Graph properties, graph limits and entropy

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Erdős and graph limits

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But graph limits are a natural continuation of Erdős’ work.

In particular, the work by Erdős and Rényi on random graphs is one way to (partially) describe the structure of very large graphs; the graph limit theory generalizes and extends this.
Quick course: (see e.g. the recent book by Lovász for details)

Some sequences of (unlabelled) graphs $G_n$ (with $|G_n| \to \infty$) are defined to be \textit{convergent}.

A set of limit objects is defined, which together with the set of unlabelled graphs forms a compact metric space.

The limit objects have (non-unique) representations as \textit{graphons}, symmetric functions $[0, 1]^2 \to [0, 1]$. 
Graph classes and graph limits

Let $\mathcal{Q}$ be a graph class, i.e., a set of (unlabelled) graphs.

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General project: Study \( \hat{Q} \), the set of graph limits that are limits of sequences of graphs in \( Q \).

Today only hereditary graph classes. (If \( G \in Q \), then \( H \in Q \) for every induced subgraph \( H \) of \( G \).)
Entropy

We define the *entropy* of a graphon $W$, and of the corresponding graph limit $\Gamma$, by

$$\text{Ent}(W) := \int_0^1 \int_0^1 h(W(x, y)) \, dx \, dy,$$

where $h$ is the binary entropy function

$$h(x) = -x \log_2(x) - (1 - x) \log_2(1 - x).$$

Note that $0 \leq \text{Ent}(W) \leq 1$.

This is related to the entropy of random graphs, see e.g. Aldous (1985); it has also previously been used by Chatterjee and Varadhan (2011) and Chatterjee and Diaconis (2011).
Let $\mathcal{Q}$ be a hereditary graph class, and let $\mathcal{Q}_n := \{ G \in \mathcal{Q} : |G| = n \}$.

The rate of growth of $|\mathcal{Q}_n|$ has been studied by Alekseev (1992), Bollobás and Thomason (1997) and many others. It is known that

$$|\mathcal{Q}_n| = 2^{(1-r^{-1}+o(1))\binom{n}{2}}$$

for some integer $r \in \{1, 2 \ldots, \infty\}$ (the colouring number).
Theorem

Let $Q$ be a hereditary class of graphs. Then

$$\lim_{n \to \infty} \frac{\log_2 |Q_n|}{\binom{n}{2}} = \max_{\Gamma \in \hat{Q}} \text{Ent}(\Gamma).$$
Corollary

\(|Q_n| = 2^{o(n^2)} \) if and only if every graphon \( W \) in \( \widehat{Q} \) satisfies 
\( W(x, y) \in \{0, 1\} \) a.e. (\( W \) is random-free).

Such properties \( Q \) are called \textit{random-free}.
Maximize entropy

For $r = 1, 2, \ldots$, let $I_i := ((i - 1)/r, i/r]$ and let $R_r$ be the sets of graphons $W$ such that

\[
W(x, y) = \frac{1}{2}, \quad (x, y) \in I_i \times I_j \text{ with } i \neq j;
\]
\[
W(x, y) \in \{0, 1\}, \quad (x, y) \in I_i \times I_i
\]

and $R_\infty := \{\frac{1}{2}\}$. (Thus $R_1$ is the set of random-free graphons.)

For $0 \leq s \leq r < \infty$, let $W_{r,s}^*$ be the graphon in $R_r$ that is 1 on $I_i \times I_i$ for $i \leq s$ and 0 for $i > s$. 
Theorem

$$\max_{\Gamma \in \hat{Q}} \text{Ent}(\Gamma) = 1 - \frac{1}{r} \quad \text{and}$$

$$|Q_n| = 2^{(1-r^{-1}+o(1))(n)}$$

where $$r \in \{1, 2 \ldots , \infty \}$$ and furthermore

$$r = \sup \left\{ t : W_{t,u}^* \in \hat{Q} \text{ for some } u \leq t \right\}$$

$$= \min \left\{ s \geq 1 : \{(x,y) : W(x,y) \notin \{0,1\}\} \text{ is } K_{s+1}-\text{free for } W \in \hat{Q} \right\}$$

