1. A process moves on the integers 1, 2, 3, 4, and 5. It starts at 1 and, on each successive step, moves to an integer greater than its present position, moving with equal probability to each of the remaining larger integers. State five is an absorbing state. Find the expected number of steps to reach state five.

2. Generalise the previous exercise, by replacing 5 by a general positive integer n. Find the expected number of steps to reach state n, when starting from state 1. Test your conjecture for several different values of n. Can you conjecture an estimate for the expected number of steps to reach state n, for large n?

3. Show that a Markov chain with transition matrix

$$\mathsf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}$$

has more than one stationary distributions. Find the matrix that P^n converges to, as $n \to \infty$, and verify that it is not a matrix all of whose rows are the same.
You should work out this exercise by direct methods, without appealing to the general limiting theory of Markov chains–see lecture notes.

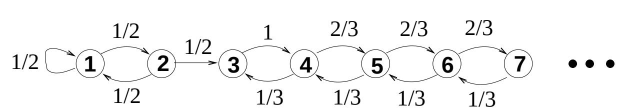
4. Toss a fair die repeatedly. Let S_n denote the total of the outcomes through the nth toss. Show that there is a limiting value for the proportion of the first n values of S_n that are divisible by 7, and compute the value for this limit.
Hint: The desired limit is a stationary distribution for an appropriate Markov chain with 7 *states.*

5. (i) Consider a Markov chain on the vertices of a triangle: the chain moves from one vertex to another with probability 1/2 at each step. Find the probability that, in n steps, the chain returns to the vertex it started from.
(ii) Suppose that we alter the transition probabilities as follows:

$$p_{12} = p_{23} = p_{31} = 2/3, \quad p_{21} = p_{32} = p_{13} = 1/3.$$

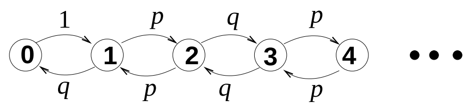
Answer the same question as above.

6. Consider a Markov chain, with state space S the set of all positive integers, whose transition diagram is as follows:



(i) Which states are essential and which inessential?
(ii) Which states are transient and which recurrent?
(iii) Discuss the asymptotic behaviour of the chain, i.e. find the limit, as $n \to \infty$, of $P_i(X_n = j)$ for each i and j.

7. Consider the following Markov chain, which is motivated by the "umbrellas problem" (see–but it's not necessary–an earlier exercise). Here, $p + q = 1$, $0 < p < 1$.



(i) Is the chain irreducible?
(ii) Does it have a stationary distribution?
Hint: Write the balance equations, together with the normalisation condition and draw your conclusions.
(iii) Find the period $d(i)$ of each state i.
(iv) Decide which states are transient and which recurrent.
Hint: Let τ_j be the first hitting time of state j. Let $N \geq 1$ As in the gambler's ruin problem, let $\varphi(i) := P_i(\tau_N < \tau_0)$. What is $\varphi(0)$? What is $\varphi(N)$? For $1 < i < N$, how does $\varphi(i)$ relate to $\varphi(i-1)$ and $\varphi(i+1)$? Solve the equations you thus obtain to find $\varphi(i)$. Let $N \to \infty$. What do you conclude?