Flajolet Lecture: The Sum of Powers of Subtrees Sizes for Random Trees

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Therefore I have chosen to speak about a problem where I and my coauthor have combined both generating functions and probabilistic methods, to show different parts of the results. (Convergence in distribution by probabilistic methods; convergence of moments by generating functions.)

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Maybe Flajolet could have proved all our results by a single method, but I cannot.

Conclusion

We mathematicians sometimes engage in friendly competitions between different methods for some problem. This is fine, but we should all try to respect, love and learn ALL methods!

Different methods are useful for different problems, and it is sometimes useful to use and combine different methods. I hope that this will be demonstrated in this lecture.

References

This lecture is mainly based on joint work with Jim Fill.
arXiv:2104.02715

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Some related papers:

Fill, Flajolet & Kapur 2005.

Hwang & Neininger 2002. Fill & Kapur 2004, 2005. Holmgren & Janson 2012. Delmas, Dhersin & Sciauveau 2018. Ralaivaosaona & Wagner 2019. Caracciolo, Erba & Sportiello 2020. Abraham, Delmas & Nassif 2022.

General problem

Additive functional: Let $f(T)$ (the toll function) be a given functional of rooted trees, and define

$$F(T) := \sum_{v \in T} f(T_v),$$

where T_v is the fringe tree rooted at v.

Problem: Study asymptotics of $F(\mathcal{T}_n)$ (mean, variance, distribution, ...) when $\mathcal{T}_n$ is some random tree of "size" n, and $n \to \infty$.

Today, the random tree $\mathcal{T}_n$ will be a conditioned Galton–Watson tree (a.k.a. simply generated tree) with $|\mathcal{T}_n| = n$; the offspring distribution ξ will be critical with finite variance $0 < \sigma^2 < \infty$. (Higher moments usually not needed.)

The toll function will be simply

$$f_\alpha(T) := |T|^\alpha$$

for a constant α.

Examples.

$\alpha = 1$ gives $F_1(T) =$ the total path length.

$\alpha = 2$ gives a functional related to the Wiener index.

We allow α to be complex, and we consider $F_\alpha(T)$ as a function of $\alpha \in \mathbb{C}$. We write

$$X_n(\alpha) := F_\alpha(\mathcal{T}_n) = \sum_{v \in \mathcal{T}_n} |(\mathcal{T}_n)_v|^\alpha$$

$$\widetilde{X}_n(\alpha) := X_n(\alpha) - \mathbb{E}\, X_n(\alpha).$$

Remark

Why complex α?

- Useful in proofs (also for real α) since powerful methods of analytic functions can be used.

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Why complex α?

- Useful in proofs (also for real α) since powerful methods of analytic functions can be used.
- Gives us new problems to study. How do the phase transitions look in the complex plane?

There are two phase transitions for real α: $\alpha = 0$ and $\alpha = \frac{1}{2}$.

Thus three phases in the complex plane:

$$\mathrm{Re}(\alpha) < 0, \qquad 0 < \mathrm{Re}(\alpha) < \tfrac{1}{2}, \qquad \mathrm{Re}(\alpha) > \tfrac{1}{2}.$$

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Thus three phases in the complex plane:

$$\mathrm{Re}(\alpha) < 0, \qquad 0 < \mathrm{Re}(\alpha) < \tfrac{1}{2}, \qquad \mathrm{Re}(\alpha) > \tfrac{1}{2}.$$

What happens at the boundaries $\mathrm{Re}(\alpha) = 0$ and $\mathrm{Re}(\alpha) = \frac{1}{2}$?

$\mathrm{Re}(\alpha) < 0$

Let $H_- := \{\alpha : \mathrm{Re}(\alpha) < 0\}$.

Theorem

- *There exists a random analytic function* $\widetilde{X}(\alpha)$, $\alpha \in H_-$, *such that, as* $n \to \infty$,

$$n^{-1/2}\widetilde{X}_n(\alpha) \xrightarrow{\mathrm{d}} \widetilde{X}(\alpha)$$

for each fixed $\alpha \in H_-$, *and uniformly on each compact subset of* H_-. *(I.e., in the space* $\mathcal{H}(H_-)$ *of analytic functions on* H_-.*)*
- $\widetilde{X}(\alpha)$ *is a complex Gaussian, for every fixed* $\alpha \in H_-$. *Also jointly.*
- *The covariance matrix of* $\widetilde{X}(\alpha)$ *depends on the offspring distribution.*

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- *$\widetilde{X}(\alpha)$ is a complex Gaussian, for every fixed $\alpha \in H_-$. Also jointly.*
- *The covariance matrix of $\widetilde{X}(\alpha)$ depends on the offspring distribution.*

In this case $X_n(\alpha) = F_\alpha(\mathcal{T}_n)$ is dominated by the many small fringe trees. Hence normality, but not universality.

