and not your political enemies, and would presumably lead to bad feelings and repercussions.

from complete; many other election methods have been proposed and many different ones have actually been used in different countries and contexts. Note also that the same method often is known under different names. The methods described below are mainly those of relevance to a discussion of Phragmén's and Thiele's methods. For further methods and further discussions, from both mathematical and political aspects, see e.g. [33; 63; 64]. We give some examples (far from exhaustive) of current or earlier use in elections. For the election methods actually used at present in parliamentary elections around the world, see [41].

Note the important difference between elections methods that are *proportional*, i.e., methods where different parties get numbers of seats that are (more or less) proportional to their numbers of votes, and other methods (that usually favour the largest party).[61] Proportional election methods are used for political elections in many countries; in most (but not all) cases using a list method with party lists, see Secction E.3 below. Recall that the idea of both Phragmén's and Thiele's methods is to have a proportional election method without formal parties.

As above, $s \geqslant 1$ is the number to be elected (in a constituency). We are (as Phragmén and Thiele) mainly interested in the case $s \geqslant 2$ (multi-member constituencies), but the case $s = 1$ is often included for comparison.

E.1. **Election methods with unordered ballots.** In these method, each ballot contains a list of names, but their order is ignored. The number of candidates on each ballot is, depending on the method, either arbitrary or limited to at most (or exactly) a given number, for example s, the number to be elected.

E.1.1. *Block Vote. Each voter votes for at most s candidates. The s candidates with the largest numbers of votes are elected.*[62]

The Block Vote is the case $s \geqslant 2$, the multi-member constituency version of the case $s = 1$, which is the widely used *Single-Member Plurality* system, also called *First-Past-The-Post*, where each voter votes for one candidate,

[61]Whether an election method is proportional or not is not precisely defined, even when there are formal parties. Obviously, there are necessarily deviations since the number of seats is an integer for each party. Moreover, in practice, whether the outcome of an election is (approximatively) proportional depends not only on the method, but also on the number of seats in the contituency, and on other factors such as the number of parties and their sizes.

[62]One version requires each voter to vote for exactly s candidates. This led to a scandalous result in Sweden in the (autumn) election in 1887: Stockholm was a single constituency with 22 seats (the by far largest constituency). After the election, it was found that one of the 22 that had been elected was not eligible, because he had unpaid back taxes. As a consequence, all 6206 ballots with his name were deemed to contain only 21 valid names, and were therefore invalid. As a consequence, all 22 elected (that had received 4898–6749 votes) were replaced by others (that had received 2585–2988 votes) [8, pp. 45, 51]. (This changed the majority in the parliament to the protectionists, and as a result the government resigned.)

and the candidate with the largest number of votes wins. Single-Member Plurality is the perhaps simplest possible election method and has a long history, and it is still used in many different contexts.

Also the Block Vote (with $s \geqslant 2$) is simple and intuitive and has been used for a very long time. (For example, it has been used since the Middle Ages in English parliamentary elections, where until 1885 most constituencies had 2 seats. [63, p. 158]) Until the late 19th century, it seems to have been the dominant election method for elections in multi-member constituencies. It was the method used in Sweden when Phragmén and Thiele proposed their methods.

The Block Vote is still used in general elections in some countries. It is also widely used in non-political elections, for example in various organizations and associations. (For example, in elections to department boards at my university).

The Block Vote is well-known for not being proportional; when there are organized parties, the largest party will get all seats. Hence, many other election methods have been invented in order to get a proportional result, among them the methods by Phragmén and Thiele discussed in the present paper.

E.1.2. *Approval Voting. Each voter votes for an arbitrary number of candidates. The s candidates with the largest numbers of votes are elected.*

Here $s \geqslant 1$. In a system with well-organized parties, Approval Voting ought to give the same result as the Block Vote (or Single-Member Plurality when $s = 1$), see Appendix E.1.1, with each party fielding s candidates and their voters voting for these. In particular, the method is not proportional. (Without parties, the result may be quite different from the Block Vote.)

This is also an old method, but seems to have been used much less frequently than the Block Vote. For example, it was used (with $s > 1$) for parliamentary elections in Greece 1864–1926, and it is used for distribution of seats within parties in some local elections in Norway.[63] As noted in Section 4.1, the method is one of the methods proposed by Thiele [78] in 1895, viz. his "strong method". The method was reinvented (and given the name Approval Voting) by Weber c. 1976 and was made widely known by for example Brams and Fishburn [16]; see further Weber [80] and Brams and Fishburn [17]. Approval Voting seems to be used mainly in non-political elections, for example it is used in several professional organizations.

