Asymptotics for the maximum of a random walk with negative increments

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Large Deviations, Lecture 1

Short Instructional Courses, September 2006

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This lecture (SF):
One-dimensional random walk, asymptotics in two main cases (the “classical” and the “heavy-tailed”) out of five.

Next lecture (TK):
The LD Principle in \( \mathbb{R}^1 \).

Lectures 3-5 (AP):
General LD Principle and Applications.

There will be 4 additional hours in LD’s in the afternoons – for lecturing an additional matherial, discussions, and exercises.
1 Notation and definitions

Consider a sequence $\xi, \xi_1, \xi_2, \ldots$ of independent identically distributed (i.i.d.) random variables (r.v.’s) with a common distribution function $F(x) = P(\xi \leq x)$. Let

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} \xi_i, \quad n \geq 1,$$

$$M_n = \max_{0 \leq i \leq n} S_i, \quad M = \sup_{n \geq 0} M_n \equiv M_\infty.$$

We are interested in the asymptotics for $P(M > x)$ when $x \to \infty$.

First, there is 0−1 Law:
Either $P(M < \infty) = 1$ (in other words, $M$ is finite a.s.) or $P(M = \infty) = 1$ (in other words, $M$ is infinite a.s.).

Further, if the mean (expectation, drift) of $\xi$ exists (i.e. $E|\xi| < \infty$) then
$$P(M < \infty) = 1 \text{ if and only if } E\xi = -a < 0 \quad (\text{se, e.g., Feller [5]).}$$

If the mean does not exist ($E|\xi| = \infty$), then $M$ is finite a.s. if and only if

$$\int_{0}^{\infty} \frac{x dF(x)}{m(x)} < \infty$$

where $m(x) = E(\min(x, -\xi))$ – see Erickson ([4]).

Let $\varphi(t) = E e^{t\xi}$ be a moment generating function of $\xi$. Clearly, $\varphi(0) = 1$. If $\varphi(t)$ is finite for some $t > 0$, then it is finite for all $0 < u < t$ and convex in this interval since $\varphi''(u) = E \left( \xi^2 e^{u\xi} \right) \geq 0$.

There are 5 cases of interest:
(I) the mean does not exist, but condition (1) holds;
(II) $a > 0$ and $\varphi(t) = \infty$ for all $t > 0$ (the case of heavy-tailed distribution);
(III) $a > 0$ and there exists $\gamma > 0$ such that

$$\varphi(\gamma) = 1$$

and $\varphi'(\gamma) < \infty$ (this is the “classical case”, here $\varphi'(\gamma)$ means the left derivative);
(IV) $a > 0$, (2) holds, but $\varphi'(\gamma) = \infty$;
(V) $a > 0$, $\varphi(t) < \infty$ for some $t > 0$, but (2) is violated.

We consider today only cases (II) and (III). For the asymptotics in other cases, see recent papers [3], [8], [6], and further references therein.
2 Applications

(a) Queueing theory. Stationary (limiting) waiting time $W$ in a single server queue coincides in distribution with the supremum $M$ of an associated random walk.

Consider a single-server queue with i.i.d. interarrival times $t_n$ and independent of them i.i.d. service times $\sigma_n$. Assume that customer 1 arrives in an empty system. Then its waiting time (before service) is $W_1 = 0$. Customer 2 arrives $t_1$ units of time later, and its waiting time is $W_2 = \max(0, \sigma_1 - t_1)$. By induction,

$$W_{n+1} = \max(0, W_n + \sigma - t_n).$$

Let $\xi_n = \sigma - t_n$. Note that

$$W_{n+1} = \max(0, W_n + \xi_n)$$

$$= \max(0, \max(0, W_{n-1} + \xi_{n-1}) + \xi_n)$$

$$= \max(0, \xi_n, \xi_n + \xi_{n-1} + W_{n-1})$$

$$= \ldots$$

$$= \max(0, \xi_n, \xi_n + \xi_{n-1}, \ldots, \xi_n + \ldots + \xi)$$

Since $\{\xi_n\}$ is an i.i.d. sequence, $W_{n+1}$ has the same distribution as

$$M_n = \max(0, \xi_1, \xi_1 + \xi_2, \ldots, \xi_1 + \ldots + \xi_n).$$

Therefore, as $n \to \infty$, distributions of $W_n$ converge weakly to that of $M$.

