

Coupling Methods  
Edinburgh, Sept. 4-9, 2006

1. set of  
(bare-bone)  
slides

Coupling has developed into one of the most powerful and beautiful tools of probability theory.

Beautiful (for probabilists) because it puts all the emphasis on realizations and sample paths, rather than on measures;

beautiful also because a judicious coupling often makes self-evident the proof of a previously difficult theorem.

Andrew Barbour  
in JASA

This course on coupling is based on the first 250 pages of the book:  
Thorisson (2000), Coupling, Stationarity and Regeneration, Springer, New York.

$(\Omega, \mathcal{F}, P)$  supports all random things

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Definition: A coupling of  
the random elements

$X_i, i \in I,$

some  
index  
set

is a family

$(\hat{X}_i : i \in I)$

satisfying

$\hat{X}_i \stackrel{D}{=} X_i$

for each  $i \in I$ .

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Def  $C$  coupling event if  
 $C \subseteq \{ \hat{X}_i = \hat{X}_j, i, j \in I \}$

Thm Coupling event inequality:

$$P(C) \leq \left\| \bigwedge_{i \in I} P(X_i, \epsilon_i) \right\|_{\text{mass}}$$

↑  
greatest common component

Proof  $P(\hat{X}_j, \epsilon_j; C) \leq P(X_j, \epsilon_j)$

$$\Rightarrow P(\hat{X}_j, \epsilon_j; C) \leq \bigwedge_{i \in I} P(X_i, \epsilon_i)$$

Thm  $\exists$  maximal coupling:

$$P(C) = \left\| \bigwedge_{i \in I} P(X_i, \epsilon_i) \right\|$$

Proof Take  $J, V, W_i$  independent

$$P(J=1) = \|V\|$$

$$P(V \in \cdot) = V / \|V\|$$

$$P(W_i \in \cdot) = (P(X_i, \epsilon_i) - V) / (1 - \|V\|)$$

Put

$$\hat{X}_i = \begin{cases} V & \text{if } J=1 \\ W_i & \text{if } J \neq 1 \end{cases}$$

Thm

$$X \stackrel{D}{\leq} Y$$

$\Rightarrow$

$\exists$  coupling:  $\hat{X} \leq \hat{Y}$

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Thm  $X_n \stackrel{D}{\rightarrow} X, n \rightarrow \infty$

$\Rightarrow$

$\exists$  coupling:  $\hat{X}_n \rightarrow \hat{X}$  a.s.,  $n \rightarrow \infty$

---

Thm. If  $X_n$  has density  $f_n$  then:

$\lim_{n \rightarrow \infty} f_n$  is a density of  $X$

$\Rightarrow$

$\exists$  coupling and an  $N$ :

$$\hat{X}_n = \hat{X}, n \geq N$$

random variable

Application: To replace  $\leq$  by  $\stackrel{D}{\leq}$  and  $\rightarrow$  by  $\stackrel{D}{\rightarrow}$  in this theorem we only have to put hats (^) on the  $X$ -s

