

HOMEWORK 5: SOLUTIONS

1.

A process moves on the integers 1, 2, 3, 4, and 5. It starts at 1 and, on each successive step, moves to an integer greater than its present position, moving with equal probability to each of the remaining larger integers. State five is an absorbing state. Find the expected number of steps to reach state five.

Solution. A Markov chain is defined and its transition probability matrix is as follows:

$$P = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We apply first step analysis for the function

$$\psi(i) := E_i S_5, \quad 1 \leq i \leq 5,$$

where $S_5 = \inf\{n \geq 0 : X_n = 5\}$. One of the equations is $\psi(5) = 0$ (obviously). Another is

$$\psi(1) = 1 + \frac{1}{4}\psi(2) + \frac{1}{4}\psi(3) + \frac{1}{4}\psi(3) + \frac{1}{4}\psi(5).$$

It's up to you to write the remaining equations and solve to find

$$\psi(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \approx 2.0833.$$

2.

Generalise the previous exercise, by replacing 5 by a general positive integer n . Find the expected number of steps to reach state n , when starting from state 1. Test your conjecture for several different values of n . Can you conjecture an estimate for the expected number of steps to reach state n , for large n ?

Solution. The answer here is

$$E_1 S_n = \sum_{k=1}^{n-1} \frac{1}{k}.$$

We here recognise the harmonic series:

$$\sum_{k=1}^n \frac{1}{k} \approx \log n,$$

for large n , in the sense that the difference of the two sides converges to a constant. So,

$$E_1 S_n \approx \log n,$$

when n is large.

3.

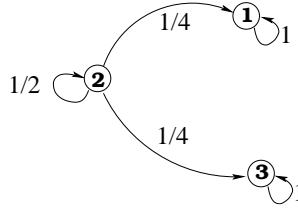
Show that a Markov chain with transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}$$

has more than one stationary distributions. Find the matrix that P^n converges to, as $n \rightarrow \infty$, and verify that it is not a matrix all of whose rows are the same.

You should work out this exercise by direct methods, without appealing to the general limiting theory of Markov chains—see lecture notes.

Solution. The transition diagram is:



Write the balance equations $\pi P = \pi$:

$$(\pi(1) \quad \pi(2) \quad \pi(3)) \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix} = (\pi(1) \quad \pi(2) \quad \pi(3))$$

or

$$\pi(1) \cdot 1 + \pi(2) \cdot (1/4) + \pi(3) \cdot 0 = \pi(1) \tag{1}$$

$$\pi(1) \cdot 0 + \pi(2) \cdot (1/2) + \pi(3) \cdot 0 = \pi(2) \tag{2}$$

$$\pi(1) \cdot 0 + \pi(2) \cdot (1/4) + \pi(3) \cdot 1 = \pi(3), \tag{3}$$

together with the normalisation condition $\sum \pi(i) = 1$, i.e.

$$\pi(1) + \pi(2) + \pi(3) = 1. \tag{4}$$

and solve for $\pi(1), \pi(2), \pi(3)$. Equation (1) gives

$$\pi(2) = 0.$$

Equation (2) gives

$$\pi(2) = \pi(2),$$

i.e. it is useless. Equation (3) gives

$$\pi(3) = \pi(3),$$

again, obviously true. Equation (4) gives

$$\pi(1) + \pi(3) = 1.$$

Therefore, equations (1)–(4) are EQUIVALENT TO:

$$\pi(2) = 0, \quad \pi(1) + \pi(3) = 1.$$

Hence we can set $\pi(1)$ to ANY value we like between 0 and 1, say, $\pi(1) \equiv p$, and then let $\pi(3) = 1 - p$. Thus there is not just one stationary distribution but infinitely many. For each value of $p \in [0, 1]$, any $\boldsymbol{\pi}$ of the form

$$\boldsymbol{\pi} = (p \quad 0 \quad 1 - p)$$

is a stationary distribution.

