Edge exchangeable random graphs

Svante Janson

Edge exchangeable random (multi)graphs were introduced by Crane and Dempsey [6, 7]. An equivalent model, using somewhat different formulations, was given by Broderick and Cai [4] and Campbell, Cai and Broderick [5].

The idea is that we have a fixed (labelled) vertex set, and add a sequence of edges (regarded as pairs of vertices). Repetions are allowed, so we construct a multigraph. The sequence of edges is supposed to be exchangeable. By De Finetti's theorem, this is equivalent to the following:

Let V be a finite or infinite set, and let μ be a deterministic or random probability measure on the edges of the complete graph on V.

(1) Given μ, take N i.i.d. edges with distribution μ.
(2) Delete all isolated vertices.

There are some similarities with the (in my opinion more important) vertex exchangeable random graphs (see e.g. [2, 3, 9, 1, 11]) with a discrete type space $\mathbb{N}$, but the two models are quite different. For example, the edge exchangeable graphs have at most one vertex of each type.

Example 1. Let (q_i) be a probability distribution on $\mathbb{N}$. For each edge, just pick its two endpoints independently with this distribution.

Thus $\mu(ij) = q_i q_j$.

Cf. similar "rank 1" cases of vertex exchangeable graphs, with $W(x, y) = \phi(x)\phi(y)$.

Example 2. Pittel [10] considered a random multigraph process with a fixed vertex set $[n]$ and N edges added one by one, with an edge ij added with probability proportional to $(d_i + \alpha)(d_j + \alpha)$, where d_i is the current degree of i. (Slightly modified for loops). Here $\alpha > 0$ is a fixed parameter.

Equivalently: choose vertices with probability proportional to $d_i + \alpha$. Then join the first two vertices to an edge, then the next two, and so on.

Thus, the vertices are chosen according to a Pólya urn process, starting with α balls of each colour (= vertex). The sequence of vertices is exchangeable, and thus so is the sequence of edges. Hence, this is an edge exchangeable random multigraph.

Remarks:

(1) Exchangeability implies that conditioned on the final degree of each vertex, all possible edge sequences have the same probability. Hence, conditioned on the degree sequence, the random multigraph is the multigraph given by the configuration model.
(2) A standard result for Pólya urn processes shows that the vector $(d_i/2N)$ converges to a Dirichlet$(\alpha, \dots, \alpha)$ distribution as $N \to \infty$.
(3) The random sequence of vertices in the construction can be seen as a two-parameter Chinese restaurant process with parameters $(-\alpha, n\alpha)$. A

Chinese restaurant process with other parameters yields a similar edge exchangeable random multigraph (on a number of vertices growing to ∞).

Simple graph version. We can merge multiple edges and ignore loops, and thus obtain a random simple graph. This gives an increasing sequence of simple graphs.

Let G_m be the resulting simple graph with m edges.

Example 3. If $P(ij) \sim ((i \vee j)!)^{-4}$, then a.s. $G_m = K_n$ when $m = \binom{n}{2}$, for all large n. Thus $G_n \to$ the graphon 1 a.s. as $n \to \infty$.

Example 4. There exists a distribution μ of edges on $V = \mathbb{N}$ such that a.s. the sequence G_n is dense in the space of graph limits, i.e., for every graph limit (graphon), there exists a subsequence G_{m_i} converging to it.

An example of everything? Or of nothing?

See further [8].

References

[1] Christian Borgs, Jennifer T. Chayes, Henry Cohn & Nina Holden. Sparse exchangeable graphs and their limits via graphon processes. Preprint, 2016. `arXiv:1601.07134v1`

[2] Christian Borgs, Jennifer T. Chayes, László Lovász, Vera T. Sós & Katalin Vesztergombi. Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing, *Advances in Math.* **219** (2008), 1801–1851.

[3] Christian Borgs, Jennifer T. Chayes, László Lovász, Vera T. Sós & Katalin Vesztergombi. Convergent sequences of dense graphs II. Multiway cuts and statistical physics. *Ann. of Math. (2)* **176** (2012), no. 1, 151–219.

[4] Tamara Broderick and Diana Cai. Edge-exchangeable graphs and sparsity. Preprint, 2016. `arXiv:1603.06898v1`

[5] Trevor Campbell, Diana Cai and Tamara Broderick. Exchangeable trait allocations. Preprint, 2016. `arXiv:1609.09147v1`

[6] Harry Crane and Walter Dempsey. Edge exchangeable models for network data. Preprint, 2016. `arXiv:1603.04571v3`

[7] Harry Crane and Walter Dempsey. Relational exchangeability. Preprint, 2016. `arXiv:1607.06762v1`

[8] Svante Janson, On edge exchangeable random graphs. *J. Statistical Physics*, to appear.

[9] László Lovász, *Large Networks and Graph Limits.* American Mathematical Society, Providence, RI, 2012.

[10] Boris Pittel. On a random graph evolving by degrees. *Adv. Math.* **223** (2010), no. 2, 619–671.

[11] Victor Veitch & Daniel M. Roy. The class of random graphs arising from exchangeable random measures. Preprint, 2015. `arXiv:1512.03099`