E.1.3. *Limited Vote. Each voter votes for at most ℓ candidates, where ℓ is some given number with $1 < \ell < s$. The s candidates with the largest numbers of votes are elected.*[64]

[63] Versions requiring a qualified majority were used alread to elect the Pope 1294–1621, and the Doge (Duke) of Venice 1268–1797, in both cases (obviously) with $s = 1$.

[64] As for the Block Vote, one version requires each voter to vote for exactly ℓ candidates.

The idea of this modification of the Block Vote is that a small party can concentrate its votes on (for example) ℓ candidates and obtain at least some seats in the competition with a larger party that spreads its votes over more candidates. This introduces a degree of proportionality, but it is far from perfect. If $\ell > s/2$ (which in practice usually is the case), and the parties are similar in size, the expected result of optimal voting strategies is that the largest party gets ℓ seats, and the second largest the remaining $s - \ell$ seats, while the other parties (if any) get none.

Moreover, the outcome of an election depends heavily on how the votes of a party are distributed inside the party. Hence the method makes it possible, and more or less necessary, for a party that wants to win many seats, to use schemes of tactical voting. See e.g. [42, Section A.9] for the optimal strategy, and its result, in an ideal situation when all voters follow the instructions from their parties, and, moreover, the parties accurately know in advance the number of votes for each party.

Phragmén [57] mentions Limited Vote as an example of an unsatisfactory election method because of the possibilities of tactical voting, saying that by dividing the votes between several lists, a majority could get all seats, also against a rather strong minority, and that this actually has happened in Spain (where the method was used since 1878). Another well-known example of this was England, where Limited Vote was used in some constituencies in 1867–1885, and the Liberal Party in Birmingham was very successful in organising the voters so that they got all seats in each election; this also gave the method a bad reputation [21, p. 193], [33, p. 27].

Limited Vote was proposed by Condorcet [22, Titre III, Section première, Article 4] already 1793 (for some elections in the French republic; the proposal was not adopted), and it is still used in a few places.

Note that Limited Vote properly means the case when $\ell < s$, but if we ignore this and allow an arbitrary ℓ, then Block Vote (Appendix E.1.1) is the special case $\ell = s$ and Approval Voting (Appendix E.1.2) is the case $\ell = \infty$.

E.1.4. *Single Non-Transferable Vote (SNTV). Each voter votes for one candidate. The s candidates with the largest numbers of votes are elected.* This can be seen as the special case $\ell = 1$ of Limited Vote (Appendix E.1.3), and it has the same weaknesses as Limited Vote. In particular, the outcome depends heavily on how the votes are distributed inside the parties; note that a party can get hurt by too much concentration of votes as well as by too little concentration. In an ideal situation where all voters belong to parties and vote as instructed by their party, and, moreover, the parties accurately know in advance the number of votes for each party, it can be shown that optimal strategies yield the same result as D'Hondt's method, see [42, Appendix A.8]. In a theoretical, and restricted, sense, SNTV can thus be regarded as a proportional election method. However, just as Limited Vote (Appendix E.1.3), it requires elaborate voting tactics, and bad tactics, e.g. based on incorrect guesses of the party's strength, can lead to disastrous results. Phragmén [59, p. 67] discusses SNTV and regards it as unsatisfactory.

SNTV is an ancient method, and was mentioned already by Plato [62, book VI]. It was analysed mathematically by Charles Dodgson[65] [27] in 1884. It was used already in 1835 in North Carolina, and it is still used in a few countries. However, it seems to have been much less common than the Block Vote.

E.1.5. *Cumulative Voting.* The idea is that a voter may divide his or her vote between several candidates. The perhaps the most common version is:

Each voter has $\ell > 1$ votes, that can be given to one or several candidates. (This seems to be the oldest version of Cumulative Voting, used e.g. in the Cape Colony 1853–1909.) See e.g. Droop [29].

