1.

The President of the United States tells person A his or her intention to run or not to run in the next election. Then A relays the news to B, who in turn relays the message to C, and so forth, always to some new person. We assume that there is a probability a that a person will change the answer from yes to no when transmitting it to the next person and a probability b that he or she will change it from no to yes. We choose as states the message, either yes or no. The transition probabilities are

$$p_{yes,no} = a, \quad p_{no,yes} = b.$$

The initial state represents the President's choice. Suppose $a = 0.5$, $b = 0.75$.
(a) Assume that the President says that he or she will run. Find the expected length of time before the first time the answer is passed on incorrectly.
(b) Find the mean recurrence time for each state. In other words, find the expected amount of time r_i, for $i = yes$ and $i = no$ required to return to that state.
(c) Write down the transition probability matrix P and find $\lim_{n\to\infty} \mathsf{P}^n$.
(d) Repeat (b) for general a and b.
(e) Repeat (c) for general a and b.

2.

Consider a Galton-Watson branching process and let X_n denote the size of the n-th generation. Assume $X_0 = 1$. Let ξ be a random variable with values in $\{0, 1, \dots\}$ representing the number of children of an individual. Suppose that $P(\xi = k) = c(2/3)^k$, $k = 1, 2, \dots$.
(i) Determine a condition on c so that the extinction probability ε is strictly less than 1.
(ii) Find a formula for EX_n as a function of c and n.
(iii) Assuming that c is such that $\varepsilon = 1$, let N be the total number of offspring of all generations, show that EN satisfies

$$EN = 1 + (E\xi)(EN),$$

and thus determine EN. Plot EN against c.

3.

Let $X_1, X_2, \dots, X_n$ be i.i.d. random variables with zero mean, variance equal to 1 and fourth moment equal to c, i.e. $EX_1^4 = c$. Let $S_n = X_1 + X_2 + \cdots + X_n$. Compute ES_n^4.

4.

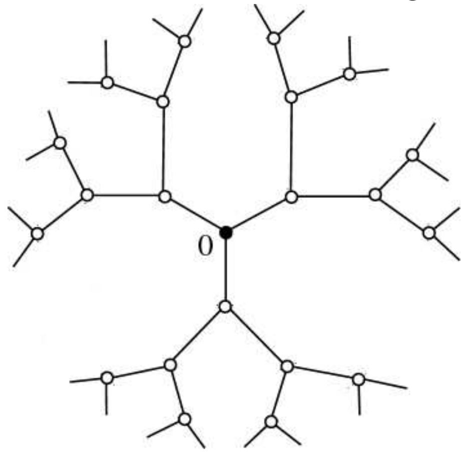
Consider the Ehrenfest chain $(X_n, n \geq 0)$ with N molecules and compute the constant c so that

$$P(\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = c) = 1.$$

Next, compute the average number of steps required for the process to return to state 0 (i.e. the quantity $E_0 T_0$). If $N = 10^{20}$ and if each step has duration 1 microsecond (i.e. one millionth of a second), then how many years will this average return time be?

5.

Consider a random walk on the following infinite graph:



The graph continues ad infinitum in the same manner.

Here, each state has exactly 3 neighbouring states (i.e. its degree is 3) and so the probability of moving to one of them is 1/3.

(i) Let 0 be the "central" state. (Actually, a closer look shows that no state deserves to be central, for they are all equivalent. So we just arbitrarily pick one and call it central.) Having done that, let $D(i)$ be the distance of a state i from 0, i.e. the number of "hops" required to reach 0 starting from i. So $D(0) = 0$, each neighbour i of 0 has $D(i) = 1$, etc. Let X_n be the position of the chain at time n. Observe that the process $Z_n = D(X_n)$ has the Markov property. (See lecture notes for criterion!) The question is:

Find its transition probabilities.

(ii) Using the results from the gambler's ruin problem, show that (Z_n) is transient.

(iii) Use (ii) to explain why (X_n) is also transient.

6.

For a simple symmetric random walk (S_n), starting from $S_0 = 0$, find the following probabilities (n is a positive integer and x is an arbitrary integer):

(i) $P(S_{2n} > 0)$

(ii) $P(S_n = 0 | S_{2n} = 0)$

(iii) $P(S_i > 0 \text{ for } i = 1, 2, \ldots, n | S_n = x)$.