## Dominated Convergence

If  $X_1, X_2, \dots, X_\infty, X \geq 0$   
are random variables such that

$$X_n \leq X, \quad 1 \leq n < \infty,$$

$$E[X] < \infty,$$

$$X_n \rightarrow X_\infty, \quad n \rightarrow \infty,$$

then

$$E[X_\infty] < \infty$$

and

$$E[X_n] \rightarrow E[X_\infty], \quad n \rightarrow \infty.$$

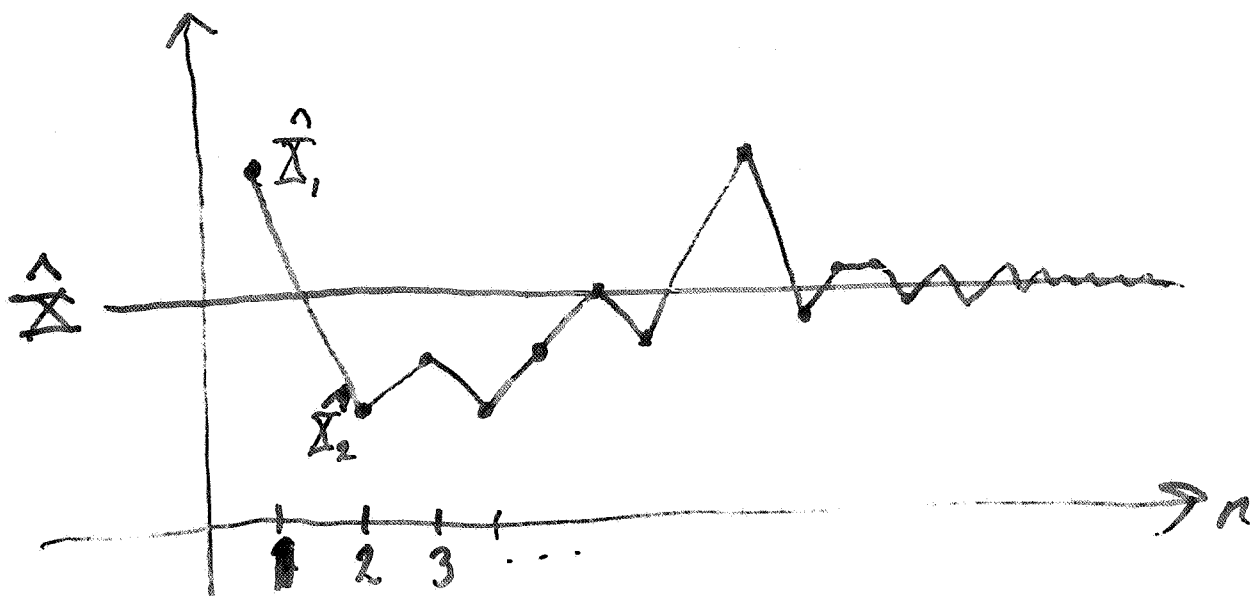
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Rmk " Extend beyond  $\geq 0$  by noting