Another version of Cumulative Voting, sometimes called *Equal and Even Cumulative Voting*[66] uses unordered ballots:

Each voter votes for an arbitrary number of candidates. A ballot with m names is counted as $1/m$ votes for each candidate. The number of names on a ballot may be restricted, for example to at most s, the number of seats. This method is at present used in Peoria in Illinois, USA (with 5 seats and at most 5 names on each ballot). In Sweden, it was proposed by Rosengren in 1896 [35, pp. 14–16], [5, pp. 23–24] and (for distribution of seats within parties) by a Royal Commission 1903 [4] and then in bills by the government in 1904 and 1905, but it was never adopted by parliament.

Phragmén [59, p. 68] discusses Cumulative Voting, saying that it is similar to SNTV and has essentially the same deficiencies, and therefore "should be seen as a theoretical experiment without practical usefulness".

E.2. **Election methods with ordered ballots.** In these method, each ballot contains an ordered list of names. The number of candidates on each ballot is usually arbitrary, but may be limited to at most, or at least, or exactly some given number, for example s, the number to be elected; some versions require each voter to rank all candidates. (In the latter case, each ballot can be seen as a permutation of the set of candidates.)[67]

E.2.1. *Single Transferable Vote (STV). First, a quota is calculated, nowadays almost always the Droop quota (rounded to an integer or not), see Section E.3 below. Each ballot is counted for its first name only (at later*

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[65]Better known by his pseudonym Lewis Carroll used when writing his famous children's classics.

[66]This version is called *Satisfaction Approval Voting* by Brams and Kilgour [18].

[67]In elections to the Australian Senate, all candidates have to be ranked. This is rather impractical, so since 1983, each party makes a list ranking all candidates (its own and all others), and the voter has the option of voting for one of the party list. This is used by the vast majority of voters, so the system in reality has become a list system. [33, pp. 140–141]

stages, ignoring candidates that have been elected or eliminated). A candidate whose number of votes is at least the quota is elected; the surplus, i.e., the ballots exceeding the quota, are transferred to the next (remaining) name on the ballot. This is repeated as long as some unelected candidate reaches the quota.

If there is no such candidate, and not enough candidates have been elected, then the candidate with the least number of votes is eliminated, and the votes for that candidate are transferred to the next name.

At the end, if the number of remaining candidates equals the number of remaining seats, then all these are elected (regardless of whether they have reached the quota or not).

This description is far from a complete definition; many details are omitted, and can be filled in in different ways, see e.g. [79]. As a result, STV is a family of election methods rather than a single method. (There seems to be hardly no two implementations that agree in all details.) The most important differences between different version are in the treatment of the surplus, but also other details, such as the order of transfer when more than one candidate has reached the quota, the treatment of ballots where all names have been elected or eliminated, and different rounding rules, can affect the final outcome. (See Example 17.2 for one example.)

The special case $s = 1$, when only a single candidate is elected, is usually called *Alternative Vote*. This case is much simpler than the general case, since no surplus has to be transferred, so the method only involves eliminations until someone has reached the quota, which in this case means a majority of the votes.

In the party list case (see Section 11), STV yields the same result as a quota method with the same quota (see Section E.3), i.e. in almost all cases Droop's method (Section E.3.5). (This is easily seen using the formulation in Footnote 77.) STV is therefore regarded as a proportional election method.

The eliminations are an important part of the method. As a result of them, even if the voters of a party split their votes by voting on the party's candidates in different orders, so that no-one of them gets many votes in the first round (when only the first name is counted), as the counting progresses and weaker candidates are eliminated the votes will concentrate on the remaining ones. Hence vote splitting within a party will typically not affect the outcome, unlike with Phragmén's and Thiele's ordered methods.[68]

STV is used, in different versions, in some (but rather few) countries, for example in all elections in Ireland and in some in Northern Ireland, Scotland and Australia; it is the most common proportional election method in English-speaking countries.

There are two main groups of different methods of handling the surplus when a candidate is elected: either some ballots (in number equal to the

[68]However, eliminations introduce also problems with non-monotonicity, see Footnote 27.

quota) are set aside, and the others are transferred, or all ballots are transferred but with reduced values, so that the total voting value is reduced by the quota.[69] In the first case it has to be decided how to chose the ballots to be transferred, and in the second case it has to be decided how to reduce the values, and there are several versions in both cases.

Some of the methods to do the transfers of surplus from an elected candidate are as follows (again omitting some details), see further e.g. [34], [79]. In the descriptions, the quota is denoted by Q, and we assume that a candidate just has been elected with v votes; the surplus is thus $v - Q$. ("The last group of ballots transferred to a candidate" means all ballots counted for the candidate if there has not yet been any transfers to the candidate.)

