Random permutations avoiding some patterns

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Patterns in a permutation

Let $\mathfrak{S}_n$ be the set of permutations of $[n] := \{1, \ldots, n\}$.

If $\sigma = \sigma_1 \cdots \sigma_k \in \mathfrak{S}_k$ and $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$, then an *occurrence* of σ in π is a subsequence $\pi_{i_1} \cdots \pi_{i_k}$, with $1 \leq i_1 < \cdots < i_k \leq n$, that has the same relative order as σ. σ is called a *pattern*.

Let $n_\sigma(\pi)$ be the number of occurrences of σ in π.
For example, $n_{21}(\pi)$ is the number of inversions in π.

A permutation π *avoids* a pattern σ if there is no occurence of σ in π, i.e., if $n_\sigma(\pi) = 0$.

Let $\mathfrak{S}_n(\tau) := \{\pi \in \mathfrak{S}_n : n_\tau(\pi) = 0\}$, the set of permutations of length n that avoid τ.

Similarly, let $\mathfrak{S}_n(\tau_1, \ldots, \tau_k) := \bigcap_i \mathfrak{S}_n(\tau_i)$, the set of permutations of length n that avoid $\tau_1, \ldots, \tau_k$.

Example

Donald Knuth, *The Art of Computer Programming, vol. 1*, Exercise 2.2.1-5:

A permutation π can be obtained by a stack if and only if π is 312-avoiding, i.e., $\pi \in \mathfrak{S}_n(312)$.

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Equivalently:

A permutation π is stack-sortable if and only if π is 231-avoiding.

Example

Donald Knuth, Exercise 2.2.1-10,11,
West (1995):

A permutation π is deque-sortable if and only if π is $\{2431, 4231\}$-avoiding, i.e., $\pi \in \mathfrak{S}_n(2431, 4231)$.

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Tarjan (1972):

A permutation π can be sorted by 2 parallel queues if and only if π is 321-avoiding, i.e., $\pi \in \mathfrak{S}_n(321)$.

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Further examples, properties and references:
See Stanley, *Enumerative Combinatorics*,
Exercises 6.19 x (321), y (312), ee (321), ff (312), ii (231), oo (132), xx (321); 6.25 g (321); 6.39 k, l ($\{2413, 3142\}$), m ($\{1342, 1324\}$); 6.47 a ($\{4231, 3412\}$); 6.48 (1342).

One fundamental question, studied by many authors, is the size of these classes $|\mathfrak{S}_n(\tau)|$ and $|\mathfrak{S}_n(\tau_1, \ldots, \tau_k)|$.

Theorem (Knuth [Exercises 2.2.1-4,5], and others)

If $|\tau| = 3$, then

$$|\mathfrak{S}_n(\tau)| = C_n = \frac{1}{n+1}\binom{2n}{n},$$

the nth Catalan number.

The cases with $|\tau| \geq 4$ are much more complicated. See e.g. Bóna (2004).

Some results are also known for $|\mathfrak{S}_n(\tau_1, \dots, \tau_k)|$ with $k \geq 2$.

Example All cases with all $|\tau_i| = 3$ are treated by Simion and Schmidt (1995). For example, several such cases yield 2^{n-1}.

Example $|\mathfrak{S}_n(2431, 4231)| = r_{n-1}$, a Schröder number, see Donald Knuth, Exercise 2.2.1-10,11,
West (1995), Stanley (1999), Exercise 6.39 l,m.

A related problem is to study properties of a random permutation chosen uniformly from a class $\mathfrak{S}_n(\tau_1, \ldots, \tau_k)$.

Several properties of such restricted random permutations have been studied by a number of authors. For example: consecutive patterns, descents, major index, number of fixed points, position of fixed points, exceedances, longest increasing subsequence, shape and distribution of individual values π_i.

I consider today instances of the following general problem:

Fix patterns $\tau_1, \ldots, \tau_k$ and σ. What is the asymptotic distribution, as $n \to \infty$, of $n_\sigma(\boldsymbol{\pi})$ for $\boldsymbol{\pi} \in \mathfrak{S}_n(\tau_1, \ldots, \tau_k)$, chosen uniformly at random?

