1. ---

The President of the United States tells person A his or her intention to run or not to run in the next election. Then A relays the news to B, who in turn relays the message to C, and so forth, always to some new person. We assume that there is a probability a that a person will change the answer from yes to no when transmitting it to the next person and a probability b that he or she will change it from no to yes. We choose as states the message, either yes or no. The transition probabilities are

$$p_{yes,no} = a, \quad p_{no,yes} = b.$$

The initial state represents the President's choice. Suppose $a = 0.5$, $b = 0.75$.
(a) Assume that the President says that he or she will run. Find the expected length of time before the first time the answer is passed on incorrectly.
(b) Find the mean recurrence time for each state. In other words, find the expected amount of time r_i, for $i = yes$ and $i = no$ required to return to that state.
(c) Write down the transition probability matrix P and find $\lim_{n\to\infty} \mathsf{P}^n$.
(d) Repeat (b) for general a and b.
(e) Repeat (c) for general a and b.

Solution. (a) The expected length of time before the first answer is passed on incorrectly, i.e. that the President will **not** run in the next election, equals the mean of a eometrically distributed random variable with parameter $1 - p_{yes,no} = 1 - a = 0.5$. Thus, the expected length of time before the first answer is passed on incorrectly is 2.

(b) The stationary distribution is unique:

$$\pi = \big(0.6, 0.4\big)\,.$$

The chain is positive recurrent. We know that the mean recurrence time E_iT_i (=average number of steps required for the chan to return to a state) of state i equals $1/\pi(i)$. Hence the mean recurrence time for the state *yes* is $\frac{5}{3}$ and for the state *no* is $\frac{5}{2}$.

(c) The transition probability matrix is

$$\mathsf{P} = \begin{pmatrix} 0.5 & 0.5 \\ 0.75 & 0.25 \end{pmatrix}$$

The corresponding chain is irreducible. It has finitely many states, hence it is positive recurrent. It is also aperiodic. For such a chain

$$\lim_{n\to+\infty} \mathsf{P}^n = \begin{pmatrix} \pi(1) & \pi(2) \\ \pi(1) & \pi(2) \end{pmatrix} = \begin{pmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{pmatrix}.$$

(d) We apply the same arguments as in (b) and find that the transition probability matrix

$$\mathsf{P} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

has the following stationary probability:

$$\pi = \left(\tfrac{b}{a+b}, \tfrac{a}{a+b}\right),$$

so that the mean recurrence time for the state *yes* is $1+\frac{a}{b}$ and for the state *no* is $1+\frac{b}{a}$.
(d) Suppose $a \neq 0$ and $b \neq 0$ to avoid absorbing states and achieve aperiodicity. Thus,

$$\lim_{n\to+\infty} \mathsf{P}^n = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix}.$$

2. ---

Consider a Galton-Watson branching process and let X_n denote the size of the n-th generation. Assume $X_0 = 1$. Let ξ be a random variable with values in $\{0, 1, \dots\}$ representing the number of children of an individual. Suppose that $P(\xi = k) = c(2/3)^k$, $k = 1, 2, \dots$.
(i) Determine a condition on c so that the extinction probability ε is strictly less than 1.
(ii) Find a formula for EX_n as a function of c and n.
(iii) Assuming that c is such that $\varepsilon = 1$, let N be the total number of offspring of all generations, show that EN satisfies

$$EN = 1 + (E\xi)(EN),$$

and thus determine EN. Plot EN against c.

Solution.

(i) Let $p = 2/3$. The condition that $\varepsilon < 1$ is $E\xi > 1$. But

$$E\xi = \sum_{k=1}^{\infty} P(\xi \geq k) = \sum_{k=1}^{\infty} \frac{cp^k}{1-p} = \frac{cp}{(1-p)^2} = 6c.$$

Hence $E\xi > 1$ if and only if $c > 1/6$.

(ii) We have $EX_{n+1} = (E\xi)EX_n$. Hence $EX_n = (E\xi)^n = (6c)^n$.