One can use also the “Loynes scheme”. Assume that i.i.d.r.v.’s $\xi_n$ are defined for all $-\infty < n < \infty$ (this extension may be done using Kolmogorov’s theorem). Clearly, $W_{n+1}$ has the same distribution as

$$\tilde{M}_n = \max(0, \xi_1, \xi_1 + \xi_2, \ldots, \xi_1 + \ldots + \xi_n)$$

Here we do not reverse numbering – we just shift all indices by $(-n-1)$. Then $\tilde{M}_{n+1} \geq \tilde{M}_n$ a.s. for all $n$ and $\tilde{M}_n$ converge monotonically to the supremum

$$\tilde{M} = \sup_{n \geq 0} \sum_{i=1}^{n} \xi_{-i}$$

of a random walk with increments $\{\xi_{-n}\}$.

Remark 1. You will meet Loynes scheme again tomorrow during the 2nd lecture on stability methods.

(b) Risk theory. In the risk theory, $P(M > x)$ may be interpreted as a probability of ruin (in the infinite time horizon).

In the risk theory setting, an insurance company has an initial capital $x$. Consequent i.i.d. claims have sizes $\sigma_n$, and independent of them i.i.d. “interclaim times” are $t_n$. There is a surplus process with rate $c > 0$. A ruin occurs if

$$\inf \{x + \sum_{i=1}^{n} (ct_i - \sigma_i)\} < 0,$$

or, equivalently,

$$\sup \sum_{n \geq 0}^{n} (\sigma_i - ct_i) > x$$

where $\{\sigma_i - ct_i\}_{i \geq 1}$ is an i.i.d. sequence.
3 Elements of renewal theory

A r.v. $\xi$ has a lattice (arithmetic) distribution with span $h > 0$ if

$$\sum_{n=-\infty}^{\infty} P(\xi = nh) = 1$$

and if $h$ is a maximal number with this property. Clearly, any r.v. has either a lattice distribution (with uniquely determined span) or a non-lattice (non-arithmetic) distribution.

Let $\xi, \xi_1, \xi_2, \ldots$ be an i.i.d. sequence of random variables with a finite positive mean $b = E\xi$. For any $x$, let

$$\tau(x) = \min\{n : S_n > x\}$$

be the first hitting time of $(x, \infty)$, and

$$\chi(x) = S_{\tau(x)} - x$$

an overshoot over $x$.

Here are some basic facts:

(i) since $b > 0$, $\tau(x)$ is a.s. finite for any $x$ and, moreover, $E\tau(x) < \infty$.

(ii) If $\xi$ has a non-arithmetic distribution, distributions of $\chi(x)$ converge (weakly), as $x \to \infty$, to a proper continuous distribution of, say, random variable $\chi(\infty)$. This means that

$$Eg(\chi(x)) \to Eg(\chi(\infty)), \quad x \to \infty$$

for any bounded continuous function, or, equivalently,

$$P(\chi(x) \leq t) \to P(\chi(\infty) \leq t), \quad x \to \infty$$

for any $t$.

(iii) If $\xi$ has an arithmetic distribution with span $h$, distributions of $\{\chi(nh), n = 1, 2, \ldots\}$ converge weakly, as $n \to \infty$, to a proper discrete distribution, of, say, random variable $\tilde{\chi}(\infty)$.

Remark 2. See, e.g., Asmussen ([1]), Feller ([5]), or/and Gut ([7]) for more detailed treatment of renewal theory and, in particular, for limiting distributions of $\chi(\infty)$ and $\tilde{\chi}(\infty)$. 