Example

Take $\sigma = 21$. (Recall that $n_{21}(\pi)$ is the number of inversions in π.) What is the asymptotic distribution of the number of inversions in a random $\boldsymbol{\pi} \in \mathfrak{S}_n(\tau_1, \ldots, \tau_k)$?

I consider today only the cases with $|\tau_i| = 3$, when I can give more or less complete results.

Also these simple cases are treated case by case, by different methods.
No general method is known for these problems, even in the comparatively simple case $|\tau| = 3$.

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PLEASE HELP!

Remark. Some impressive results for $\mathfrak{S}_n(2413, 3142)$ (separable permutations) are recently given by Bassino, Bouvel, Féray, Gerin, Pierrot (2017), with generalizations by Bassino, Bouvel, Féray, Gerin, Maazoun, Pierrot (2017).

There are many cases, even with all $|\tau_i| = 3$, but the number is reduced by obvious symmetries:

inverse: 25341 $\leftrightarrow$ 51342

reflection left-right: 25431 $\leftrightarrow$ 13452

reflection up-down: 25431 $\leftrightarrow$ 41235

(These generate a dihedral group of 8 symmetries.)

Unrestricted permutations

As a background, consider random permutations without restrictions.

Theorem (Bóna (2007, 2010), Janson, Nakamura and Zeilberger (2015))

Consider a random unrestricted permutation $\boldsymbol{\pi} \in \mathfrak{S}_n$. Then $n_\sigma(\boldsymbol{\pi})$ is asymptotically normally distributed, for any σ: if $k := |\sigma|$ then

$$\frac{n_\sigma(\boldsymbol{\pi}) - n^k/k!^2}{n^{k-1/2}} \xrightarrow{\mathrm{d}} N(0, c_\sigma^2)$$

for some constant $c_\sigma > 0$.

Proof.

Represent $n_\sigma(\boldsymbol{\pi})$ as a U-statistic, and use the central limit theorem by Hoeffding (1948). $\square$

231-avoiding permutations (or 132, 213, 312)

Theorem

Suppose $\sigma \in \mathfrak{S}_*(231)$. Let

$$\lambda(\sigma) := |\sigma| + 1 + \#\textit{ascents}.$$

Then, as $n \to \infty$, for a random $\boldsymbol{\pi} \in \mathfrak{S}_n(231)$,

$$n_\sigma(\boldsymbol{\pi})/n^{\lambda_\sigma/2} \xrightarrow{\mathrm{d}} \Lambda_\sigma,$$

for some random variable $\Lambda_\sigma > 0$.

Example The number of inversions.

$$n_{21}(\boldsymbol{\pi})/n^{3/2} \xrightarrow{\mathrm{d}} \Lambda_{21} = \sqrt{2} \int_0^1 e(t)\,\mathrm{d}t,$$

where the random function $e(t)$ is a Brownian excursion.

In general, Λ_σ can be expressed using a Brownian excursion $e(t)$, but in general in a complicated way.

Proof.

- A natural bijection between $\mathfrak{S}_n(231)$ and binary trees of order n.
- the standard bijection between the latter and Dyck paths.
- A random Dyck path converges (after scaling) in distribution to a Brownian excursion.

$\square$

321-avoiding permutations (or 123)

Theorem

Suppose $\sigma \in \mathfrak{S}_*(321)$. Let m be the number of blocks in σ. Then, as $n \to \infty$, for a random $\boldsymbol{\pi} \in \mathfrak{S}_n(321)$,

$$n_\sigma(\boldsymbol{\pi})/n^{(|\sigma|+m)/2} \xrightarrow{\mathrm{d}} W_\sigma,$$

for some random variable $W_\sigma > 0$.

Example The number of inversions.

$$n_{21}(\boldsymbol{\pi})/n^{3/2} \xrightarrow{\mathrm{d}} \Lambda_{21} = 2^{-1/2} \int_0^1 e(t)\,\mathrm{d}t,$$

where the random function $e(t)$ is a Brownian excursion.