To find the limit of \mathbf{P}^n as $n \rightarrow \infty$, we compute the entries of the matrix \mathbf{P}^n . Notice that the (i, j) -entry of \mathbf{P}^n equals

$$p_{i,j}^{(n)} = P_i(X_n = j).$$

If $i = 1$ we have

$$P_1(X_n = 1) = 1, \quad P_1(X_n = 2) = 0, \quad P_1(X_n = 3) = 0,$$

because state 1 is absorbing. Similarly, state 3 is absorbing:

$$P_3(X_n = 1) = 0, \quad P_3(X_n = 2) = 0, \quad P_3(X_n = 3) = 1.$$

We thus know the first and third rows of \mathbf{P}^n :

$$\mathbf{P}^n = \begin{pmatrix} 1 & 0 & 0 \\ p_{2,1}^{(n)} & p_{2,2}^{(n)} & p_{2,3}^{(n)} \\ 0 & 0 & 1 \end{pmatrix}.$$

We now compute the missing entries of the second row by simple observations, based on the fact that the chain, started in state 2, will remain at 2 for some time and then will leave it and either go to 1 or 3:

$$P_2(X_n = 2) = P_2(\text{chain has stayed in state 2 for } n \text{ consecutive steps}) = (1/2)^n.$$

$$\begin{aligned} P_2(X_n = 1) &= \sum_{m=1}^n P_2(X_{m-1} = 2, X_m = 1) \\ &= \sum_{m=1}^n (1/2)^{m-1} \cdot (1/4) \\ &= \frac{1 - (1/2)^n}{1 - (1/2)} \cdot \frac{1}{4} = \frac{1 - 0.5^n}{2}. \end{aligned}$$

$$P_2(X_n = 3) = 1 - P_2(X_n = 2) - P_2(X_n = 1) = \frac{1 - 0.5^n}{2}.$$

Therefore,

$$\mathbf{P}^n = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1-0.5^n}{2} & (0.5)^n & \frac{1-0.5^n}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $0.5^n \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\mathbf{P}^n \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{as } n \rightarrow \infty.$$

4.

Toss a fair die repeatedly. Let S_n denote the total of the outcomes through the n th toss. Show that there is a limiting value for the proportion of the first n values of S_n that are divisible by 7, and compute the value for this limit.

Hint: The desired limit is a stationary distribution for an appropriate Markov chain with 7 states.

Solution. An integer $k \geq 1$ is divisible by 7 if it leaves remainder 0 when divided by 7. When we divide an integer $k \geq 1$ by 7, the possible remainders are

$$0, 1, 2, 3, 4, 5, 6.$$

Let X_1, X_2, \dots be the outcomes of a fair die tossing. These are i.i.d. random variables uniformly distributed in $\{1, 2, 3, 4, 5, 6\}$. We are asked to consider the sum

$$S_n = X_1 + \dots + X_n.$$

Clearly, S_n is an integer. We are interested in the remainder of S_n when divided by 7. Call this R_n . So:

$$R_n := \text{the remainder of the division of } S_n \text{ by } 7.$$

Note that the random variables R_1, R_2, R_3, \dots form a Markov chain because if we know the value of R_n , all we have to do to find the next value R_{n+1} is to add X_n to R_n , divide by 7, and take the remainder of this division, as in elementary-school arithmetic:

$$R_{n+1} = \text{the remainder of the division of } R_n + X_n \text{ by } 7.$$

We need to find the transition probabilities

$$\begin{aligned} p_{i,j} &:= P(R_{n+1} = j | R_n = i) \\ &= P(\text{the remainder of the division of } i + X_n \text{ by } 7 \text{ equals } j) \end{aligned}$$

for this Markov chain, for all $i, j \in \{0, 1, 2, 3, 4, 5, 6\}$. But X_n takes values in $\{1, 2, 3, 4, 5, 6\}$ with equal probabilities $1/6$. If to an i we add an x chosen from $\{1, 2, 3, 4, 5, 6\}$ and then divide by 7 we are going to obtain *any* j in $\{0, 1, 2, 3, 4, 5, 6\}$. Therefore,

$$p_{i,j} = 1/6, \quad \text{for all } i \text{ and all } j \in \{0, 1, 2, 3, 4, 5, 6\}.$$

We are asked to consider the proportion of the first n values of S_n that are divisible by 7, namely the quantity

$$\frac{1}{n} \sum_{k=1}^n \mathbf{1}(R_k = 0).$$

This quantity has a limit from the Strong Law of Large Numbers for Markov chains and the limit is the stationary distribution at state 0:

$$P \left(\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbf{1}(R_k = 0) = \pi(0) \right) = 1$$

Therefore we need to compute π for the Markov chain (R_n) . This is very easy. From symmetry, all states i must have the same $\pi(i)$. Therefore

$$\pi(i) = 1/7, \quad i = 0, 1, 2, 3, 4, 5, 6.$$

Hence

$$P \left(\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbf{1}(R_k = 0) = 1/7 \right) = 1.$$

In other words, *if you toss a fair die 10,000 times then approximately 1667 times n you had a sum S_n that was divisible by 7, and this is true with probability very close to 1.*

5.