$\mathrm{Re}(\alpha) > 0$

Let $H_+ := \{\alpha : \mathrm{Re}(\alpha) > 0\}$.

Theorem

- *There exists a random analytic function* $\widetilde{Y}(\alpha)$, $\alpha \in H_+$, *such that, as* $n \to \infty$,

  $$\widetilde{Y}_n(\alpha) := n^{-\alpha-\frac{1}{2}} \widetilde{X}_n(\alpha) \xrightarrow{\mathrm{d}} \sigma^{-1} \widetilde{Y}(\alpha)$$

  for each fixed $\alpha \in H_+$, *and uniformly on each compact subset of* H_+. *(I.e., in the space* $\mathcal{H}(H_+)$ *of analytic functions on* H_+.*)*
- $\widetilde{Y}(\alpha)$ *is not Gaussian.*
- $\widetilde{Y}(\alpha)$ *does not depend on the offspring distribution.*

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  for each fixed $\alpha \in H_+$, and uniformly on each compact subset of H_+. (I.e., in the space $\mathcal{H}(H_+)$ of analytic functions on H_+.)
- *$\widetilde{Y}(\alpha)$ is not Gaussian.*
- *$\widetilde{Y}(\alpha)$ does not depend on the offspring distribution.*

In this case $\widetilde{X}_n(\alpha) = F_\alpha(\mathcal{T}_n) - \mathbb{E}\, F_\alpha(\mathcal{T}_n)$ is dominated by the large fringe trees. Therefore universality but not normality.

Let $\mathcal{T}$ be an (unconditioned) Galton–Watson tree with the given offspring distribution. Recall that

$$\mathbb{P}(|\mathcal{T}| = n) \sim cn^{-3/2}.$$

A random fringe tree has asymptotically the distribution of $\mathcal{T}$, i.e., the number of fringe trees of size k in $\mathcal{T}_n$ is $\approx cnk^{-3/2}$. Let

$$\mu(\alpha) := \mathbb{E}\,|\mathcal{T}|^\alpha = \sum_{n=1}^{\infty} n^\alpha\,\mathbb{P}(|\mathcal{T}| = n).$$

This converges for $\mathrm{Re}(\alpha) < \frac{1}{2}$, and defines an analytic function in this half-plane.

Theorem

(i). *If* $\mathrm{Re}(\alpha) < \frac{1}{2}$, *then*

$$\mathbb{E}\, X_n(\alpha) = \mu(\alpha) n + o(n)$$

(ii). *If* $\mathrm{Re}(\alpha) > \frac{1}{2}$, *then*

$$\mathbb{E}\, X_n(\alpha) = \frac{1}{\sqrt{2}\sigma} \frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)} n^{\alpha + \frac{1}{2}} + o(n^{\alpha + \frac{1}{2}})$$

(iii). *If* $\alpha = \frac{1}{2}$, *then*

$$\mathbb{E}\, X_n(1/2) = \frac{1}{\sqrt{2\pi\sigma^2}} n \log n + o(n \log n).$$

Let

$$Y(\alpha) := \widetilde{Y}(\alpha) + \frac{1}{\sqrt{2}\sigma} \frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}.$$

Theorem

(i). *If* $\mathrm{Re}(\alpha) > \frac{1}{2}$, *then*

$$Y_n(\alpha) := n^{-\alpha-\frac{1}{2}} X_n(\alpha) \xrightarrow{\mathrm{d}} \sigma^{-1} Y(\alpha).$$

(ii). *If* $\mathrm{Re}(\alpha) < \frac{1}{2}$, *then*

$$n^{-\alpha-\frac{1}{2}} [X_n(\alpha) - n\mu(\alpha)] \xrightarrow{\mathrm{d}} \sigma^{-1} Y(\alpha).$$

