HOMEWORK 4

1.

A Markov chain with state space $\{1, 2, 3\}$ has transition probability matrix

$$\mathsf{P} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

Show that state 3 is absorbing and, starting from state 1, find the expected time until absorption occurs.

2.

Smith is in jail and has 3 dollars; he can get out on bail if he has 8 dollars. A guard agrees to make a series of bets with him. If Smith bets A dollars, he wins A dollars with probability 0.4 and loses A dollars with probability 0.6. Find the probability that he wins 8 dollars before losing all of his money if
(a) he bets 1 dollar each time (timid strategy).
(b) he bets, each time, as much as possible but not more than necessary to bring his fortune up to 8 dollars (bold strategy).
(c) Which strategy gives Smith the better chance of getting out of jail?

3.

A fair coin is tossed repeatedly and independently. Find the expected number of tosses till the pattern HTH appears.

4.

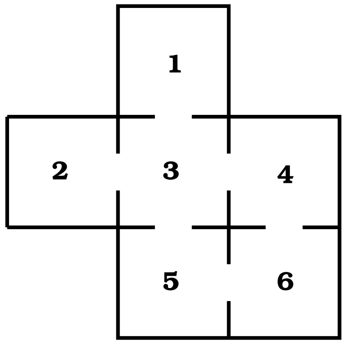
A. Assume that an experiment has m equally probable outcomes. Show that the expected number of independent trials before the first occurrence of k consecutive occurrences of one of these outcomes is

$$\frac{m^k - 1}{m - 1}.$$

Hint: Form an absorbing Markov chain with states $1, 2, \ldots, k$ *with state* i *representing the length of the current run. The expected time until a run of* k *is 1 more than the expected time until absorption for the chain started in state 1.*
B. How many tosses of a fair die are required, on the average, until 10 consecutive sixes are obtained?
C. It has been found that, in the decimal expansion of $\pi = 3.14159\ldots$, starting with the 24,658,601st digit, there is a run of nine 7's. What would your result say about the expected number of digits necessary to find such a run if the digits are produced randomly?

5.

A rat runs through the maze shown below. At each step it leaves the room it is in by choosing at random one of the doors out of the room.



(a) Find the transition matrix for this Markov chain.
(b) Show that it is irreducible and find its period.
(c) Find the stationary distribution.
(d) Now suppose that a piece of mature cheddar is placed on a deadly trap in Room 5. The rat starts in Room 1. Find the expected number of steps before reaching Room 5 for the first time, starting in Room 1.
(e) Supppose again that the rat starts in Room 1 and moves until it dies in Room 5. Every time the rat is in room i it gnaws an amount of wood equal to i^2. Find the expected amount of wood gnawed by the rat before its death. (Assume when it enters room 5 it dies immediately before chewing any wood at all.)
(f) Supppose again that the rat starts in Room 1 but there is no trap in Room 5. Explain why it will, for sure, return to room 1. Find the average time until it returns to room 1.

6.

Two players, A and B, play the game of matching pennies: at each time n, each player has a penny and must secretly turn the penny to heads or tails. The players then reveal their choices simultaneously. If the pennies match (both heads or both tails), Player A wins the penny. If the pennies do not match (one heads and one tails), Player B wins the penny. Suppose the players have between them a total of 5 pennies. If at any time one player has all of the pennies, to keep the game going, he gives one back to the other player and the game will continue.
(a) Show that this game can be formulated as a Markov chain.
(b) Is the chain irreducible? Is it aperiodic?
(c) If Player A starts with 3 pennies and Player B with 2, what is the probability that A will lose his pennies first?