Moreover, every graph limit in $$\hat{Q}$$ with maximal entropy $$1 - 1/r$$ can be represented by a graphon $$W \in R_r$$.

$$r = 1$$ if and only if $$Q$$ is random-free, and $$r = \infty$$ if and only if $$Q$$ is the class of all graphs.
Random graphs in $\mathcal{Q}$

**Theorem**

*Suppose that $\max_{\Gamma \in \hat{\mathcal{Q}}} \text{Ent}(\Gamma)$ is attained by a unique graph limit $\Gamma_\mathcal{Q}$. Let $G_n$ be a uniformly random (unlabelled or labelled) element of $\mathcal{Q}_n$. Then $G_n$ converges to $\Gamma_\mathcal{Q}$ in probability as $n \to \infty$.***
Example: Bipartite graphs

Let $Q$ be the class of *bipartite graphs*.

It is easy to characterize all graph limits in $\hat{Q}$.

There is a unique graph limit with maximum entropy, represented by the graphon $W_{2,0}^{*} \in R_2$.

Thus the colouring number $r = 2$ and $|Q_n| = 2\frac{1}{2} \binom{n}{2} + o(n^2)$ (which can be easily proved directly).

If $G_n$ is a uniformly random (labelled or unlabelled) bipartite graph, then $G_n \rightarrow W_{2,0}^{*}$ in probability.

\[
\begin{array}{|c|c|}
\hline
\frac{1}{2} & 0 \\
0 & \frac{1}{2} \\
\hline
\end{array}
\]

$W_{2,0}^{*}$
Example: Triangle-free graphs

Let $Q$ be the class of *triangle-free* graphs.

This class is strictly larger than the class of bipartite graphs. The set $\widehat{Q}$ of triangle-free graph limits is strictly larger than the set of bipartite graph limits, but the graph limit represented by $W_{2,0}^*$ is still a unique graph limit of maximum entropy.

Thus the colouring number $r = 2$ and $|Q_n| = 2^{\frac{1}{2}}(\binom{n}{2}) + o(n^2)$ (as is well-known).

If $G_n$ is a uniformly random triangle-free graph, then $G_n \to W_{2,0}^*$ in probability.

\[
\begin{array}{|c|c|}
\hline
\frac{1}{2} & 0 \\
\hline
0 & \frac{1}{2} \\
\hline
\end{array}
\]

$W_{2,0}^*$
This extends to $K_t$-free graphs, for any $t \geq 2$. The colouring number is $t - 1$ and the unique graph limit of maximum entropy is represented by $W_{t-1,0}^*$.

Thus, a uniformly random $K_t$-free graph converges (in probability) to the graphon $W_{t-1,0}^*$.

\[
\begin{array}{cccc}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{array}
\]

$W_{4,0}^* (t = 5)$
Example: String graphs

Let $\mathcal{Q}$ be the class of string graphs.

Then $|\mathcal{Q}_n| = 2^{\frac{3}{4} \binom{n}{2} + o(n^2)}$ (Pach and Tóth (2006)). Thus the maximum entropy in $\hat{\mathcal{Q}}$ is $\frac{3}{4}$ and the colouring number $r = 4$.

One graph limit in $\hat{\mathcal{Q}}$ with maximum entropy is $W_{4,4}^*$. However, this is not unique.

Hence we do not know the limit of a uniformly random string graph of order $n$, as $n \to \infty$. 

\[
\begin{array}{cccc}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \\
\frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\
\frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\
1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{array}
\]

$W_{4,4}^*$
Proofs use the graphon version of the weak regularity lemma and approximation of graphons by step graphons, together with elementary counting.