4 The classical case

4.1 Exponential change of measure (Cramer transform)

Note that
\[ \phi(\gamma) \equiv E e^{\gamma \xi} = \int_{-\infty}^{\infty} e^{\gamma x} dF(x) = 1. \]

Introduce a new distribution function
\[ dF^*(x) = e^{\gamma x} dF(x). \]

Thus, \( F^* \) is a probability distribution function.

For any \( n \) and for any bounded function \( g : \mathbb{R}^n \rightarrow \mathbb{R} \), let \( E^* g(\xi_1, \ldots, \xi_n) \) be an expectation with respect to distribution \( F^* \):

\[
E^* g(\xi_1, \ldots, \xi_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_n) dF^*(x_1) \cdots dF^*(x_n)
\]

and, in particular,

\[
P^*(g(\xi_1, \ldots, \xi_n) \in A) = \int_{g^{-1}(A)} \cdots \int_{g^{-1}(A)} dF^*(x_1) \cdots dF^*(x_n).
\]

Then

\[
E g(\xi_1, \ldots, \xi_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_n) dF(x_1) \cdots dF(x_n)
\]

\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_n) e^{-\gamma x_1} dF^*(x_1) \cdots e^{-\gamma x_n} dF^*(x_n)
\]

\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\gamma \sum_{i=1}^{n} x_i} g(x_1, \ldots, x_n) dF^*(x_1) \cdots dF^*(x_n)
\]

\[
= E^* (e^{-\gamma S_n} g(\xi_1, \ldots, \xi_n))
\]

and, similarly,

\[
E (g(\xi_1, \ldots, \xi_n) e^{\gamma S_n}) = E^* g(\xi_1, \ldots, \xi_n).
\]

In particular,

\[
E^* \xi = \int_{-\infty}^{\infty} x dF^*(x) = \int_{-\infty}^{\infty} x e^{\gamma x} dF(x) = E \left( \xi e^{\gamma \xi} \right),
\]

and this number is positive (!) Therefore

\[
P^*(\tau(x) < \infty) = 1
\]

for all \( x \).
Exercise 1. In the classical case, the following are equivalent:
(a) $F$ has a lattice distribution with span $h$;
(b) $F^*$ has a lattice distribution with span $h$.

Exercise 2. Assume that $t > 0$ is such that $\varphi(t) < \infty$. Let
\[
dF_t(x) = \frac{e^{tx} dF(x)}{\varphi(t)}
\]
and denote by $E_t$ the corresponding expectation operator. Find an expression for $E_t g(\xi_1, \ldots, \xi_n)$ in this case.
4.2 The asymptotics for $P(M > x)$

Let $I$ be an indicator function, i.e., for any (random) event $B$,

$$I(B) = 1 \text{ if the event } B \text{ occurs and } I(B) = 0 \text{ otherwise.}$$

In particular, for any $x \in \mathbb{R}^n$ and for any measurable set $A \subset \mathbb{R}^n$,

$$I(x \in A) = 1 \text{ if } x \in A \text{ and } I(x \in A) = 0 \text{ otherwise.}$$

Let $\tau(x) = \min\{n \geq 0: S_n > x\}$ if there exists such $n$, and $\tau(x) = \infty$ otherwise. Then, for $x > 0$,

$$P(M > x) = P(\tau(x) < \infty) = \sum_{n=1}^{\infty} P(\tau(x) = n) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} I((x_1, \ldots, x_n) \in A_n(x)) dF(x_1) \cdots dF(x_n)$$

where

$$A_n(x) = \{(z_1, \ldots, z_n): z_1 \leq x, \ldots, z_1 + \ldots + z_{n-1} \leq x, z_1 + \ldots + z_n > x\}.$$ 

Further,

$$P(M > x) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\gamma \sum_{i=1}^{n} x_i} I((x_1, \ldots, x_n) \in A_n(x)) dF^*(x_1) \cdots dF^*(x_n)$$

$$= \sum_{n=1}^{\infty} E^* (e^{-\gamma S_n} I(\tau(x) = n))$$

$$= E^* (e^{-\gamma S_{\tau(x)}} I(\tau(x) < \infty))$$

$$= E^* e^{-\gamma S_{\tau(x)}}.$$ 

The very last equality follows from (3).