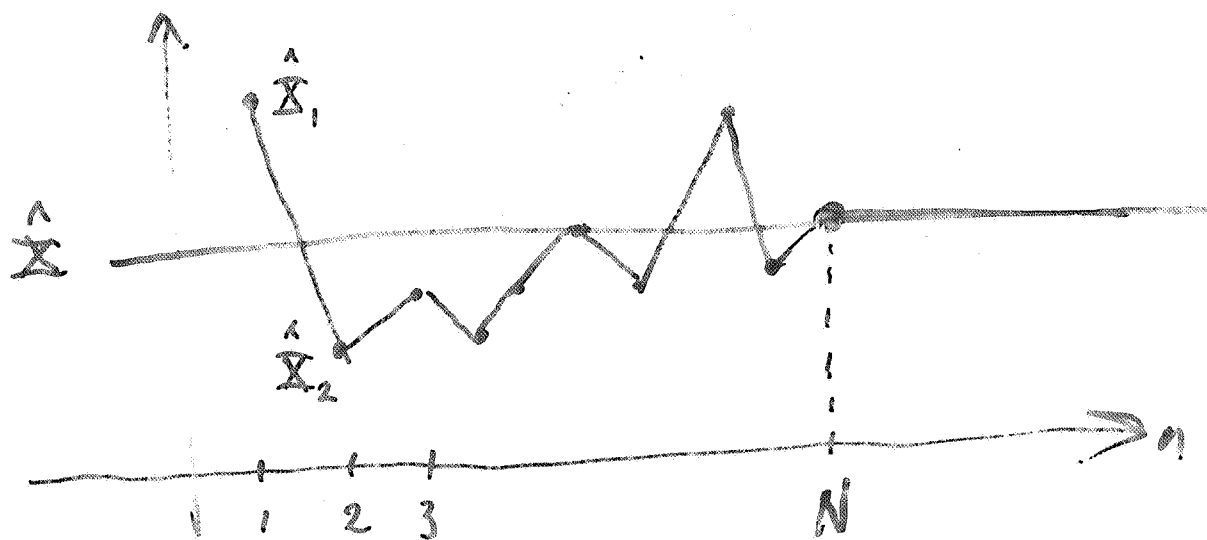
$$X_n = X_n^+ - X_n^-$$

and applying the theorem to  $X_n^+$  and  $X_n^-$

$\hat{D}$



$f_n \rightarrow f$



Thm

$$X \stackrel{D}{\leq} Y$$

p.o. Polish  
spaces



$\exists$  coupling:  $\hat{X} \leq \hat{Y}$

Thm

$$X_n \xrightarrow{D} X, \quad n \rightarrow \infty$$

separable



$\exists$  coupling:  $\hat{X}_n \rightarrow \hat{X}$  a.s.,  $n \rightarrow \infty$

Thm. If  $X_n$  has density  $f_n$  then:

$\lim_{n \rightarrow \infty} f_n$  is a density of  $X$



$\exists$  coupling and an  $N$ :

$$\hat{X}_n = \hat{X}, \quad n \geq N$$

random variable

Thm 1: Let  $X_1, X_2, \dots; X$  be r.v. in a countable set  $E$ . If

$P(X_n = x) \rightarrow P(X = x), n \rightarrow \infty, x \in E$ ,  
then there exists a coupling  
 $(\hat{X}_1, \hat{X}_2, \dots; \hat{X})$  and an  $N$ -valued r.v.  $N$   
such that  $\hat{X}_n = \hat{X}, n \geq N$ .

Notation: With  $x^1 \in E^1, x^2 \in E^2, \dots$  and  $k \in \mathbb{N}$   
put  $\underline{x} := (x^1, x^2, \dots), \underline{x}^k := (x^1, \dots, x^k), E^k := E^1 \times \dots \times E^k$

Thm 2: For  $k \in \mathbb{N}$ , let  $X_1^k, X_2^k, \dots; X^k$  be r.v. in a countable set  $E^k$ . If

(1)  $\forall k \in \mathbb{N}: P(\underline{X}_n^k = \underline{x}^k) \rightarrow P(\underline{X}^k = \underline{x}^k), n \rightarrow \infty, \underline{x}^k \in E^k$

then there exists a coupling

$(\hat{\underline{X}}_1, \hat{\underline{X}}_2, \dots; \hat{\underline{X}})$  of  $\underline{X}_1, \underline{X}_2, \dots; \underline{X}$  and

finite  $N$ -valued r.v.s  $N^1, N^2, \dots$  such that

(2)  $\forall k \in \mathbb{N}: \hat{\underline{X}}_n^k = \hat{\underline{X}}^k, n \geq N^k$ .

Thm 3: Let  $X_1, X_2, \dots; X$  be r.v. in a measurable space  $(E, \mathcal{E})$  where  $E$  is a metric separable space and  $\mathcal{E}$  its Borel subsets. If

$$X_n \xrightarrow{d} X, n \rightarrow \infty,$$

then there exists a coupling  $(\hat{X}_1, \hat{X}_2, \dots; \hat{X})$   
such that  $\hat{X}_n \rightarrow \hat{X}$  a.s.,  $n \rightarrow \infty$ .



Corollary to Thm 2: There exists a finite r.v.  $N$  and integers  $0 < k_1 \leq k_2 \leq \dots \leq k_n \rightarrow \infty, n \rightarrow \infty$ , such that

$$\underline{\hat{\Sigma}}_n^{k_n} = \underline{\hat{\Sigma}}^{k_n}, \quad n \geq N.$$