(i) *Random selection (the Cincinnati method). $v - Q$ ballots are selected at random from the ballots counted for the elected candidate.*

(ii) *Random stratified selection (Ireland). The ballots to be transferred are taken from the last group of ballots transferred to the elected candidate. These ballots are sorted according to their next name, and a proportional number is selected for each next name. The transferred ballots are selected as the ones last added to their group, i.e., in practice they are selected randomly in each group. (The ballots are put in random order before the counting starts.)* The surplus is thus transferred proportionally to the next name, but if also that candidate is elected, the next name to transfer to has been chosen by random selection.

 This method is used in Ireland for the lower house (Dáil) of the parliament.

(iii) *The Gregory method. All ballots in the last group of ballots transferred to the elected candidate are transferred. If there are N ballots in this group, each of them is now counted as $(v - Q)/N$ votes.* This version was introduced in Tasmania in 1907 and has been used there since then. The Irish Senate uses a minor variation where N is the numebr of ballots in the group that are transferable, i.e., have some remaining candidate on them.

(iv) *The inclusive Gregory method. All ballots counted for the elected candidate are transferred. If there are N ballots in this group, each of them is now counted as $(v - Q)/N$ votes.* This is used e.g. for the Australian Senate.

(v) *The weighted inclusive Gregory method. Each ballot has an initial value of 1 vote. All ballots counted for the elected candidate are transferred. Each of them has its current value multiplied by $(v - Q)/N$.*

 This method was invented by Phragmén in c. 1893 (see also Footnote 38), although for a version of STV with unordered ballots and no eliminations, see Section 18.5. It seems to have been reinvented in 1986 by the Proportional Representation Society of Australia, see [34].

[69] Methods of the latter type are often called *Gregory methods*, after J.B. Gregory who proposed one such method in 1880.

The method is used in Western Australia and in local elections in Scotland.

The oldest version of STV, proposed by Carl Andræ in 1855 and used in some elections in Denmark 1856–1953, is called *Andræ's method*. It can be seen as a primitive version; it differs from later versions in that there are no eliminations.[70]

ANDRÆ'S METHOD. *The Hare quota*[71] *(see Appendix E.3) is used. The votes are counted in random order. Any candidate that reaches the quota is elected, and ignored in the sequel. If not enough are elected when all ballots are counted, then the candidates that now have the largest number of votes are elected, provided they have more than half the quota; if there still are remaining seats, the votes are counted again, and if there are m remaining seats, then each ballot is now counted as a vote for the m first candidates that not already have been elected. The m candidates with most votes are elected.*

This method was proposed for Sweden in 1896 by the government, but it was not adopted.

Phragmén proposed a modification of Andræ's method, which (unlike all other versions of STV) used *unordered ballots*, see Section 18.5.

E.2.2. *Bottoms-up. Only the first name on each ballot is counted. The candidate with the least number of votes is eliminated. This is repeted (with eliminated names ignored) until only the desired number s of candidates remain; these are elected.*

This can be seen as STV (Section E.2.1) with the quota ∞, so that no-one reaches the quota. (Hence there is never any surplus to consider.) When only one is elected, this gives the same result as Alternative Vote (see Section E.2.1).

This very minor method was used in local elections in some municipalities in South Australia 1985–1999.

Bottoms-up is mentioned here because it might be viewed as an ordered version of Thiele's elimination method (Section 4.3). However, when $s > 1$, it will in the party list case (see Section 11) *not* give the same result as D'Hondt's method. Actually, in the party list case where everone from the same party votes for the same ordered list, Bottoms-up will behave strangely, since if there are at least s parties, everyone except the first candidate from each party will be eliminated, and thus the seats will go to the s largest parties, with one seat each. This is the same outcome as Thiele's elimination

[70]Eliminations were introduced in STV by Thomas Hare in 1865, who had proposed earlier forms of STV in 1857 and 1859. Thomas Hare was not the first to propose STV, since Andræ's method then already was in use, but he developed and promoted STV and is seen as the founder of STV in its modern forms. For the early history of STV, see further [79].

[71]In this case a misnomer, since Andræ invented the method, with the quota, before Hare's writings.

method with the weak satisfaction function (4.3). On the other hand, if instead the voters for each party vote are uniformly split between all possible orderings of the candidates of their party, then Bottoms-up will give the same result as Thiele's elimination method (which has unordered ballots and thus ignores the order).