Since $S_{\tau(x)} = x + \chi(x)$,

we get finally

**Theorem 1.** (1) In the classical case,

$$P(M > x) = e^{-\gamma x} E^* e^{-\gamma \chi(x)}.$$ 

(2) Cramer upper bound. Since $\chi(x) \geq 0$, for any $x > 0$,

$$P(M > x) \leq e^{-\gamma x}.$$ 

(3) From the basic renewal theory,

(a) if $F$ is non-lattice, then

$$P(M > x)e^{\gamma x} \rightarrow E^* e^{-\gamma \chi(\infty)} \in (0, \infty), \quad x \rightarrow \infty,$$ 

(b) If $F$ is lattice with span $h$, then

$$P(M > nh)e^{\gamma nh} \rightarrow E^* e^{-\gamma \tilde{\chi}(\infty)} \in (0, \infty), \quad n \rightarrow \infty.$$
Exercise 3. Show that (in both lattice and non-lattice cases)
\[ \sup_{x \geq 0} P(M > x)e^{\gamma x} < \infty \quad \text{and} \quad \inf_{x \geq 0} P(M > x)e^{\gamma x} > 0. \]

Corollary 1. In the classical case,
\[ \lim_{x \to \infty} \frac{1}{x} \log P(M > x) = -\gamma. \]

Remark 3. This is a particular example of LDP (to be introduced in Lecture 2.)
4.3 “Typical” sample paths

Since \( E^* \xi = b > 0 \), the SLLN says that

\[
P^* \left( \frac{S_n}{n} \to b \right) = 1, \quad \text{or} \quad \frac{S_n}{n} \to b \quad \text{P}^* \text{ a.s.} \quad (6)
\]

This also means that, for any \( \varepsilon > 0 \),

\[
P^* \left( \sup_{m \geq n} \left| \frac{S_n}{n} - b \right| > \varepsilon \right) \to 0 \quad \text{as} \quad n \to \infty.
\]

**Exercise 4.** Show that (6) is equivalent to the following:

for any \( \varepsilon \in (0, 1) \) and any \( \delta \in (0, 1) \), there exists \( R > 0 \) such that

\[
P^* (B) \equiv P^* (-R + n(b - \delta) \leq S_n \leq R + n(b + \delta)) \quad \text{for all} \quad n \geq 1 - \varepsilon,
\]

and also to the following:

for any \( \varepsilon \in (0, 1) \), there exist \( R > 0 \) and a sequence \( \delta_n \to 0 \) such that

\[
P^* (B) \equiv P^* (-R + n(b - \delta_n) \leq S_n \leq R + n(b + \delta_n)) \quad \text{for all} \quad n \geq 1 - \varepsilon.
\]

Let

\[
B(x) = \{-R + n(b - \delta) \leq S_n \leq R + n(b + \delta) \quad \text{for all} \quad n \leq \tau(x)\}.
\]

**Theorem 2.** For any \( \varepsilon, \delta \in (0, 1) \), one can choose \( R > 0 \) such that, for any \( x > 0 \),

\[
P (B(x) \mid M > x) \geq 1 - \varepsilon/C
\]

where \( C = E^* e^{-\gamma \chi(x)} \) or \( C = \inf_{x > 0} E^* e^{-\gamma \chi(x)} \).

**Corollary 2.** For any \( \varepsilon > 0 \), as \( x \to \infty \),

\[
P \left( \frac{\tau(x)}{x} - \frac{1}{b} > \varepsilon \mid M > x \right) \to 0 \quad \text{and} \quad P \left( \sup_{k \leq \tau(x)} \left| \frac{S_k}{x} - \frac{kb}{x} \right| > \varepsilon \mid M > x \right) \to 0.
\]

**Proof of Theorem 2.** From Bayes formula,

\[
P (B(x) \mid M > x) = \frac{P (B(x), M > x)}{P (M > x)}.
\]