(i) Consider a Markov chain on the vertices of a triangle: the chain moves from one vertex to another with probability $1/2$ at each step. Find the probability that, in n steps, the chain returns to the vertex it started from.

(ii) Suppose that we alter the transition probabilities as follows:

$$p_{12} = p_{23} = p_{31} = 2/3, \quad p_{21} = p_{32} = p_{13} = 1/3.$$

Answer the same question as above.

Solution. (i) The transition matrix is

$$P = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial is

$$\det(xI - P) = x^3 - \frac{3}{4}x - \frac{1}{4}$$

whose roots are

$$x_1 := 1, \quad x_2 := -1/2 \text{ (double)} .$$

Therefore,

$$p_{11}^{(n)} = C_1 x_1^n + C_2 x_2^n = C_1 + C_2 (-1/2)^n,$$

where C_1, C_2 are constants. Since, clearly, $p_{11}^{(0)} = 1$, $p_{11}^{(1)} = p_{11} = 0$, we have

$$C_1 + C_2 = 1$$

$$C_1 - (1/2)C_2 = 0.$$

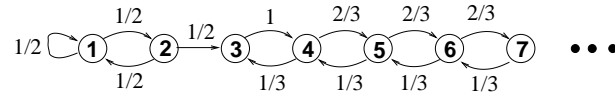
Solving, we find $C_1 = 1/3$, $C_2 = 2/3$. So

$$p_{11}^{(n)} = \frac{1}{3} + \frac{2}{3}(-1/2)^n.$$

(ii) TRY THIS BY YOURSELVES.

6.

Consider a Markov chain, with state space S the set of all positive integers, whose transition diagram is as follows:



- (i) Which states are essential and which inessential?
- (ii) Which states are transient and which recurrent?
- (iii) Discuss the asymptotic behaviour of the chain, i.e. find the limit, as $n \rightarrow \infty$, of $P_i(X_n = j)$ for each i and j .

Solution. (i) The states $3, 4, 5, \dots$ communicate with one another. So they are all essential. However state 1 leads to 3 but 3 does not lead to 1. Hence 1 is inessential. Likewise, 2 is inessential.

(ii) Every inessential state is transient. Hence both 1 and 2 are transient. On the other hand, the Markov chain will eventually take values only in the set $\{3, 4, 5, \dots\}$. We observe that the chain on this set is the same type of chain we discussed in gambler's ruin problem with $p = 2/3$, $q = 1/3$. Since $p > q$ the chain is transient. Therefore all states of the given chain are transient.

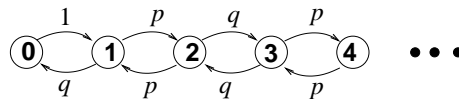
(iii) Since the states are transient, we have that $X_n \rightarrow \infty$ as $n \rightarrow \infty$, with probability 1. Therefore,

$$P_i(X_n = j) \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

for all i and j .

7.

Consider the following Markov chain, which is motivated by the “umbrellas problem” (see—but it’s not necessary—an earlier exercise). Here, $p + q = 1$, $0 < p < 1$.



- (i) Is the chain irreducible?
- (ii) Does it have a stationary distribution?

Hint: Write the balance equations, together with the normalisation condition and draw your conclusions.

(iii) Find the period $d(i)$ of each state i .

(iv) Decide which states are transient and which recurrent.

Hint: Let τ_j be the first hitting time of state j . Let $N \geq 1$. As in the gambler's ruin problem, let $\varphi(i) := P_i(\tau_N < \tau_0)$. What is $\varphi(0)$? What is $\varphi(N)$? For $1 < i < N$, how does $\varphi(i)$ relate to $\varphi(i-1)$ and $\varphi(i+1)$? Solve the equations you thus obtain to find $\varphi(i)$. Let $N \rightarrow \infty$. What do you conclude?