In general,

$$W_\sigma = w_\sigma \int_{t_1 < \cdots < t_m} e(t_1)^{\ell_1 - 1} \cdots e(t_m)^{\ell_m - 1} \, \mathrm{d}t_1 \cdots \mathrm{d}t_m$$

where $\ell_1, \ldots, \ell_m$ are the lengths of the blocks in σ, and w_σ is a curious combinatorial constant.

A block in σ is a minimal interval $[i, j]$ such that π maps $[1, i-1]$, $[i, j]$ and $[j+1, n]$ to themselves.

Proof.

- A bijection with Dyck paths by Billey, Jockush and Stanley (1993).
- Further developments by Hoffman, Rizzolo and Slivken (2017).

$\square$

$\{231, 312\}$-avoiding permutations

Theorem

Suppose $\sigma \in \mathfrak{S}_*(231, 312)$. Then, for a random $\boldsymbol{\pi} \in \mathfrak{S}_n(231, 312)$, $n_\sigma(\boldsymbol{\pi})$ is asymptotically normally distributed: if m is the number of blocks in σ, then

$$\frac{n_\sigma(\boldsymbol{\pi}) - a_\sigma n^m}{n^{m-1/2}} \xrightarrow{\mathrm{d}} N(0, b_\sigma^2)$$

for some constants a_σ, b_σ.

Example The number of inversions.

$$\frac{n_{21}(\boldsymbol{\pi}) - \frac{1}{2}n}{n^{1/2}} \xrightarrow{\mathrm{d}} N(0, \tfrac{1}{4}).$$

Proof.

- $\pi \in \mathfrak{S}_*(231, 312) \iff$ each block is decreasing: $\ell(\ell-1)\cdots 21$ [Simion and Schmidt, 1995]. (Hence, $|\mathfrak{S}_n(231, 312)| = 2^{n-1}$.)
- If the block lengths are $\ell_1, \dots, \ell_m$, then $n_{21}(\boldsymbol{\pi}) = \sum_{i=1}^{m} \binom{\ell_i}{2}$. Similar in general, with multiple sum.
- $(\ell_1, \dots, \ell_m)$ is a random composition of n.
- Can be realized as the first elements, up to sum n, of an i.i.d. sequence $L_1, L_2, \dots$ of random variables with a Geometric $\mathrm{Ge}(1/2)$ distribution.
- Renewal theory, in a U-statistics version by Janson (2018+).

$\square$

$\{132, 321\}$-avoiding permutations

Theorem

For a random $\pi \in \mathfrak{S}_n(132, 321)$, the number of inversions has the asymptotic distribution

$$n^{-2} n_{21}(\pi) \xrightarrow{\mathrm{d}} W := XY,$$

where (X, Y) is uniformly distributed in the triangle $\{x, y \geq 0,\ x + y \leq 1\}$. The limit variable W has density function

$$2\log\left(1 + \sqrt{1 - 4x}\right) - 2\log\left(1 - \sqrt{1 - 4x}\right), \qquad 0 < x < 1/4,$$

and moments

$$\mathbb{E}\, W^r = 2\frac{r!^2}{(2r + 2)!}, \qquad r > 0.$$

Proof.

- $\mathfrak{S}_n(132, 321)$ has only $\binom{n}{2} + 1$ elements:
  the identity and $\{\pi_{k,\ell,n-k-\ell} : k, \ell \geq 1,\ k + \ell \leq n\}$,
  where $\pi_{k,\ell,m}$ is the permutation
  $(\ell+1, \dots, \ell+k, 1, \dots, \ell, k+\ell+1, \dots, k+\ell+m) \in \mathfrak{S}_{k+\ell+m}$,
  consisting of three increasing runs of lengths k, ℓ, m (where the third run is empty when $m = 0$).
- $n_{21}(\pi_{k,\ell,n-k-\ell}) = kl$.

$\square$