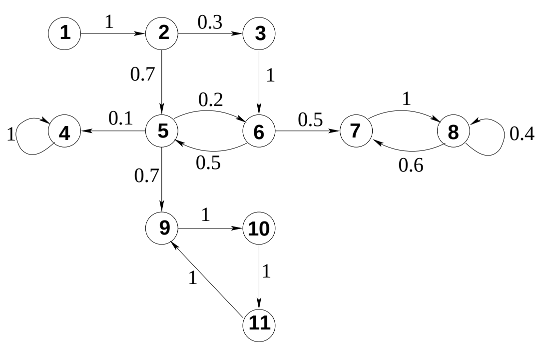
1.

Consider a Markov chain whose transition diagram is as below:



(i) Which (if any) states are inessential?
(ii) Which (if any) states are absorbing?
(iii) Find the communicating classes.
(iv) Is the chain irreducible?
(v) Find the period of each essential state. Verify that essential states that belong to the same communicating class have the same period.
(vi) Are there any aperiodic communicating classes?
(vii) Will your answers to the questions (i)–(vi) change if we replace the positive transition probabilities by other positive probabilities and why?

Solution. (i) The inessential states are: 1, 2, 3, 5, 6, because each of them leads to a state from which it is not possible to return.

(ii) 4 is the only absorbing state.

(iii) As usual, let $[i]$ denote the class of state i i.e. $[i] = \{j \in S : \; j \leftrightsquigarrow i\}$. We have:
$[1] = \{1\}$.
$[2] = \{2\}$.
$[3] = \{3\}$.
$[4] = \{4\}$.
$[5] = \{5, 6\}$.
$[6] = \{5, 6\}$.
$[7] = \{7, 8\}$.
$[8] = \{7, 8\}$.
$[9] = \{9, 10, 11\}$
$[10] = \{9, 10, 11\}$
$[11] = \{9, 10, 11\}$
Therefore there are 7 communication classes:

$$\{1\}, \quad \{2\}, \quad \{3\}, \quad \{4\}, \quad \{5, 6\}, \quad \{7, 8\}, \quad \{9, 10, 11\}$$

(iv) No because there are many communication classes.

(v) Recall that for each essential state i, its period $d(i)$ is the gcd of all n such that $p_{i,i}^{(n)} > 0$. So:

$$\begin{aligned} d(4) &= \gcd\{1, 2, 3, \ldots\} = 1 \\ d(7) &= \gcd\{2, 3, \ldots\} = 1 \\ d(8) &= \gcd\{1, 2, 3, \ldots\} = 1 \\ d(9) &= \gcd\{3, 6, 9, \ldots\} = 3 \\ d(10) &= \gcd\{3, 6, 9, \ldots\} = 3 \\ d(11) &= \gcd\{3, 6, 9, \ldots\} = 3 \end{aligned}$$

Observe $d(7) = d(8) = 1$, and $d(10) = d(11) = d(9) = 3$.

(vi) Yes: $\{4\}$ and $\{7, 8\}$ are aperiodic communication classes (each has period 1).

(vii) No the answers will not change. These questions depend only on whether, for each i, j, $p_{i,j}$ is positive or zero.

2.

Discuss the topological properties of the graphs of the Markov chains defined by the following transition probability matrices:

(a) $\mathsf{P} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$ (b) $\mathsf{P} = \begin{pmatrix} 0.5 & 0.5 \\ 1 & 0 \end{pmatrix}$ (c) $\mathsf{P} = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ 0 & 1/5 & 4/5 \end{pmatrix}$

(d) $\mathsf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (e) $\mathsf{P} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$

Solution. Draw the transition diagram for each case.

(a) Irreducible? YES because there is a path from every state to any other state. Aperiodic? YES because the times n for which $p_{1,1}^{(n)} > 0$ are $1, 2, 3, 4, 5, \ldots$ and their gcd is 1.

(b) Irreducible? YES because there is a path from every state to any other state. Aperiodic? YES because the times n for which $p_{1,1}^{(n)} > 0$ are $1, 2, 3, 4, 5, \ldots$ and their gcd is 1.

(c) Irreducible? NO because starting from state 2 it remains at 2 forever. However, it can be checked that all states have period 1, simply because $p_{i,i} > 0$ for all $i = 1, 2, 3$.

(d) Irreducible? YES because there is a path from every state to any other state. Aperiodic? NO because the times n for which $p_{1,1}^{(n)} > 0$ are $2, 4, 6, \ldots$ and their gcd is 2.