Solution. (i) Yes because all states communicate with one another. (There is just one communication class).

(ii) Let us write balance equations in the form of equating flows (see handout). We have

$$\begin{aligned}\pi(0) &= \pi(1)q \\ \pi(1)p &= \pi(1)p \\ \pi(2)q &= \pi(2)q \\ &\dots\end{aligned}$$

Let $\pi(1) = c$. Then $\pi(0) = cq$ and

$$\pi(1) = \pi(2) = \pi(3) = \dots = c.$$

The normalisation condition is $\sum_{i=0}^{\infty} \pi(i) = 1$. This implies that $c = 0$. Hence $\pi(i) = 0$ for all i . This is NOT a probability distribution. Hence there is no stationary distribution.

(iii) We only have to find the period of one state, since all states communicate with one another. Pick state 0. We have $d(0) = \gcd\{2, 4, 6, \dots\} = 2$. Hence $d(i) = 2$ for all i .

(iv) Let $\varphi(i) := P_i(\tau_N < \tau_0)$. We have

$$\varphi(0) = 0, \quad \varphi(N) = 1.$$

Indeed, if $X_0 = 0$ then $\tau_0 = 0$ and so $\varphi(0) = P_0(\tau_N < 0) = 0$. On the other hand, if $X_0 = N$ then $\tau_N = 0$ and $\tau_0 \geq 1$, so $\varphi(N) = P_N(0 < \tau_0) = 1$.

Now, from first-step analysis, for each $i \in [1, N-1]$, we have

$$\varphi(i) = p_{i,i+1}\varphi(i+1) + p_{i,i-1}\varphi(i).$$

But $p_{i,i+1} = p_{i,i-1} = p$ if i is odd and $p_{i,i+1} = p_{i,i-1} = q$ if i is even and positive. So

$$\begin{aligned}p[\varphi(i+1) - \varphi(i)] &= q[\varphi(i) - \varphi(i-1)], & i \text{ odd} \\ q[\varphi(i+1) - \varphi(i)] &= p[\varphi(i) - \varphi(i-1)], & i \text{ even.}\end{aligned}$$

Hence

$$\begin{aligned}\varphi(2) - \varphi(1) &= \frac{q}{p}[\varphi(1) - \varphi(0)] = \frac{q}{p}\varphi(1) \\ \varphi(3) - \varphi(2) &= \frac{p}{q}[\varphi(2) - \varphi(1)] = \varphi(1) \\ \varphi(4) - \varphi(3) &= \frac{q}{p}[\varphi(3) - \varphi(2)] = \frac{q}{p}\varphi(1) \\ \varphi(5) - \varphi(4) &= \frac{p}{q}[\varphi(4) - \varphi(3)] = \varphi(1),\end{aligned}$$

and, in general,

$$\begin{aligned}\varphi(i) - \varphi(i-1) &= \frac{q}{p}\varphi(1) & i \text{ even} \\ \varphi(i) - \varphi(i-1) &= \varphi(1) & i \text{ odd.}\end{aligned}$$

Next, use the “fundamental theorem of (discrete) calculus”:

$$\varphi(i) = [\varphi(i) - \varphi(i-1)] + [\varphi(i-1) - \varphi(i-2)] + \cdots + [\varphi(2) - \varphi(1)] + \varphi(1).$$

If i is even then, amongst $1, 2, \dots, i$ there are $i/2$ even numbers and $i/2$ odd numbers.

$$\varphi(i) = \left(\frac{q}{p}\right)^{i/2} \varphi(1) + \frac{i}{2} \varphi(1) \quad i \text{ even}$$

Suppose N is even. Use $\varphi(N) = 1$ to get that, if both i and N are even,

$$\varphi(i) = \frac{\left(\frac{q}{p}\right)^{i/2} + \frac{i}{2}}{\left(\frac{q}{p}\right)^{N/2} + \frac{N}{2}} = P_i(\tau_N < \tau_0).$$

Taking the limit as $N \rightarrow \infty$, we find

$$P_i(\tau_0 = \infty) = 0, \quad i \text{ even.}$$

This implies that $P_i(\tau_0 < \infty) = 1$. The same conclusion holds for i odd. (After all, all states communicate with one another.) Therefore all states are recurrent.