E.2.3. *Scoring rules (Borda methods). Each ballot is counted as p_1 votes for the first name, p_2 for the second name, and so on, according to some given non-increasing sequence $(p_k)_{k=1}^{\infty}$ of non-negative numbers.*

Several different sequences $(p_k)_k$ are used. The method was suggested (for the election of a single person) by Jean-Charles de Borda in 1770 [15] with $p_k = n - k + 1$ where n is the number of candidates, i.e., the sequence $n, n-1, \dots, 1$ (his proposal required each voter to rank all candidates); the same method was proposed already in 1433 by Nicolas Cusanus for election of the king (and thus emperor, after being crowned by the pope) of the Holy Roman Empire [50; 37]. This scoring rule is called the *Borda method*; more generally, any scoring rule is sometimes called a Borda method.

Another common choice of scoring rule uses the harmonic series, $p_k = 1/k$. This was proposed in 1857 by Thomas Hare (who later instead developed and proposed STV, Section E.2.1); it was used for parliamentary elections in Finland 1906–1935 (within parties), and is currently used in Nauru. It was one of several methods discussed (and rejected) by the Swedish Royal Commission that instead proposed Phragmén's ordered method [6], see Appendix D. This version gives in the party-list case (see Section 11) the same result as D'Hondt's method, so it may be regarded as a proportional method. (See further [65].) However, tactical voting where the voters of a party vote on the same name in different orders may give quite different results.

An important difference from Phragmén's and Thiele's methods and STV is that with a scoring rule, also the order of the candidates below the first unelected may influence the outcome. In particular, with a scoring rule, adding names after the favourite candidate may decrease the chances of the favourite.

Moreover, scoring rules invite to tactical voting, especially since the ordering of all of the candidates is important.[72]

E.3. **Election methods with party lists.** As said above, among the proportional election methods, the ones that are most often used are party list methods, where the voter votes for a party and the seats are distributed among the parties according to their numbers of votes.[73] The seats a party obtains then are distributed to candidates, either according to a list made in advance by the party (*closed list*), or in some way that more or less is decided by the voters (*open list*), see Appendix D for examples from Sweden.

[72]Borda is said to have commented this with: "My scheme is only intended for honest men". [14, pp. 182, 238].

[73]The most important proportional method of another type is STV, Section E.2.1.

Conversely, most (but not all) party list methods that are used in practice, including the ones below, are proportional, so that each party gets a proportion of the seats that approximates its proportion of votes.

Many different list methods are described in detail in e.g. Pukelsheim [64]; see also Balinski and Young [13] for the mathematically (but not politically) equivalent problem of allocating the seats in the US House of Representatives proportionally among the states.[74]

There are two major types of party list methods: *divisor methods* and *quota methods*.

In one traditional formulation of divisor methods, seats are allocated sequentially. For each seat, the number of votes of each party is reduced by division with a *divisor* that depends on the number of seats already given to the party; the party with the highest quotient gets the seat.[75] [76]

In a quota method, first a *quota* Q is calculated; this is roughly the number of votes required for each seat. The two main quotas that are used are the *Hare quota* V/s and the *Droop quota* $V/(s+1)$ [29], where V is the total number of (valid) votes and s is the number of seats (cf. Remark 16.5); the quota is often rounded to an integer, either up, or down, or to the nearest integer (see e.g. [64] and [42] for examples with different, or no, rounding). The method then gives a party with v_i votes first $\lfloor v_i/Q \rfloor$ seats (i.e., the integer part of v_i/Q); the remaining seats, if any, are given to the parties with largest remainder in these divisions.[77]

E.3.1. *D'Hondt's method. The divisor method with the sequence of divisors* $1, 2, 3, \dots$. Thus, the reduced vote of a party equals the number of votes divided by 1+ the number of seats already obtained. (The method is sometimes called *highest average*, since the reduced vote is the average number

[74]There, until 1941, the method was decided after each census, so the choice of method was heavily influenced by its result. Moreover, the number of seats was not fixed in advance and thus also open to negotiations.

[75]The quotient may be called *reduced vote*; the Swedish term is *jämförelsetal* (comparative figure).