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Proof: Let  $n' < n^2 < \dots < n^k \rightarrow \infty, k \rightarrow \infty$ , be such that  $P(N^k > n^k) \leq 1/k^2, k \in \mathbb{N}$ . Then by Borel-Cantelli

$$\mathbb{K} := \sup \{k \in \mathbb{N} : N^k > n^k\} + 1 < \infty \text{ a.s.}$$

and

$$n^k \geq N^k, \quad k \geq \mathbb{K}. \quad (3)$$

Define for  $n \in \mathbb{N}$ ,

$$k_n = k \quad \text{if} \quad n_k \leq n < n_{k+1}$$

and

$$N := n_{\mathbb{K}}$$

and note that

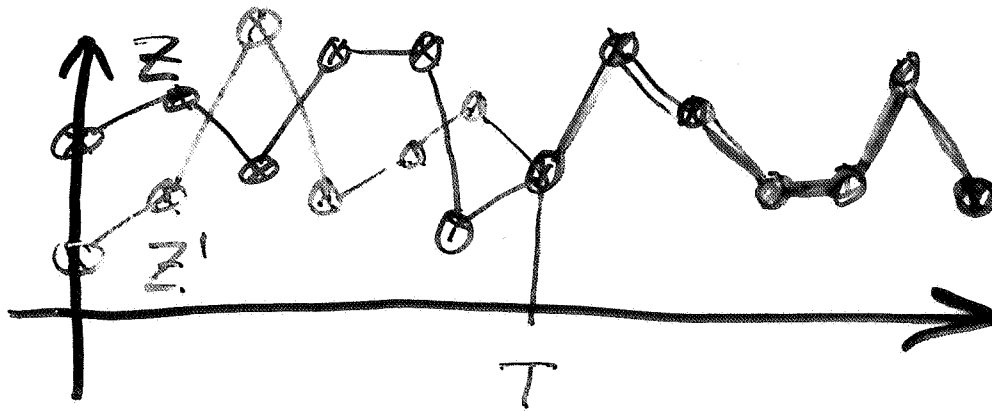
$$n \geq N \iff k_n \geq \mathbb{K} \quad (4)$$

Now

$$\begin{aligned} n \geq N &\stackrel{(4)}{\implies} k_n \geq \mathbb{K} \stackrel{(3)}{\implies} n^{k_n} \geq N^{k_n} \\ &\stackrel{n \geq n^{k_n}}{\implies} n \geq N^{k_n} \stackrel{(2)}{\implies} \underline{\hat{\Sigma}}_n^{k_n} = \underline{\hat{\Sigma}}^{k_n} \end{aligned}$$



# The Coupling Proof $Z$ and $Z'$ Markov



Positive recurrent case:

$$\left. \begin{array}{l} P(Z'_n = i) = \pi_i \quad \forall n \\ Z_n \approx Z'_n \quad n \text{ large} \end{array} \right\} \Rightarrow P(Z_n = i) \approx \pi_i \quad n \text{ large}$$

Null recurrent case:

$$\left. \begin{array}{l} P(Z'_n = i) \approx 0 \quad \forall n \\ Z_n \approx Z'_n \quad n \text{ large} \end{array} \right\} \Rightarrow P(Z_n = i) \approx 0 \quad n \text{ large}$$

$Z = (Z_k)_{k=0}^{\infty}$  irreducible, aperiodic, recurrent

$E$  state space,  $P$  transition matrix

$\pi$  stationary vector, that is,

$$\pi P = \pi$$

$$0 < \pi_i < \infty, \quad i \in E,$$

$Z$  positive recurrent  $\Rightarrow \sum_{i \in E} \pi_i = 1$

$Z$  null recurrent  $\Rightarrow \sum_{i \in E} \pi_i = \infty$

**Lemma** With  $B \subseteq E$  finite

define  $\pi^B$  by

$$\pi_i^B = \begin{cases} \frac{\pi_i}{\sum_{j \in B} \pi_j} & \text{if } i \in B \\ 0 & \text{if } i \notin B \end{cases}$$