Here

\[
P (M > x) = e^{-\gamma x} E^* e^{-\gamma \chi(x)}
\]

and, similarly,

\[
P (B(x), M > x) = \ldots = E^* (e^{-\gamma S_{\tau(x)} \cdot I(B(x))}) I(\tau(x) < \infty)
\]

\[
\geq E^* (e^{-\gamma S_{\tau(x)}} \cdot I(B(x))) = e^{-\gamma x} \left( E^* e^{-\gamma \chi(x)} - E^* \left( e^{-\gamma \chi(x)} I(B) \right) \right).
\]

Here

\[
0 \leq E^* \left( e^{-\gamma \chi(x)} I(B) \right) \leq E^* I(B) = P^* (B) \leq \varepsilon,
\]

and the result follows.

**Exercise 5.** Prove Corollary 2.
4.4 Distributional asymptotics for the cycle maxima

Let \( \theta = \min \{ n \geq 1 : S_n \leq 0 \} \) be the first hitting time of the negative half-line. Note that \( \theta < \infty \) a.s. Let

\[
M_\theta = \max_{n \leq \theta} S_n
\]

**Theorem 3.** In the classical case, as \( x \to \infty \),

\[
P(M_\theta > x)e^{-\gamma x} \to (1 - Ee^{\gamma S_\theta}) E^*e^{-\gamma \chi(x)}.
\]

**Proof of Theorem 3.** Let \( \tilde{M} = \sup_{n \geq \theta} (S_n - S_\theta) \).

Then \( \tilde{M} \) does not depend on \( \theta, S_1, \ldots, S_\theta \) and, in particular, on \( M_\theta \) and \( S_\theta \). Further,

\[
P(M > x) = P(M_\theta > x) + P(M_\theta \leq x, S_\theta + \tilde{M} > x)
\]

\[
= P(M_\theta > x) + P(S_\theta + \tilde{M} > x) - P(M_\theta > x, S_\theta + \tilde{M} > x).
\]

Here

\[
0 \leq P(M_\theta > x, S_\theta + \tilde{M} > x) \leq P(M_\theta > x, \tilde{M} > x)
\]

\[
= P(M_\theta > x)P(\tilde{M} > x) = o(P(M_\theta > x)).
\]

Therefore

\[
P(M_\theta > x) \sim P(M > x) - P(M > x - \tilde{S}_\theta)
\]

\[
= \int_0^\infty P(-S_\theta \in dt)P(M \in (x, x + t])
\]

where \( \tilde{S}_\theta \) scindates in distribution with \( S_\theta \) and does not depend on \( M \). Here \( f(x) \sim g(x) \) means that \( f(x)/g(x) \to 1 \) as \( x \to \infty \).

From Theorem 2,

\[
P(M \in (x, x + t]) = e^{-\gamma x}E^*e^{-\gamma \chi(x)} - e^{-\gamma(x+t)}E^*e^{-\gamma \chi(x+t)}
\]

\[
= e^{-\gamma x} \left( E^*e^{-\gamma \chi(x)} - e^{-\gamma t}E^*e^{-\gamma \chi(x+t)} \right).
\]

Thus, by dominate convergence theorem,

\[
P(M_\theta > x) \sim e^{-\gamma x}E^*e^{-\gamma \chi(\infty)} \int_0^\infty P(-S_\theta \in dt) (1 - e^{-\gamma t})
\]

\[
= e^{-\gamma x}E^*e^{-\gamma \chi(\infty)} \left( 1 - Ee^{-\gamma S_\theta} \right).
\]
5 The heavy tail case

In the heavy tail case, we need some specific assumptions to derive the asymptotics for $P(M > x)$.