(e) Irreducible? YES because there is a path from every state to any other state. Aperiodic? YES because the times n for which $p_{1,1}^{(n)} > 0$ are $1, 2, 3, 4, 5, \ldots$ and their gcd is 1.

3.

I have 4 umbrellas, some at home, some in the office. I keep moving between home and office. I take an umbrella with me only if it rains. If it does not rain I leave the umbrella behind (at home or in the office). It may happen that all umbrellas are in one place, I am at the other, it starts raining and must leave, so I get wet.

1. If the probability of rain is p, what is the probability that I get wet?
2. Current estimates show that $p = 0.6$ in Edinburgh. How many umbrellas should I have so that, if I follow the strategy above, the probability I get wet is less than 0.1?

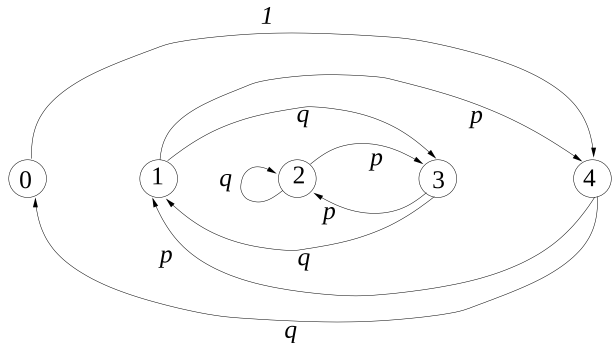
Solution. To solve the problem, consider a Markov chain taking values in the set $S = \{i : i = 0, 1, 2, 3, 4\}$, where i represents the number of umbrellas in the place where I am currently at (home or office). If $i = 1$ and it rains then I take the umbrella, move to the other place, where there are already 3 umbrellas, and, including the one I bring, I have next 4 umbrellas. Thus,

$$p_{1,4} = p,$$

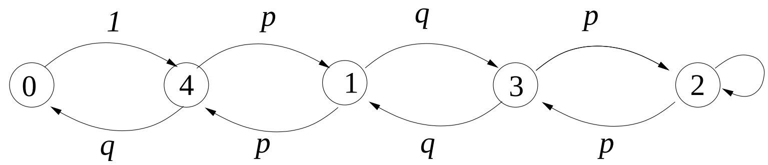
because p is the probability of rain. If $i = 1$ but does not rain then I do not take the umbrella, I go to the other place and find 3 umbrellas. Thus,

$$p_{1,3} = 1 - p \equiv q.$$

Continuing in the same manner I form a Markov chain with the following diagram:



But this does not look very nice. So let's redraw it:



Let us find the stationary distribution. By equating fluxes, we have:

$$\pi(2) = \pi(3) = \pi(1) = \pi(4)$$
$$\pi(0) = \pi(4)q.$$

Also,

$$\sum_{i=0}^{4} \pi(i) = 1.$$

Expressing all probabilities in terms of $\pi(4)$ and inserting in this last equation, we find

$$\pi(4)q + 4\pi(4) = 1,$$

or

$$\pi(4) = \frac{1}{q+4} = \pi(1) = \pi(2) = \pi(3), \quad \pi(0) = \frac{q}{q+4}.$$

I get wet every time I happen to be in state 0 and it rains. The chance I am in state 0 is $\pi(0)$. The chance it rains is p. Hence

$$P(WET) = \pi(0) \cdot p = \frac{qp}{q+4}.$$

With $p = 0.6$, i.e. $q = 0.4$, we have

$$P(WET) \approx 0.0545,$$

less than 6%. That's nice.

If I want the chance to be less than 1% then, clearly, I need more umbrellas. So, suppose I need N umbrellas. Set up the Markov chain as above. It is clear that

$$\pi(N) = \pi(N-1) = \cdots = \pi(1),$$
$$\pi(0) = \pi(N)q.$$

Inserting in $\sum_{i=0}^{N} \pi(i)$ we find

$$\pi(N) = \frac{1}{q+N} = \pi(N-1) = \cdots = \pi(1), \quad \pi(0) = \frac{q}{q+N},$$

and so

$$P(WET) = \frac{pq}{q+N}.$$

We want $P(WET) = 1/100$, or $q + N > 100pq$, or

$$N > 100pq - q = 100 \times 0.4 \times 0.6 - 0.4 = 23.6.$$

So to reduce the chance of getting wet from 6% to less than 1% I need 24 umbrellas instead of 4. That's too much. I'd rather get wet.