(iii). *If* $\alpha = \frac{1}{2}$, *then*

$$\mathbb{E}\, X_n(1/2) = \frac{1}{\sqrt{2\pi\sigma^2}} n \log n + o(n \log n).$$

Moment convergence

Theorem

All moments converge in the limit theorems above for $\operatorname{Re}\alpha > 0$. *If* $\alpha \neq \frac{1}{2}$, *the limiting moments* $\kappa_\ell := \mathbb{E}\, Y(\alpha)^\ell$ *satisfy the recursion*

$$\kappa_1 = \frac{\Gamma(\alpha - \frac{1}{2})}{\sqrt{2}\,\Gamma(\alpha)},$$

and, for $\ell \geq 2$, *with* $\alpha' := \alpha + \frac{1}{2}$,

$$\kappa_\ell = \frac{\ell\Gamma(\ell\alpha' - 1)}{\sqrt{2}\,\Gamma(\ell\alpha' - \frac{1}{2})}\kappa_{\ell-1} + \frac{1}{4\sqrt{\pi}}\sum_{j=1}^{\ell-1}\binom{\ell}{j}\frac{\Gamma(j\alpha' - \frac{1}{2})\Gamma((\ell-j)\alpha' - \frac{1}{2})}{\Gamma(\ell\alpha' - \frac{1}{2})}\kappa_j\kappa_{\ell-j}.$$

Remark. For $\alpha = \frac{1}{2}$, our proof requires that the offspring distribution ξ satisfies $\mathbb{E}\,\xi^{2+\delta} < \infty$ for some $\delta > 0$.

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The case $\operatorname{Re}\alpha < 0$?

Mean and higher moments

The proofs of these results are based on the methods of Fill, Flajolet and Kapur (2005), see also Flajolet and Sedgewick, Section VI.10. (Singularity analysis with Hadamard products.)

Let $q_n := \mathbb{P}(|\mathcal{T}| = n)$, and define the generating functions

$$y(z) := \mathbb{E}[z^{|\mathcal{T}|}] = \sum_{n=0}^{\infty} q_n z^n,$$

$$M_\ell(z) := \mathbb{E}\big[F(\mathcal{T})^\ell z^{|\mathcal{T}|}\big] = \sum_{n=0}^{\infty} q_n \, \mathbb{E}[F(\mathcal{T}_n)^\ell] z^n,$$

$$B(z) := \sum_{n=0}^{\infty} n^\alpha z^z.$$

Denote Hadamard products by $\odot$.

Then, for every $\ell \geq 1$,

$$M_\ell(z) = \frac{zy'(z)}{y(z)} \sum_{m=0}^{\ell} \frac{1}{m!} \sum{}^{**} \binom{\ell}{\ell_0, \dots, \ell_m} B(z)^{\odot \ell_0} \odot \left[zM_{\ell_1}(z) \cdots M_{\ell_m}(z) \Phi^{(m)}(y(z)) \right],$$

where $\sum^{**}$ is the sum over all $(m+1)$-tuples $(\ell_0, \dots, \ell_m)$ of non-negative integers summing to ℓ such that $1 \leq \ell_1, \dots, \ell_m < \ell$.

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Use singularity analysis and find the leading terms ...

Brownian excursion, $\operatorname{Re}\alpha > 1$

Let $\mathbf{e}$ be a standard Brownian excursion. Recall that this is a random continuous function $[0,1] \to [0,\infty)$. For a function g and $s < t$, define

$$m(g; s, t) := \inf_{u \in [s,t]} g(u).$$

Theorem

If $\operatorname{Re}\alpha > 1$, *we can represent the limit* $Y(\alpha)$ *as*

$$Y(\alpha) = 2\alpha(\alpha - 1) \iint_{0<s<t<1} (t - s)^{\alpha-2} m(\mathbf{e}; s, t)\, \mathrm{d}s\, \mathrm{d}t.$$

Proof. If we replace $\mathbf{e}$ by a suitably scaled version of the contour process of $\mathcal{T}_n$, then a calculation shows that the integral equals $Y_n(\alpha) = n^{-\alpha-\frac{1}{2}} X_n(\alpha) + o(1)$. The contour process converges to $\mathbf{e}$, and the integral is a continuous functional.