5.1 Definition and basic properties of long-tailed and subexponential distributions

5.1.1 Long-tailed distributions

Definition. A distribution $F$ with is long-tailed (belong to the class $\mathcal{L}$) if $F(x) > 0$ for all $x$ and $F(x+1) \sim F(x)$, i.e.

$$\frac{F(x+1)}{F(x)} \to 1 \quad \text{as} \quad x \to \infty.$$

Exercise 6. Show that if $F \in \mathcal{L}$, then, for any fixed $-\infty < y < \infty$,

$$\frac{F(x+y)}{F(x)} \to 1 \quad \text{as} \quad x \to \infty.$$

Exercise 7. Show that if $F \in \mathcal{L}$, then there exists a function $h(x) \geq 0$, $h(x) \to \infty$ as $x \to \infty$ such that

$$\frac{F(x+h(x))}{F(x)} \to 1 \quad \text{as} \quad x \to \infty.$$

Exercise 8. Show that if $F \in \mathcal{L}$ and if $F(x) \sim G(x)$, then $G \in \mathcal{L}$.

Examples of long-tailed distributions:

1. Pareto distribution:
   $$F(x) = x^{-\alpha} \quad \text{for} \quad x \geq 1.$$

2. Weibull distribution with parameters $\beta \in (0, 1)$ and $c > 0$:
   $$F(x) = e^{-cx^\beta} \quad \text{for} \quad x \geq 0.$$

3. Log-normal distribution:
   A r.v. $X$ has a log-normal distribution with parameters $(a, \sigma^2)$ if $X = e^Y$ where $Y$ has a distribution $N(a, \sigma^2)$. In other words, $X$ has a density function
   $$f(x) = \frac{1}{\sqrt{2\pi}x} e^{-\frac{(\log x - a)^2}{2\sigma^2}}.$$

5.1.2 Subexponential distributions

For any distribution $F$ on $[0, \infty)$ with unbounded support,

$$\liminf_{x \to \infty} \frac{F(x)}{F(x)} \geq 2.$$

Indeed, for i.i.d. r.v.’s $\xi_1$ and $\xi_2$ with common distribution $F$,

$$F \ast F(x) = P\{\xi_1 + \xi_2 > x\} \geq P\{\xi_1 > x\} \cup \{\xi_2 > x\} = 2P(\xi_1 > x) - P(\xi_1 > x)^2 = F(x)(2 - F(x)) \sim 2F(x).$$
Now we introduce a special class of distributions for which a limit of the latter ratio exists and equals 2.

**Definition.** Let $F$ be a distribution on $[0, \infty)$ with unbounded support. We say that $F$ is *subexponential* and write $F \in \mathcal{S}$ if

$$F = F(x) \sim 2F(x) \quad \text{as } x \to \infty.$$  

Equivalently, a non-negative random variable $\xi$ has a subexponential distribution if, for two independent copies $\xi_1$ and $\xi_2$ of $\xi$,

$$P\{\xi_1 + \xi_2 > x\} \sim 2P\{\xi > x\} \quad \text{as } x \to \infty.$$

Since the equivalence

$$P\{\max(\xi_1, \xi_2) > x\} = 1 - (1 - P\{\xi > x\})^2 \sim 2P\{\xi > x\}$$

always holds as $x \to \infty$, we can say that $F$ is a subexponential distribution iff

$$P\{\xi_1 + \xi_2 > x\} \sim P\{\max(\xi_1, \xi_2) > x\} \quad \text{as } x \to \infty.$$  

Moreover, since the event $\{\max(\xi_1, \xi_2) > x\}$ implies the event $\{\xi_1 + \xi_2 > x\}$, in subexponential case we have the following relation:

$$P\{\xi_1 + \xi_2 > x, \max(\xi_1, \xi_2) \leq x\} = o(P\{\xi > x\}) \quad \text{as } x \to \infty. \quad (9)$$

**Examples** of SE distributions: Pareto, Weibull, Log-normal.

**Exercise 9.** Show (by induction) that if $F \in \mathcal{S}$, then, for any $n = 3, 4, \ldots$, 

$$F^{*n}(x) \sim nF(x).$$

**Exercise 10.** Show that $\mathcal{S} \subset \mathcal{L}$, i.e. any subexponential distribution on the positive half-line is long-tailed.

