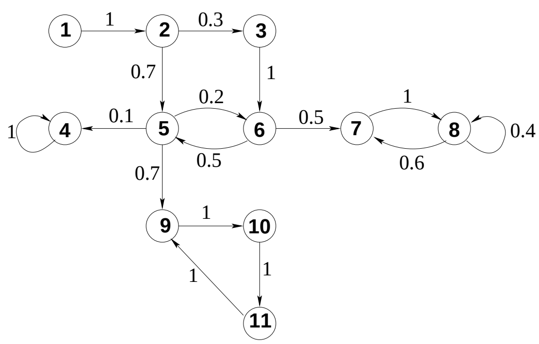
HOMEWORK 3

1.

Consider a Markov chain whose transition diagram is as below:



(i) Which (if any) states are inessential?
(ii) Which (if any) states are absorbing?
(iii) Find the communicating classes.
(iv) Is the chain irreducible?
(v) Find the period of each essential state. Verify that essential states that belong to the same communicating class have the same period.
(vi) Are there any aperiodic communicating classes?
(vii) Will your answers to the questions (i)–(vi) change if we replace the positive transition probabilities by other positive probabilities and why?

2.

Discuss the topological properties of the graphs of the Markov chains defined by the following transition probability matrices:

(a) $\mathsf{P} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$ (b) $\mathsf{P} = \begin{pmatrix} 0.5 & 0.5 \\ 1 & 0 \end{pmatrix}$ (c) $\mathsf{P} = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ 0 & 1/5 & 4/5 \end{pmatrix}$

(d) $\mathsf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (e) $\mathsf{P} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$

3.

I have 4 umbrellas, some at home, some in the office. I keep moving between home and office. I take an umbrella with me only if it rains. If it does not rain I leave the umbrella behind (at home or in the office). It may happen that all umbrellas are in one place, I am at the other, it starts raining and must leave, so I get wet.

1. If the probability of rain is p, what is the probability that I get wet?
2. Current estimates show that $p = 0.6$ in Edinburgh. How many umbrellas should I have so that, if I follow the strategy above, the probability I get wet is less than 0.1?

4.

Find the n-step transition probability matrix when the (one-step) transition probability matrix P equals

(i) $\mathsf{P} = \begin{pmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{9}{10} & \frac{1}{10} \end{pmatrix}$, (ii) $\mathsf{P} = \begin{pmatrix} \frac{1}{10} & \frac{9}{10} \\ \frac{9}{10} & \frac{1}{10} \end{pmatrix}$.

Hint: A basic Linear Algebra theorem (Cayley-Hamilton theorem) states that if $g(x)$ *is the characteristic polynomial of* P *then* $g(\mathsf{P}) = 0$.

5.

Consider a Markov chain with states $S = \{0, \ldots, N\}$ and transition probabilities $p_{i,i+1} = p$, $p_{i,i-1} = q$, for $1 < i < N-1$, where $p + q = 1$, $0 < p < 1$; assume $p_{0,1} = 1$, $p_{N,N-1} = 1$.
1. Draw the graph (= transition diagram).
2. Is the chain irreducible? How many communicating classes are there?
3. What are the periods of the essential states?
4. Find the stationary distribution.

6.

A fair die is tossed repeatedly. Which of the following are Markov chains? For those that are, find the (one-step) transition probabilities and the limit of their n-step transition probabilities, as $n \to \infty$:
(i) The largest number shown up to time n.
(ii) The time until the next six after time n.
(iii) The number of sixes shown by time n.