1. ---

Consider three events A_+, A_-, A_0 in the same probability space Ω with a probability P on it. Suppose that A_+ and A_- are conditionally independent given A_0. Show that $P(A_+|A_0 \cap A_-) = P(A_+|A_0)$.

2. ---

Suppose that the sequence $X_0, X_1, X_2, \ldots$ of random variables with values in $\mathbb{Z}$ have the Markov property.
(i) Show that the sequence $X_0^3, X_1^3, X_2^3, \ldots$ also has the Markov property.
(ii) Given an example where the sequence $X_0^2, X_1^2, X_2^2, \ldots$ does not have the Markov property.
Hint: If $x^3 = y$ then $x = y^{1/3}$. For example, $x^3 = -8$ means that $x = -2$. But if $x^2 = y$ then x equals $\sqrt{|y|}$ or $-\sqrt{|y|}$. For example, $x^2 = 100$ means that $x = 10$ or $x = -10$.

3. ---

In the Dark Ages, Harvard, Dartmouth, and Yale admitted only male students. Assume that, at that time, 80 percent of the sons of Harvard men went to Harvard and the rest went to Yale, 40 percent of the sons of Yale men went to Yale, and the rest split evenly between Harvard and Dartmouth; and of the sons of Dartmouth men, 70 percent went to Dartmouth, 20 percent to Harvard, and 10 percent to Yale.
(i) Find the probability that the grandson of a man from Harvard went to Harvard.
(ii) Modify the above by assuming that the son of a Harvard man always went to Harvard. Again, find the probability that the grandson of a man from Harvard went to Harvard.
(iii) Find the stationary distribution(s) of the Markov chain.

4. ---

Consider an experiment of mating rabbits. We watch the evolution of a particular gene that appears in two types, G or g. A rabbit has a pair of genes, either GG (dominant), Gg (hybrid–the order is irrelevant, so gG is the same as Gg) or gg (recessive). In mating two rabbits, the offspring inherits a gene from each of its parents with equal probability. Thus, if we mate a dominant (GG) with a hybrid (Gg), the offspring is dominant with probability 1/2 or hybrid with probability 1/2.
Start with a rabbit of given character (GG, Gg, or gg) and mate it with a hybrid. The offspring produced is again mated with a hybrid, and the process is repeated through a number of generations, always mating with a hybrid.
(i) Write down the transition probabilities of the Markov chain thus defined.
(ii) Assume that we start with a hybrid rabbit. Let μ_n be the probability distribution of the character of the rabbit of the n-th generation. In other words, $\mu_n(GG), \mu_n(Gg), \mu_n(gg)$ are the probabilities that the n-th generation rabbit is

GG, Gg, or gg, respectively. Compute μ_1, μ_2, μ_3. Can you do the same for μ_n for general n?
(iii) Find the stationary distribution(s) of the Markov chain.

5.

A certain calculating machine uses only the digits 0 and 1. It is supposed to transmit one of these digits through several stages. However, at every stage, there is a probability p that the digit that enters this stage will be changed when it leaves and a probability $q = 1 - p$ that it won't. Form a Markov chain to represent the process of transmission by taking as states the digits 0 and 1.
(i) What is the matrix of transition probabilities?
(ii) For $n = 2, 3$ find the probability that the machine, after n stages of transmission produces the digit 0 (i.e., the correct digit).
(iii) Find the stationary distribution(s) of the Markov chain.

6.

Mr Baxter's bank account (see Example in lecture notes) evolves, from month to month, according to the rule

$$X_{n+1} = \max(X_n + D_n - S_n, 0), \quad n = 0, 1, 2, \ldots$$

where $X_0, D_0, S_0, D_1, S_1, D_2, S_2, \ldots$ are independent random variables. Suppose that the S_n have the common distribution

$$P(S_n = 0) = p, \quad P(S_n = 1) = q = 1 - p.$$

Suppose that the D_n have the common distribution

$$P(D_n = k) = \alpha^k(1 - \alpha), \quad k = 0, 1, 2, \ldots,$$

where $0 < p, \alpha < 1$.
(i) Compute the one-step transition probabilities
(ii) (Attempt to) draw the transition diagram.
(iii) What is the form of the transition probability matrix?
(iv) Write down the equation that you need to solve in order to compute the stationary distribution (if it exists!) of the Markov chain.

7.

Definition: A Markov chain is called an **actuarial chain** if:
(a) It has a finite number of states (typically less than 10).
(b) It is an absorbing chain.
(c) It has time-varying one-step transition probabilities. [1]

Consider an urn with n balls, out of which m are red and $n - m$ blue. (See exercise of earlier homework.) Assign the value 1 to a red and 0 to a blue ball. Start picking the balls, one by one (without replacement), and let S_t be the sum

[1]Strictly speaking, an actuarial chain is a continuous-time chain, but we shall waive this requirement for the purposes of this part of the course.

of the values of the balls you have picked up to the t-th step. Start with $S_0 = 0$ and observe that, obviously, $S_n = m$. Define $S_t = n$ for all $t > n$.
(i) Show that $(S_t, t = 0, 1, \ldots)$ is a Markov chain.
(ii) Compute $P(S_{t+1} = j | S_t = i)$ for all values of states i, j and 'times' t. Observe that the result depends on t.
(iii) Conclude that the chain is an actuarial one.

8.

Consider the Ehrenfest chain with N molecules.
(i) Compute its stationary distribution π.
(ii) Suppose that $N = 10^{20}$ molecules. Show that

$$\pi(N/2) \approx 10^{-10} = 0.0000000001$$

but

$$\pi(N/2.0001) \approx 10^{-60,000,000,000}$$

which is the number 0.0.......01 (i.e. $60,000,000,000$ zeros after the decimal point).
Hint: You may use Stirling's approximation, i.e. $n! \approx n^n e^{-n}\sqrt{2\pi n}$ *for large* n.
(iii) If it takes about 1 millimetre to write the symbol 0 show that you need about 5 Earths to write the last number explicitly.
Hint: The diameter of the Earth is about 12 thousand km.
(iv) If you wanted a faithful plot of $\pi(i)$ as a function of i, what shape would the plot have? Sketch it.