**Proposition 1.** Let $F$ be a distribution on $[0, \infty)$ and $\xi_1, \xi_2$ be two independent random variables with distribution $F$. Then the following assertions are equivalent:

(i) $F$ is subexponential;

(ii) $F$ is long-tailed and, for every function $h(x) \to \infty$,

$$P\{\xi_1 + \xi_2 > x, \xi_1 > h(x), \xi_2 > h(x)\} = o(F(x)) \quad \text{as } x \to \infty; \quad (10)$$

(iii) there exists a function $h(x) < x/2$ such that $h(x) \to \infty$, $F(x - h(x)) \sim F(x)$ as $x \to \infty$, and (10) holds.

**Proof of (i)⇒(ii).** $F$ is long-tailed by Exercise 10. Note that if (10) is valid for some $h(x)$, then it follows for any $h_1 \geq h$. So without loss of generality we assume that $h(x) < x/2$. For $h(x) < x/2$, the probability of the event $B = \{\xi_1 + \xi_2 > x\}$ is equal to

$$P\{B, \xi_1 \leq h(x)\} + P\{B, \xi_2 \leq h(x)\} + P\{B, \xi_1 > h(x), \xi_2 > h(x)\}.$$

Since

$$P\{B, \xi_1 \leq h(x)\} = P\{B, \xi_2 \leq h(x)\}$$

$$= \int_0^{h(x)} F(x - y)F(dy) \sim \int_0^{h(x)} F(dy) \sim F(x), \quad (11)$$

$$P\{B, \xi_1 > h(x), \xi_2 > h(x)\}$$
the conclusion follows from the relation $P\{B\} \sim 2\overline{F}(x)$.

(ii)$\Rightarrow$(iii). By Exercise 7, if $F$ is long-tailed then there exists a function $h$ such that $h(x) \to \infty$ and $\overline{F}(x-h(x)) \sim \overline{F}(x)$ as $x \to \infty$.

(iii)$\Rightarrow$(i). Substituting (11) and (10) into decomposition of the probability of the event $B$, we get the desired equivalence $P\{B\} \sim 2\overline{F}(x)$. The proof is complete.

Let $\{\xi_n\}$ be a sequence of i.i.d. non-negative random variables with common distribution $F(B) = P\{\xi_1 \in B\}$. Put $S_n = \xi_1 + \ldots + \xi_n$.

**Proposition 2.** Assume that $F \in S$. Then, for any $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ such that, for any $x \geq 0$ and $n \geq 1$,

$$F^{**n}(x) \leq c(\varepsilon)(1 + \varepsilon)^n \overline{F}(x).$$

**Proof.** For $x_0 > 0$ and $k \geq 1$, put

$$A_k \equiv A_k(x_0) = \sup_{x > x_0} \frac{F^k(x)}{\overline{F}(x)}.$$

Take any $\varepsilon > 0$. It follows from (9) that there exists $x_0$ such that, for any $x > x_0$,

$$P\{\xi_1 + \xi_2 > x, \xi_2 \leq x\} \leq (1 + \varepsilon/2)\overline{F}(x).$$

We have the following decomposition

$$P\{S_n > x\} = P\{S_n > x, \xi_n \leq x - x_0\} + P\{S_n > x, \xi_n > x - x_0\} \equiv P_1(x) + P_2(x).$$

By the definition of $A_{n-1}$ and $x_0$, for any $x > x_0$,

$$P_1(x) = \int_0^{x-x_0} P\{S_{n-1} > x-y\} P\{\xi_n \in dy\} \leq A_{n-1} \int_0^{x-x_0} \overline{F}(x-y) P\{\xi_n \in dy\} = A_{n-1} P\{\xi_1 + \xi_n > x, \xi_n \leq x - x_0\} \leq A_{n-1}(1 + \varepsilon/2)\overline{F}(x). \tag{12}$$

Further,

$$P_2(x) \leq P\{\xi_n > x - x_0\} \leq L\overline{F}(x),$$

where

$$L = \sup_y \frac{\overline{F}(y-x_0)}{\overline{F}(y)}.$$

Since $F \in L$, $L$ is finite. Then, for any $x > x_0$,

$$P_2(x) \leq L\overline{F}(x) \tag{13}.$$

It follows from (12) and (13) that $A_n \leq A_{n-1}(1 + \varepsilon/2) + L$ for $n > 1$. Therefore, an induction argument yields:

$$A_n \leq A_1(1 + \varepsilon/2)^{n-1} + L \sum_{l=0}^{n-2} (1 + \varepsilon/2)^l \leq Ln(1 + \varepsilon/2)^{n-1}.$$
This implies the conclusion of the proposition.