4.

Find the n-step transition probability matrix when the (one-step) transition probability matrix P equals

(i) $\mathsf{P} = \begin{pmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{9}{10} & \frac{1}{10} \end{pmatrix}$, (ii) $\mathsf{P} = \begin{pmatrix} \frac{1}{10} & \frac{9}{10} \\ \frac{9}{10} & \frac{1}{10} \end{pmatrix}$.

Hint: A basic Linear Algebra theorem (Cayley-Hamilton theorem) states that if $g(x)$ is the characteristic polynomial of P then $g(\mathsf{P}) = 0$.

Solution.

(i) Let $X_0, X_1, X_2, \dots$ be the Markov chain with transition probabilities those of the matrix P. Since the rows of the matrix are identical, it follows that $X_1, X_2, \dots$ are i.i.d. and so

$$\mathsf{P}^n = \mathsf{P}.$$

(ii) We first find the characteristic polynomial of P:

$$g(x) := \det(xI - \mathsf{P}) = \det \begin{pmatrix} x - \frac{1}{10} & -\frac{9}{10} \\ -\frac{9}{10} & x - \frac{1}{10} \end{pmatrix} = (x - \tfrac{1}{10})^2 - (-9/10)^2$$

We solve the equation

$$g(x) = 0$$

to find the eigenvalues. This gives

$$x = \tfrac{1}{10} \pm \tfrac{9}{10}.$$

So the eigenvalues are

$$\lambda_1 = 1, \quad \lambda_2 = -8/10.$$

Since $g(\mathsf{P}) = 0$, and since g has degree 2, we have that P^2 is expressed as a linear combination of P and the identity I. Therefore P^3 is expressed as a linear combination of P^2 and P and so P^3 is expressed as a linear combination of P and the identity I. Continuing, we obtain that P^n is expressed as a linear combination of P and the identity I. Let's say

$$\mathsf{P}^n = c_1 \mathsf{P} + c_2 I.$$

We need to identify c_1 and c_2. But the same equation must be satisfied by the eigenvalues. (This follows by changing the basis and passing on to the basis conmprising of the eigenvectors.) Hence

$$\lambda_1^n = c_1 \lambda_1 + c_2$$
$$\lambda_2^n = c_1 \lambda_2 + c_2.$$

This gives two linear equations with two unknowns, c_1, c_2. If we subtract the equations, we have

$$\lambda_1^n - \lambda_2^n = c_1(\lambda_1 - \lambda_2),$$

whence

$$c_1 = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} = \frac{1-(-0.8)^n}{1+0.8}.$$

If we multiply the first equation by λ_2 and the second by λ_1, we have

$$\lambda_2\lambda_1^n = c_1\lambda_1\lambda_2 + c_2\lambda_2$$
$$\lambda_1\lambda_2^n = c_1\lambda_1\lambda_2 + c_2\lambda_1.$$

Subtracting,

$$\lambda_1\lambda_2^n - \lambda_2\lambda_1^n = c_2(\lambda_1 - \lambda_2),$$

whence

$$c_2 = \frac{\lambda_1\lambda_2^n - \lambda_2\lambda_1^n}{\lambda_1 - \lambda_2} = \frac{(-0.8)^n + 0.8}{1+0.8}.$$

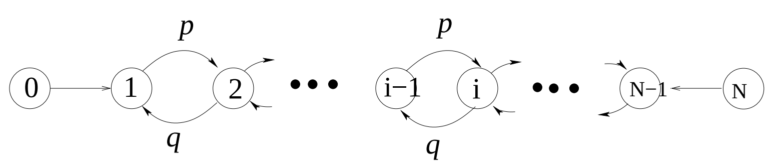
So

$$\mathsf{P}^n = c_1\mathsf{P} + c_2 I = \frac{1}{2}\begin{pmatrix} 1+(-0.8)^n & 1-(-0.8)^n \\ 1-(-0.8)^n & 1+(-0.8)^n \end{pmatrix}$$

5.

Consider a Markov chain with states $S = \{0, \ldots, N\}$ and transition probabilities $p_{i,i+1} = p$, $p_{i,i-1} = q$, for $1 \le i \le N-1$, where $p+q=1$, $0<p<1$; assume $p_{0,1} = 1$, $p_{N,N-1} = 1$.
1. Draw the graph (= transition diagram).
2. Is the chain irreducible? How many communicating classes are there?
3. What are the periods of the essential states?
4. Find the stationary distribution.