Brownian excursion, $\operatorname{Re}\alpha > 1/2$

Theorem

If $\operatorname{Re}\alpha > 1/2$, we can represent the limit $Y(\alpha)$ as

$$Y(\alpha) = 2\alpha \int_0^1 t^{\alpha-1}\mathbf{e}(t)\,\mathrm{d}t - 2\alpha(\alpha-1) \iint\limits_{0<s<t<1} (t-s)^{\alpha-2}[\mathbf{e}(t) - m(\mathbf{e}; s, t)]\,\mathrm{d}s\,\mathrm{d}t.$$

Example. $\alpha = 1$ yields

$$Y(1) = 2\int_0^1 \mathbf{e}(t)\,\mathrm{d}t,$$

the *Brownian excursion area*, with a distribution called by Flajolet *the Airy distribution*.

Tightness

Lemma

(i). *If* $\operatorname{Re}\alpha < 0$, *then* $\mathbb{E}\,|\widetilde{X}_n(\alpha)|^2 \le C(\alpha)n$.

(ii). *If* $\operatorname{Re}\alpha > 0$, *then* $\mathbb{E}\,|\widetilde{X}_n(\alpha)|^2 \le C(\alpha)n^{2\operatorname{Re}\alpha+1}$, *and thus* $\mathbb{E}\,|\widetilde{Y}_n(\alpha)|^2 \le C(\alpha)$.

In both cases $C(\alpha) = O(1 + |\alpha|^{-2})$.

This shows tightness at each fixed α.

Magic of analytic functions

Lemma

Let D be a domain in $\mathbb{C}$ and let $(Y_n(z))$ be a sequence of random analytic functions in $\mathcal{H}(D)$. Suppose that there exists a function $\gamma : D \to (0, \infty)$, bounded on each compact subset of D, such that

$$\mathbb{E}\,|Y_n(z)| \le \gamma(z)$$

for every $z \in D$. Then the sequence (Y_n) is tight in the space $\mathcal{H}(D)$ of analytic functions on D.

Proof. Cauchy's integral formula, together with $\mathbb{E} \int = \int \mathbb{E}$.

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Proof. Cauchy's integral formula, together with $\mathbb{E} \int = \int \mathbb{E}$.

Hence, the random functions $\widetilde{Y}_n(\alpha)$ are tight in $\mathcal{H}(H_+)$.

More magic of analytic functions

Lemma

Let D be a domain in $\mathbb{C}$ and let E be a subset of D that has a limit point in D. (I.e., there exists a sequence $z_n \in E$ of distinct points and $z_\infty \in D$ such that $z_n \to z_\infty$.) Suppose that (Y_n) is a tight sequence of random elements of $\mathcal{H}(D)$ and that there exists a family of random variables $\{Y_z : z \in E\}$ such that for each $z \in E$, $Y_n(z) \xrightarrow{\mathrm{d}} Y_z$ and, moreover, this holds jointly for any finite set of $z \in E$. Then $Y_n \xrightarrow{\mathrm{d}} Y$ in $\mathcal{H}(D)$, for some random function $Y(z) \in \mathcal{H}(D)$.

Proof. Subsequences converge, and limits are determined by the restriction to E, and therefore unique.

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Proof. Subsequences converge, and limits are determined by the restriction to E, and therefore unique.

Hence, the random functions $\widetilde{Y}_n(\alpha)$ converge in distribution in $\mathcal{H}(H_+)$.

Critical line $\mathrm{Re}(\alpha) = \frac{1}{2}$

We have different types of results for $\mathrm{Re}(\alpha) < \frac{1}{2}$ and $\mathrm{Re}(\alpha) > \frac{1}{2}$. What happens for $\mathrm{Re}(\alpha) = \frac{1}{2}$?

Theorem

$\mu(\alpha) \to \infty$ as $\alpha \nearrow \frac{1}{2}$. However, $\mu(\alpha)$ extend to a continuous function on $H' := \{\mathrm{Re}(\alpha) \le \frac{1}{2}\} \setminus \{\frac{1}{2}\}$. Furthermore,

$$\mathbb{E}\, X_n(\alpha) = \mu(\alpha) n + o(n), \qquad \alpha \in H' \tag{1}$$

$$n^{-\alpha-\frac{1}{2}}[X_n(\alpha) - n\mu(\alpha)] \xrightarrow{\mathrm{d}} \sigma^{-1} Y(\alpha), \qquad \alpha \in H'. \tag{2}$$

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(In some sense: Continuity to the left, but not to the right.)