Then

$$\pi^B P^n \leq \frac{\pi}{\sum_{j \in B} \pi_j}$$

**Proof**

$$(\pi^B \leq \pi / \sum_B \pi_j)$$

$$\pi^B P^n \leq \frac{\pi P^n}{\sum \pi_j} = \frac{\pi}{\sum \pi_j}$$

**Basic Limit Theorem**  $\forall \lambda, i:$

$Z$  positive recurrent  $\Rightarrow \lambda P_i^n \rightarrow \pi_i, n \rightarrow \infty$

$Z$  null recurrent  $\Rightarrow \lambda P_i^n \rightarrow 0, n \rightarrow \infty$

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Proof Use coupling to establish:

$\textcircled{*} \forall \lambda, \mu, i: |\lambda P_i^n - \mu P_i^n| \rightarrow 0, n \rightarrow \infty$

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If  $Z$  is positive recurrent put  $\mu = \pi$  to obtain

$$|\lambda P_i^n - \pi_i| \rightarrow 0, n \rightarrow \infty$$

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If  $Z$  is null recurrent put

$\mu = \pi^B$  to obtain

$$\lambda P_i^n \leq |\lambda P_i^n - \pi^B P_i^n| + \pi^B P_i^n$$

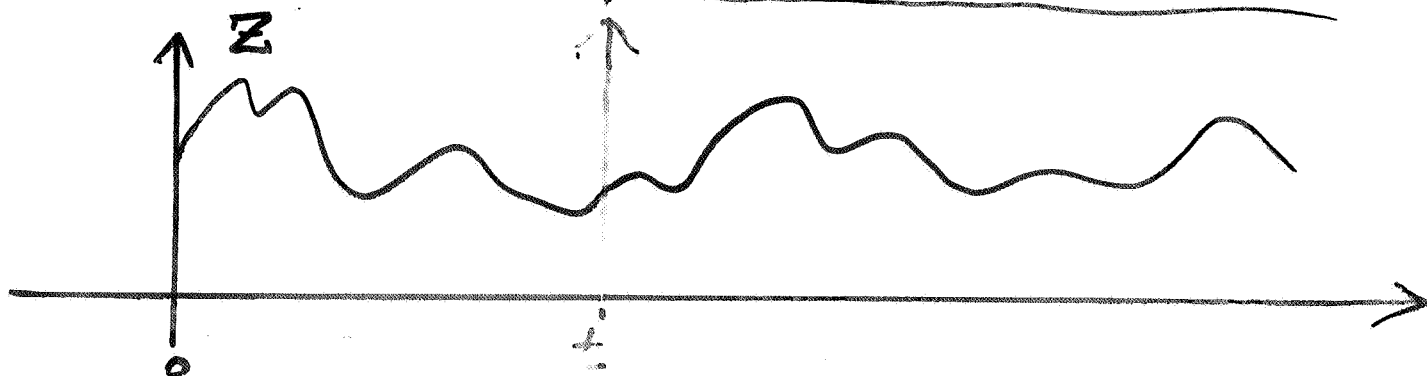
$\textcircled{\text{Lemma}}$   $\leq |\lambda P_i^n - \pi^B P_i^n| + \frac{\pi_i}{\sum_{j \in B} \pi_j}$

$\textcircled{*} \rightarrow \frac{\pi_i}{\sum_{j \in B} \pi_j}, n \rightarrow \infty$

$$\rightarrow 0, B \uparrow E$$

$Z = (Z_s)_{s \in \mathbb{R}_+}$  } processes on a Polish  
 $Z' = (Z'_s)_{s \in \mathbb{R}_+}$  } state space with  
 paths in  $(D, \mathcal{D})$

$$\theta_t Z := (Z_{t+s})_{s \in \mathbb{R}_+}$$



Def: Coupling:  $\hat{Z} \stackrel{D}{=} Z, \quad \hat{Z}' \stackrel{D}{=} Z'$

Def:  $C \in \mathcal{F}$  is coupling event if  $C \subseteq \{\hat{Z} = \hat{Z}'\}$