(iii) If $\varepsilon = 1$ the process will become extinct in finite time and so the total number N of offspring of all generations will be finite as well. We obviously have

$$N = 1 + \sum_{j=1}^{X_1} N'_j,$$

where N'_j is the total number offspring of all descendants of the j-th individual of the first generation. Using the Markov property at time 1, we have

$$E(N'_j | X_1 = k) = kEN.$$

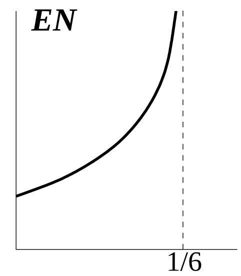
Hence

$$\begin{aligned} EN &= 1 + \sum_{k=1}^{\infty} c(2/3)^k \sum_{j=1}^{\infty} E(N'_j | X_1 = k) \\ &= 1 + \sum_{k=1}^{\infty} c(2/3)^k \; kEN \\ &= 1 + (EX)(EN). \end{aligned}$$

Thus,

$$EN = \frac{1}{1-EX} = \frac{1}{1-6c}.$$

The plot is a hyperbola with asymptote at $c = 1/6$:



3.

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with zero mean, variance equal to 1 and fourth moment equal to c, i.e. $EX_1^4 = c$. Let $S_n = X_1 + X_2 + \cdots + X_n$. Compute ES_n^4.

Solution. We have

$$ES_n^4 = E\{(X_1 + \cdots + X_n)\ (X_1 + \cdots + X_n)\ (X_1 + \cdots + X_n)\ (X_1 + \cdots + X_n)\}$$

When we multiply these 4 parentheses we choose one variable from each and the expectation of the product is a sum of monomials of the form $E(X_i X_j X_k X_\ell)$. We can do this in many ways.

If all variables are the same we get $EX_1^4 + EX_2^4 + \cdots = nc$.

If at least one variable appears singly, e.g. $X_1^2 X_2 X_3$, then the expectation equals the product of expectations and so it is zero.

To get something nonzero, we need variables to appear in pairs. For example, $X_1^2 X_2^2$ has expectation $E(X_1^2 X_2^2) = EX_1^2 EX_2^2 = 1$. We can now form $X_1^2 X_2^2$, by picking X_1 from the first two parentheses and X_2 from the last 2; or X_1 from the first and third and X_2 from the second and fourth; etc. In total, we have $\binom{4}{2}$ ways to form $X_1^2 X_2^2$. Now we can also form $X_1^2 X_3^2$, or $X_2^2 X_5^2$, and so on. We have $\binom{n}{2}$ ways to pick 2 distinct variables out of n. Hence the contribution to the expectation of terms of the form $E(X_i^2 X_j^2)$ where $i \neq j$ is equal to $\binom{4}{2}\binom{n}{2}$.

we thus have

$$ES_n^4 = cn + \binom{4}{2}\binom{n}{2} = cn + 3n(n-1) = 3n^2 + (c-3)n.$$

4.

Consider the Ehrenfest chain $(X_n, n \geq 0)$ with N molecules and compute the constant c so that

$$P(\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = c) = 1.$$

Next, compute the average number of steps required for the process to return to state 0 (i.e. the quantity $E_0 T_0$). If $N = 10^{20}$ and if each step has duration 1 microsecond (i.e. one millionth of a second), then how many years will this average return time be?

Solution. The stationary distribution is:

$$\pi(x) = \binom{N}{x} 2^{-N}, \quad x = 0, \ldots, N.$$

if $f(x) = x$ is a "reward" function, then the Theorem of Large numbers tells us that

$$c = \sum_{x=0}^{N} f(x)\pi(x) = \sum_{x=0}^{N} x\pi(x) = \frac{N}{2}.$$

Next, we have

$$E_0 T_0 = 1/\pi(0) = 2^N = 2^{10^{20}} = 2^{100000000000000000000} \approx 10^{3.011\times 10^{19}} \text{ microseconds.}$$

This is a big number. The age of the universe is

$$\begin{aligned} U &= 14 \text{ billion years } = 14 \times 10^9 \text{ years} \\ &= 14 \times 10^9 \times 365 \times 24 \times 60 \times 60 \times 10^6 \text{ microseconds} \\ &= 441504000000000000000000 \text{ microseconds} \\ &\approx 10^{23.6} \text{ microseconds.} \end{aligned}$$

