Graph properties, graph limits and entropy

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But graph limits are a natural continuation of Erdős' work.

In particular, the work by Erdős and Rényi on random graphs is one way to (partially) decribe the structure of very large graphs; the graph limit theory generalizes and extends this.

Graph limits

Quick course: (see e.g. the recent book by Lovász for details)

Some sequences of (unlabelled) graphs G_n (with $|G_n| \to \infty$) are defined to be *convergent*.

A set of limit objects is defined, which together with the set of unlabelled graphs forms a compact metric space.

The limit objects have (non-unique) representations as *graphons*, symmetric functions $[0, 1]^2 \to [0, 1]$.

Graph classes and graph limits

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Today only *hereditary graph classes*. (If $G \in \mathcal{Q}$, then $H \in \mathcal{Q}$ for every induced subgraph H of G.)

Entropy

We define the *entropy* of a graphon W, and of the corresponding graph limit Γ, by

$$\mathrm{Ent}(W) := \int_0^1 \int_0^1 h(W(x,y))\,dx\,dy,$$

where h is the binary entropy function

$$h(x) = -x\log_2(x) - (1-x)\log_2(1-x).$$

Note that $0 \leq \mathrm{Ent}(W) \leq 1$.

This is related to the entropy of random graphs, see e.g. Aldous (1985); it has also previously been used by Chatterjee and Varadhan (2011) and Chatterjee and Diaconis (2011).

Speed

Let $\mathcal{Q}$ be a hereditary graph class, and let
$\mathcal{Q}_n := \{G \in \mathcal{Q} : |G| = n\}$.

The rate of growth of $|\mathcal{Q}_n|$ has been studied by Alekseev (1992), Bollobás and Thomason (1997) and many others. It is known that

$$|\mathcal{Q}_n| = 2^{(1-r^{-1}+o(1))\binom{n}{2}}$$

for some integer $r \in \{1, 2 \ldots, \infty\}$ (the *colouring number*).

Theorem

Let $\mathcal{Q}$ be a hereditary class of graphs. Then

$$\lim_{n\to\infty} \frac{\log_2 |\mathcal{Q}_n|}{\binom{n}{2}} = \max_{\Gamma\in\widehat{\mathcal{Q}}} \mathrm{Ent}(\Gamma).$$

Corollary

$|\mathcal{Q}_n| = 2^{o(n^2)}$ if and only if every graphon W in $\widehat{\mathcal{Q}}$ satisfies $W(x,y) \in \{0,1\}$ a.e. (W is random-free).

Such properties $\mathcal{Q}$ are called *random-free*.

Maximize entropy

For $r = 1, 2, \ldots$, let $I_i := ((i-1)/r, i/r]$ and let R_r be the sets of graphons W such that

$$\begin{aligned} W(x,y) &= \tfrac{1}{2}, \quad (x,y) \in I_i \times I_j \text{ with } i \neq j; \\ W(x,y) &\in \{0,1\}, \qquad (x,y) \in I_i \times I_i \end{aligned}$$

and $R_\infty := \{\frac{1}{2}\}$. (Thus R_1 is the set of random-free graphons.)

For $0 \leq s \leq r < \infty$, let $W^*_{r,s}$ be the graphon in R_r that is 1 on $I_i \times I_i$ for $i \leq s$ and 0 for $i > s$.

Theorem

$\max_{\Gamma \in \widehat{\mathcal{Q}}} \mathrm{Ent}(\Gamma) = 1 - \frac{1}{r}$ and

$$|\mathcal{Q}_n| = 2^{(1-r^{-1}+o(1))\binom{n}{2}}$$

where $r \in \{1, 2 \ldots, \infty\}$ and furthermore

$$r = \sup\left\{t : W^*_{t,u} \in \widehat{\mathcal{Q}} \text{ for some } u \leq t\right\}$$
$$= \min\left\{s \geq 1 : \{(x,y) : W(x,y) \notin \{0,1\}\} \text{ is } K_{s+1}\text{-free for } W \in \widehat{\mathcal{Q}}\right\}$$

Moreover, every graph limit in $\widehat{\mathcal{Q}}$ with maximal entropy $1 - 1/r$ can be represented by a graphon $W \in R_r$.

$r = 1$ if and only if $\mathcal{Q}$ is random-free, and $r = \infty$ if and only if $\mathcal{Q}$ is the class of all graphs.

Random graphs in $\mathcal{Q}$

Theorem

Suppose that $\max_{\Gamma \in \widehat{\mathcal{Q}}} \mathrm{Ent}(\Gamma)$ *is attained by a unique graph limit* $\Gamma_{\mathcal{Q}}$*. Let* G_n *be a uniformly random (unlabelled or labelled) element of* $\mathcal{Q}_n$*.*

Then G_n *converges to* $\Gamma_{\mathcal{Q}}$ *in probability as* $n \to \infty$*.*

Example: Bipartite graphs

Let $\mathcal{Q}$ be the class of *bipartite graphs*.

It is easy to characterize all graph limits in $\widehat{\mathcal{Q}}$.

There is a unique graph limit with maximum entropy, represented by the graphon $W^*_{2,0} \in R_2$.

Thus the colouring number $r = 2$ and $|\mathcal{Q}_n| = 2^{\frac{1}{2}\binom{n}{2}+o(n^2)}$ (which can be easily proved directly).

If G_n is a uniformly random (labelled or unlabelled) bipartite graph, then $G_n \to W^*_{2,0}$ in probability.

$\frac{1}{2}$	0
0	$\frac{1}{2}$

$W^*_{2,0}$

Example: Triangle-free graphs

Let $\mathcal{Q}$ be the class of *triangle-free* graphs.

This class is strictly larger than the class of bipartite graphs. The set $\widehat{\mathcal{Q}}$ of triangle-free graph limits is strictly larger than the set of bipartite graph limits, but the graph limit represented by $W^*_{2,0}$ is still a unique graph limit of maximum entropy.

Thus the colouring number $r = 2$ and $|\mathcal{Q}_n| = 2^{\frac{1}{2}\binom{n}{2}+o(n^2)}$ (as is well-known).

If G_n is a uniformly random triangle-free graph, then $G_n \to W^*_{2,0}$ in probability.

$\frac{1}{2}$	0
0	$\frac{1}{2}$

$W^*_{2,0}$

K_t-free graphs

This extends to K_t-free graphs, for any $t \geq 2$. The colouring number is $t-1$ and the unique graph limit of maximum entropy is represented by $W^*_{t-1,0}$.

Thus, a uniformly random K_t-free graph converges (in probability) to the graphon $W^*_{t-1,0}$.

$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$W^*_{4,0}$ $(t=5)$

Example: String graphs

Let $\mathcal{Q}$ be the class of *string graphs*.

Then $|\mathcal{Q}_n| = 2^{\frac{3}{4}\binom{n}{2}+o(n^2)}$ (Pach and Tóth (2006)). Thus the maximum entropy in $\widehat{\mathcal{Q}}$ is $\frac{3}{4}$ and the colouring number $r = 4$.

One graph limit in $\widehat{\mathcal{Q}}$ with maximum entropy is $W^*_{4,4}$. However, this is not unique.

Hence we do not know the limit of a uniformly random string graph of order n, as $n \to \infty$.

$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$W^*_{4,4}$

Proofs

Proofs use the graphon version of the weak regularity lemma and approximation of graphons by step graphons, together with elementary counting.