Prop: Coupling event inequality:

$$\|P(\hat{Z} \in \cdot) - P(\hat{Z}' \in \cdot)\| \leq 2 P(C^c)$$

$= 2 \sup_{A \in \mathcal{D}} |P(Z \in A) - P(Z' \in A)| \leq 2 P(C^c)$

$$\mathcal{F}_t := \theta_t^{-1} \mathcal{D} \quad \text{post-}t \quad \sigma\text{-algebra} \quad \bigg| \quad \mathcal{F} := \bigcap_{t \in \mathbb{R}_+} \mathcal{F}_t \quad \text{tail} \quad \sigma\text{-algebra}$$

$$\mathcal{I} := \{A \in \mathcal{D} : \theta_t^{-1} A = A, t \in \mathbb{R}_+\} \quad \text{invariant} \quad \sigma\text{-algebra}$$

$$\mathcal{P} := \{f \in \mathcal{F} : f(\theta_t Z) \rightarrow f(Z) \text{ as } t \downarrow 0, Z \in D\} \quad \text{smooth tail} \quad \sigma\text{-algebra}$$

Call  $T$  coupling time if  
 $\theta_T \hat{Z} = \theta_T \hat{Z}'$  on  $\{T < \infty\}$

Thm 1 coupling time inequality: for  $t \geq 0$   
 $\|P(\theta_t Z \in \cdot) - P(\theta_t Z' \in \cdot)\| \leq 2P(T > t)$   
 $T < \infty \implies \epsilon \rightarrow 0$

Thm 2  $\exists$  maximal coupling:  
 $\|P(\theta_t Z \in \cdot) - P(\theta_t Z' \in \cdot)\| = 2P(T > t), t \geq 0$   
 i.e.  $\|P(Z \in \cdot)_{\mathcal{F}_t} - P(Z' \in \cdot)_{\mathcal{F}_t}\| = 2P(T > t), t \geq 0$

Thm 3  $\exists$   $\mathcal{T}$ -maximal coupling  
 $\|P(Z \in \cdot)_{\mathcal{T}} - P(Z' \in \cdot)_{\mathcal{T}}\| = 2P(T = \infty)$   
 $1+3 \implies$

Thm 4 The following claims are equivalent:  
 (a)  $\exists$  coupling with a finite  $T$   
 (b)  $\|P(\theta_t Z \in \cdot) - P(\theta_t Z' \in \cdot)\| \rightarrow 0, t \rightarrow \infty$   
 (c)  $P(Z \in A) = P(Z' \in A), A \in \mathcal{T}$

Thm 5 For the class of Markov processes with transition probab.  $P^t$  we have:  
 (c) holds for all  $Z, Z'$  from the class  
 $\iff \forall \lambda: P_\lambda(Z \in A) = 0$  or  $1$  for  $A \in \mathcal{T}$

Note that if  $\theta_t Z' \stackrel{d}{=} Z', t \geq 0$ , then  
 (b) becomes:  $\theta_t Z \xrightarrow{tv} Z', t \rightarrow \infty$

Call  $T, T'$  <sup>finite</sup> coupling times if  $\{T < \infty\}$   
 $\theta_T \hat{Z} = \theta_{T'} \hat{Z}'$  on  $\{T < \infty\} = \{T' < \infty\}$

Thm <sup>shift</sup> coupling ~~time~~ inequality: for  $t \geq 0$   
 $\|P(\theta_{t_0} Z \in \cdot) - P(\theta_{t_0} Z' \in \cdot)\| \leq 2P(T \vee T' > t_0)$   
 $T < \infty \implies \xrightarrow{t \rightarrow \infty} 0$

Thm  $\exists$  ~~shift~~-maximal coupling  
 $\|P(Z \in \cdot) - P(Z' \in \cdot)\| = 2P(T = \infty)$

Thm The following claims are equivalent:  
 $\exists$  <sup>shift</sup> coupling with a finite  $T, T'$   
 (a)  $\|P(\theta_{t_0} Z \in \cdot) - P(\theta_{t_0} Z' \in \cdot)\| \rightarrow 0, t \rightarrow \infty$   
 (b)  $P(Z \in A) = P(Z' \in A), A \in \mathcal{F}$   
 (c)