Solution. 1. The transition diagram is:



2. No, because states 0 and 1 are inessential. There are three communicating classes:

$$\{0\}, \quad \{N\}, \quad \{1, \ldots, N-1\}.$$

4. We have

$$\{n \in \mathbb{N} : \ p_{1,1}^{(n)} > 0\} = \{2, 4, 6, \ldots\}$$

and the g.c.d. of this set of integers is 2. So the period of state 1 is 2. Since all states $i = 2, \ldots, N-1$ communicate with state 1, it follows that they all have the same period.

4. Cut the state space into $\{0, 1, \ldots, i-1\}$ and $\{i, \ldots, N\}$ for some $1 \le i \le N-1$ and equate the flows:

$$\pi(i)q = \pi(i-1)p,$$

as long as $1 \le i \le N-1$. Hence

$$\pi(i) = \frac{p}{q}\pi(i-1) = \left(\frac{p}{q}\right)^2 \pi(i-2) = \cdots = \left(\frac{p}{q}\right)^{i-1} \pi(1), \quad 1 \le i \le N-1.$$

Cut into $\{0\}$ and $\{1, \ldots, N\}$ and equate the flows:

$$\pi(0) \cdot 1 = \pi(1) \cdot 0.$$

Cut into $\{N\}$ and $\{0, \ldots, N-1\}$ and equate the flows again:

$$\pi(N) \cdot 1 = \pi(N-1) \cdot 0.$$

We thus find

$$\pi(0) = \pi(N) = 0.$$

But

$$\begin{aligned} 1 &= \pi(0) + \pi(1) + \ldots + \pi(N-1) + \pi(N) \\ &= \pi(1) \left[1 + \frac{p}{q} + \left(\frac{p}{q}\right)^2 + \cdots + \left(\frac{p}{q}\right)^{N-2} \right] \\ &= \pi(1) \frac{(p/q)^{N-1} - 1}{(p/q) - 1}, \end{aligned}$$

as long as $p \neq q$. Hence, if $p \neq q$,

$$\pi(i) = \frac{(p/q) - 1}{(p/q)^{N-1} - 1} \left(\frac{p}{q}\right)^{i-1}, \quad 1 \leq i \leq N-1.$$

If $p = q = 1/2$, then

$$1 = \pi(1) \left[1 + \frac{p}{q} + \left(\frac{p}{q}\right)^2 + \cdots + \left(\frac{p}{q}\right)^{N-2} \right] = \pi(1)(N-1)$$

and so

$$\pi(i) = \frac{1}{N-1}, \text{ for all } 1 \leq i \leq N-1.$$

6.

A fair die is tossed repeatedly. Which of the following are Markov chains? For those that are, find the (one-step) transition probabilities and the limit of their n-step transition probabilities, as $n \to \infty$:
(i) The largest number shown up to time n.
(ii) The time until the next six after time n.
(iii) The number of sixes shown by time n.

Solution In each case observe that the process Z_n under question satisfies a recursion of the form

$$Z_{n+1} = f(Z_n, X_{n+1})$$

where X_i is the outcome at the i-th toss and f is a certain deterministic function.

In such cases, the process (Z_n) has the Markov property because, given Z_n, the conditional probability $P(Z_{n+1} = w | Z_n = z, Z_{n-1}, \ldots)$ equals $P(f(z, X_{n+1}) = w | Z_n = z, Z_{n-1}, \ldots)$ and since X_{n+1} is independent of $X_n, X_{n-1}, \ldots$, it follows that X_{n+1} is independent of $Z_n, Z_{n-1}, \ldots$. Therefore, $P(f(z, X_{n+1}) = w | Z_n = z, Z_{n-1}, \ldots) = P(f(z, X_{n+1}) = w)$.

For example, in (i), if we let $Z_n = \max(X_1, \ldots, X_n)$ we have

$$Z_{n+1} = \max(Z_n, X_{n+1}),$$

and so (Z_n) does have the Markov property. The transition probabilities are

$$P(Z_{n+1} = w | Z_n = z) = \begin{cases} 1/6, & \text{if } z < w \leq 6, \\ z/6, & \text{if } 1 \leq w \leq z. \end{cases}.$$

You should work out the remaining cases by yourselves, as we did in the tutorials.