[76]A different, but equivalent, formulation is that the number of seats for each party is obtained by selecting a number D and then giving a party with v_i votes v_i/D seats, rounded to an integer by some general rounding rule (different for different methods), where D is chosen such that the total number of seats given to the parties is s. See Pukelsheim [64] for a details and examples. In fact, this vas the formulation used by D'Hondt [25] when he proposed his method. (He then developed formulations that led to the equivalent sequential formulation, see [26] and [38].) Similar formulations have also been used in the United States for allocating the seats in the House of Representatives to the states [13]. The number D is also called *divisor*; it can be interpreted as roughly the number of votes required for each seat. (Just as the quota in quota methods; the difference is that the number D is not determined in advance by a simple formula, but comes out as a result of the calculations.)

[77]An equivalent definition is that seats are distributed sequentially, as just described for divisor methods, but now the number of vote for a party is reduced by subtracting Q for each seat that the party already has got. (This shows the close connection with STV, Section E.2.1).

of votes per seat for the party, if the party receives the next seat. However, this name is also sometimes used as a term for all divisor methods.)

D'Hondt's method is one of the oldest and most important proportional list methods. It was proposed by Victor D'Hondt in 1882 [25].[78] D'Hondt's method is equivalent to *Jefferson's method*, proposed by Thomas Jefferson in 1792 for allocating the seats in the US House of Representatives proportionally among the states, see [13].[79]

It is easy to see that D'Hondt's method slightly favours larger parties (see [43] for a detailed calculation of the average bias). In particular, the method is *superadditive* in the sense that if two parties merge to one (bringing all their voters with them to the new party, while all other parties keep the same number of votes), then the combined party will get at least as many seats as the two parties had together.[80] Moreover (and as a corollary), a party that gets a majority of the votes will get at least half the seats.

In Sweden, D'Hondt's method was used 1909–1951, see Appendix D.

E.3.2. *Sainte-Laguë's method. The divisor method with the sequence of divisors* $1, 3, 5, \dots$. This makes it easier that D'Hondt's method for small parties to get a seat, and the method is perfectly proportional in an average sense. (See e.g. [43].)

The method was proposed in 1910 by Sainte-Laguë [67]. It is equivalent to *Webster's method*, proposed by Daniel Webster in 1832 (and used at least in 1842 and 1911) for allocating the seats in the US House of Representatives proportionally among the states, see [13].[81] Sainte-Laguë's method is used in several countries, e.g. Germany.

E.3.3. *Modified Sainte-Laguë's method. The divisor method with the sequence of divisors* $x, 3, 5, \dots$*, where* $x > 1$ *is a constant, usually* 1.4. This differs from Sainte-Laguë's method only in that the first divisor is larger than 1. The method was introduced in Sweden in 1952 with the first divisor 1.4, and this has also been used in a few other countries. (From the next general election in Sweden, in 2018, the first divisor will be 1.2 instead [1].)

The modification makes it more difficult than the standard Sainte-Laguë's method for small parties to get the first seat, but it does not affect the distribution of seats between parties that all have seats.

In Sweden, the modified Sainte-Laguë method has been used since 1952, see Appendix D.

[78] D'Hondt [24] had earlier promoted the method of largest remainder (Appendix E.3.4).

[79] Jefferson's method is formulated as in Footnote 76, with rounding downwards. However, for the allocation of seats in the House of Representatives, the total number was not fixed in advance; a suitable divisor D was chosen by congress, which makes the method somewhat different.

[80] The method thus discourages party splits, which was seen as an important positive feature of it in the discussions in Sweden in the early 20th century.

[81] Webster's method is formulated as in Footnote 76, with rounding to the nearest integer. However, Footnote 79 applies here too.

There is no special mathematical reason for the divisor 1.4; it was chosen in Sweden 1952 because it was expected to give a politically desired result, see [74]. In the present Swedish system with adjustment seats, the first divisor does not affect the total distribution of seats, but it affects the distribution between constituencies within the parties, and the divisor was changed to 1.2 because simulations shows that this would give the best result with the present constituencies and party structure.

E.3.4. *Method of largest remainder (Hare's method). The quota method using the Hare quota, see above.*

The method is perfectly proportional in an average sense. (See e.g. [43].)

In Sweden, the method has not been used in elections, but it is since 1894 used for the distribution of seats between constituencies (before the election, based on the population or on the number of eligible voters).

E.3.5. *Droop's method. The quota method using the Droop quota, see above.*

The method slightly favours larger parties (but not as much as D'Hondt's method). (See e.g. [43].)

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