Thm For the class of Markov processes  
 with transition probab.  $P^t$  we have:  
 (c) holds for all  $Z, Z'$  from the class  
 $\iff \forall \lambda: P_\lambda(Z \in A) = 0 \text{ or } 1 \text{ for } A \in \mathcal{F}$

Note that if  $\theta_t Z' \equiv Z', t \geq 0$ , then  
 (b) becomes:  $\theta_{t_0} Z \xrightarrow{t \rightarrow \infty} Z', t \rightarrow \infty$

Call  $T, T'$   $\varepsilon$ -coupling times if  $|T - T'| \leq \varepsilon$   
 $\theta_T \hat{Z} = \theta_{T'} \hat{Z}'$  on  $\{T < \infty\}$

Thm coupling time inequality: for  $t \geq 0$   
 $\|P(\theta_t Z \in \cdot) - P(\theta_t Z' \in \cdot)\| \leq 2P(T > t) + 2\frac{\varepsilon}{h}$   
 $t \geq 0$   $T < \infty \Rightarrow \varepsilon \rightarrow 0$

Thm  $\exists$   $\mathcal{F}$ -maximal coupling  
 $\|P(Z \in \cdot) - P(Z' \in \cdot)\| = 2 \sup_{\varepsilon > 0} P(T_\varepsilon = \infty)$

Thm The following claims are equivalent:  
 (a)  $\forall \varepsilon > 0 \exists \varepsilon$ -coupling with a finite  $T_\varepsilon, T'_\varepsilon$   
 (b)  $\lim_{t \rightarrow \infty} \|P(\theta_t Z \in \cdot) - P(\theta_t Z' \in \cdot)\| = 0$ ,  $t \rightarrow \infty$   
 (c)  $P(Z \in A) = P(Z' \in A)$ ,  $A \in \mathcal{F}$

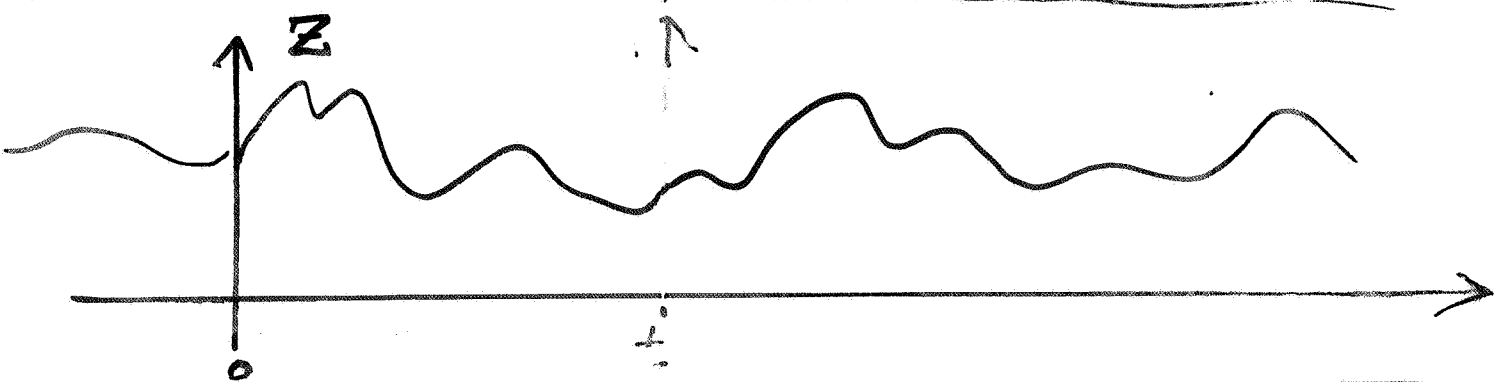
Thm For the class of Markov processes  
 with transition probab.  $P^t$  we have:  
 (c) holds for all  $Z, Z'$  from the class  
 $\Leftrightarrow \forall \lambda: P_\lambda(Z \in A) = 0$  or  $1$  for  $A \in \mathcal{F}$