More details for critical line $\mathrm{Re}(\alpha) = \frac{1}{2}$

Theorem

(i). *If the offspring distribution* ξ *has* $\mathbb{E}\,\xi^{2+\delta} < \infty$ *for some* $\delta \in (0,1)$, *then* $\mu(\alpha)$ *extends meromorphically to* $\{\mathrm{Re}(\alpha) < \frac{1}{2} + \frac{\delta}{2}\}$, *with a single pole at* $\frac{1}{2}$.

(ii). *If* ξ *has moments of all orders,* $\mu(\alpha)$ *extends to a meromorphic function in the entire complex plane.*

(iii). *There exists an offspring distribution such that* $\mu(\alpha)$ *cannot be extended beyond* $\mathrm{Re}(\alpha) = \frac{1}{2}$; *i.e.,* $\mathrm{Re}(\alpha) = \frac{1}{2}$ *is a natural boundary.*

Critical point $\alpha = 0$

Obviously, $X_n(0) = n$ is non-random. The derivative $X_n'(0)$ is the shape functional.

Theorem

At least in $\mathbb{E}\,\xi^{2+\delta} < \infty$ for some $\delta > 0$, we have

$$\frac{X_n'(0) - \mu'(0)n}{\sqrt{n \log n}} \xrightarrow{\mathrm{d}} N\big(0, 4(1 - \log 2)\sigma^{-2}\big).$$

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Let instead first $n \to \infty$, then $\alpha \searrow 0$:

Theorem

(i). *$Y(\alpha) \xrightarrow{\mathrm{p}} 0$ as $\alpha \searrow 0$.*

(ii). *In fact, $\alpha^{-1/2} Y(\alpha) \xrightarrow{\mathrm{d}} N(0, 2(1 - \log 2))$ as $\alpha \searrow 0$.*

(iii). *If $\alpha \to 0$ along other lines in H_+, then $\alpha^{-1/2} Y(\alpha)$ converges in distribution to other limits, all complex normal distributions.*

Critical line $\mathrm{Re}(\alpha) = 0$

Conjecture

If $t \neq 0$*, then*

$$\frac{X_n(it) - \mu(it)n}{\sqrt{n \log n}} \xrightarrow{\mathrm{d}} N\big(0, \sigma^2(t)\big)$$

for some $\sigma^2(t) > 0$*.*

Theorem

If $\alpha \to \mathrm{i}t$ *with* $\mathrm{Re}(\alpha) > 0$ *and* $t \neq 0$*, then* $|Y(\alpha)| \xrightarrow{\mathrm{p}} \infty$*, and*

$$\mathrm{Re}(\alpha)^{1/2} Y(\alpha) \xrightarrow{\mathrm{d}} \zeta$$

where ζ *is a symmetric complex normal variable with*

$$\mathbb{E}\,|\zeta|^2 = \frac{1}{2\sqrt{\pi}} \,\mathrm{Re}\, \frac{\Gamma(it - \frac{1}{2})}{\Gamma(it - 1)}.$$

Brownian excursion, $\operatorname{Re}\alpha \le 1/2$

For $\operatorname{Re}\alpha > 1/2$, we have seen above explicit representations of $\widetilde{Y}(\alpha)$ using a Brownian excursion $\mathbf{e}(t)$.

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We know that almost surely, this extends to an analytic function in the halfplane $\operatorname{Re}\alpha > 0$.

Thus there exists a measurable function $\Psi : H_+ \times C[0,1] \to \mathbb{C}$ such that

$$Y(\alpha) = \Psi(\alpha, \mathbf{e}), \qquad \operatorname{Re}\alpha > 0.$$

Is there an explicit formula giving $Y(\alpha)$ in terms of $\mathbf{e}(t)$ also for $0 < \operatorname{Re}\alpha < \frac{1}{2}$?

Some other random trees (partial results known)

Binary Search Tree.

Expected number of fringe trees of size k is $\approx cn/k^2$. Phase transitions at $\alpha = 1/2$ and $\alpha = 1$. Normal for $\mathrm{Re}(\alpha) < 1/2$. (Hwang & Neininger 2002; Fill, Flajolet & Kapur 2005; Holmgren & Janson 2012)

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Similar, but more remains to be done. (Holmgren & Janson, 2015)

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d-ary increasing tree

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Phase transitions not known(?).

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Lots of open questions in all cases!!!

THE END