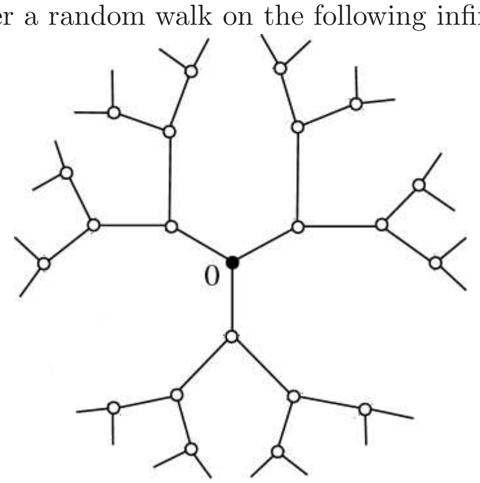
Hence

$$E_0 T_0 / U = 10^{3.011\times 10^{19} - 23.6} > 10^{3\times 10^{19}}.$$

In other words, one has to wait more than $10^{30000000000000000000}$ times the age of the universe to see find the molecules again in one of the boxes.

5.

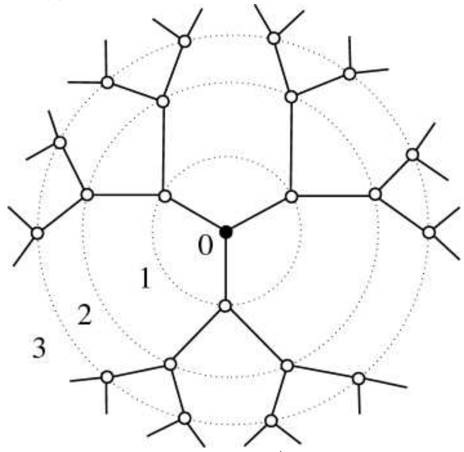
Consider a random walk on the following infinite graph:



The graph continues ad infinitum in the same manner.

Here, each state has exactly 3 neighbouring states (i.e. its degree is 3) and so the probability of moving to one of them is 1/3.
(i) Let 0 be the "central" state. (Actually, a closer look shows that no state deserves to be central, for they are all equivalent. So we just arbitrarily pick one and call it central.) Having done that, let $D(i)$ be the distance of a state i from 0, i.e. the number of "hops" required to reach 0 starting from i. So $D(0) = 0$, each neighbour i of 0 has $D(i) = 1$, etc. Let X_n be the position of the chain at time n. Observe that the process $Z_n = D(X_n)$ has the Markov property. (See lecture notes for criterion!) The question is:
Find its transition probabilities.
(ii) Using the results from the gambler's ruin problem, show that (Z_n) is transient.
(iii) Use (ii) to explain why (X_n) is also transient.

Solution. (i) First draw a figure:



The states with the same distance from 0 *are shown in this figure as belonging to the same circle.*

Next observe that if $Z_n = k$ (i.e. if the distance from 0 is k) then, no matter where X_n is actually located the distance Z_{n+1} of the next state X_{n+1} from 0 will either be $k+1$ with probability 2/3 or $k-1$ with probability 1/3. And, of course, if $Z_n = 0$ then $Z_{n+1} = 1$. So

$$\begin{aligned} P(Z_{n+1} = k+1 | Z_n = k) &= 2/3, \quad k \geq 0 \\ P(Z_{n+1} = k-1 | Z_n = k) &= 1/3, \quad k \geq 1 \\ P(Z_{n+1} = 1 | Z_n = 0) &= 1. \end{aligned}$$

(ii) Since $2/3 > 1/3$, the chain (Z_n) is transient.

(iii) We have that $Z_n \to \infty$ as $n \to \infty$, with probability 1. This means that for any k, there is a time n_0 such that for all $n \geq n_0$ we have $D(X_n) \geq k$, and this happens with probability 1. So, with probability 1, the chain (X_n) will visit states with distance from 0 less than k only finitely many times. This means that the chain (X_n) is transient.

6.

For a simple symmetric random walk (S_n), starting from $S_0 = 0$, find the following probabilities (n is a positive integer and x is an arbitrary integer):
(i) $P(S_{2n} > 0)$
(ii) $P(S_n = 0 | S_{2n} = 0)$
(iii) $P(S_i > 0 \text{ for } i = 1, 2, \ldots, n | S_n = x)$.