Note that if  $\theta_t Z' \stackrel{d}{=} Z'$ ,  $t \geq 0$ , then  
 (b) becomes:  $\theta_t Z \xrightarrow{t \rightarrow \infty} Z'$ ,  $t \rightarrow \infty$



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$\mathcal{I}_t := \theta_t^{-1} \mathcal{D}$	$\mathcal{I} := \bigcap_{t \in \mathbb{R}_+} \mathcal{I}_t$
post-t $\sigma$ -algebra	tail $\sigma$ -algebra

$\mathcal{I} := \{A \in \mathcal{D} : \theta_t^{-1} A = A, t \in \mathbb{R}_+\}$

$\mathcal{P} := \sigma\{f \in \mathcal{I} : f(\theta_t Z) \rightarrow f(Z) \text{ as } t \downarrow 0, Z \in D\}$

$\mathcal{J} \subseteq \mathcal{P} \subseteq \mathcal{I}$

smooth  
tail  
 $\sigma$ -algebra

$$d=2$$

$+$   
 $\uparrow$  zero of  $\hat{Z}'$

$+$   
 $\uparrow$  zero of  $\hat{Z}$

---


$$\theta_T \hat{Z} = \theta_{T'} \hat{Z}'$$

$$\Leftrightarrow$$

$$\theta_{T-T'} \hat{Z} = \hat{Z}'$$


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that is, with  $S = T - T'$ :  $\theta_S \hat{Z} = \hat{Z}'$

## Transformation Coupling

Let  $G$  be locally compact second countable group acting jointly measurably on some space  $(E, \mathcal{E})$ .

Let  $X, X', \hat{X}, \hat{X}'$  be random elements in  $(E, \mathcal{E})$ .

Let  $T$  be a random transformation in  $(G, \mathcal{G})$ .

Let  $C$  be an event. the Borel sets

Definition:

$(\hat{X}, \hat{X}', T, C)$  is a transformation coupling of  $X$  and  $X'$  if

$$T \hat{X} \equiv \hat{X}' \text{ on } C.$$

$(\hat{X}, \hat{X}', T)$  is a successful transformation coupling of  $X$  and  $X'$  if

$$T \hat{X} \equiv \hat{X}'.$$

Equivalences:

$\exists$  a successful transformation coupling of  $X$  and  $X'$

$\iff$

$$P(X \in A) = P(X' \in A), \quad A \in \mathcal{I} := \{A \in \mathcal{E} : \gamma A = A, \gamma \in G\}$$

where is the limit claim?

Let  $\lambda$  be Haar measure on  $(G, \mathcal{G})$ .

Key Inequality: for  $B \in \mathcal{G}$  with  $0 < \lambda(B) < \infty$ ,

$$\begin{aligned} & \|P(U_B X \in \cdot) - P(U_B X' \in \cdot)\| \\ & \leq 2E\left[\frac{\lambda(B \setminus BT')}{\lambda(B)}\right] + 2P(C^c) \end{aligned}$$

where  $U_B$  is uniform on  $B$  and independent of  $X$  and  $X'$ .

Assume there exist Følner sets  $B_t, 0 < t < \infty$ , namely,

$$\forall g \in G: \frac{\lambda(B_t \setminus B_t g)}{\lambda(B_t)} \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Remark: this means that  $G$  is amenable

The Equivalences:

$\exists$  a successful transformation coupling of  $X$  and  $X'$



$$\|P(U_{B_t} X \in \cdot) - P(U_{B_t} X' \in \cdot)\| \rightarrow 0 \text{ as } t \rightarrow \infty$$



$$P(X \in A) = P(X' \in A), \quad A \in \mathcal{Y}$$

# Working Hypothesis

every

meaningful

distributional

relation

should

have

a

coupling

counterpart