Let us consider now some random time \( \tau \) with distribution \( p_n = \mathbb{P}\{\tau = n\}, \ n \geq 0 \) which is independent of \( \{\xi_n\} \). Then the distribution of the randomly stopped sum \( S_{\tau} \) is equal to

\[
\mathbb{P}\{S_{\tau} \in B\} = \sum_{n \geq 0} p_n F^n(B).
\]

**Theorem 4.** Assume \( F(0, \infty) = 1 \) and \( \mathbb{E}\tau < \infty \).

If \( F \in \mathcal{S} \) and \( \mathbb{E}(1 + \delta)^\tau < \infty \) for some \( \delta > 0 \), then

\[
\frac{\mathbb{P}\{S_{\tau} > x\}}{F(x)} \to \mathbb{E}\tau \quad \text{as} \ x \to \infty.
\] (14)

A proof of Theorem 4 follows from Proposition 2 and the dominated convergence theorem.
5.2 Asymptotics for $P(M > x)$ in the heavy tail case

**Definition.** A distribution $F$ on the whole line is subexponential if a distribution $FI(x > 0)$ is subexponential.

**Theorem 5.** Let $S_n = \sum_1^n \xi_i$ be a random walk with negative mean $-a$. Assume that a distribution $F^s$ with the tail

$$F^s(x) = \min(1, \int_x^\infty F(t)dt)$$

is subexponential. Then, as $x \to \infty$,

$$P(M > x) \sim \frac{1}{a} \int_x^\infty F(t)dt.$$

**Sketch of Proof.**

**Step 1.** Let

$$\eta = \min\{n \geq 1 : S_n > 0\} \leq \infty$$

and let a random variable $\psi$ have a distribution

$$P(\psi \leq x) = P(S_\eta \leq x | \eta < \infty).$$

Direct probabilistic calculations show that if $F^s \in \mathcal{L}$, then

$$\overline{G}(x) \sim \frac{p}{qa} F^s(x)$$

where $p = P(M = 0)$ and $q = 1 - p$.

Further, if $F^s$ subexponential distribution, then $G$ is subexponential too.

**Step 2.** We use the following fact (see, e.g., Feller).

A random variable $M$ has the same distribution with

$$\sum_1^{\nu_1} \psi_i$$

where $\psi$’s are i.i.d. and have the same distribution with $\psi$ and $\nu$ does not depend on them and has a distribution

$$P(\nu = k) = pq^k \quad \text{for} \quad k = 0, 1, 2, \ldots$$

Then we can apply Theorem 4:

$$P(M > x) \sim E\nu \overline{G}(x)$$

where

$$E\nu = \sum_{k \geq 1} P(\nu \geq k) = q + q^2 + \ldots = \frac{q}{p}.$$
5.3 "Typical" sample paths

Use again the SLLN:

\[ P \left( \frac{S_n}{n} \to -a \right) = 1, \]

or for any \( \epsilon \in (0, 1) \) and any \( \delta \in (0, 1) \), there exists \( R > 0 \) such that

\[ P(B) \equiv P \left( -R - n(a + \delta) \leq S_n \leq R - n(a - \delta) \right) \text{ for all } n \geq 1 - \epsilon. \quad (15) \]

For \( m = 0, 1, \ldots \), introduce events

\[ B_m = \{ R - n(a + \delta) \leq S_n \leq R - n(a - \delta) \text{ for all } n \leq m \}. \]

Clearly, \( B \subset B_m \) for all \( m \). Then

\[
P(M > x) \geq \sum_{n=1}^{\infty} P \left( B_{n-1} \cap \{ S_n > x \} \right)
\geq \sum_{n=1}^{\infty} P \left( B_{n-1} \cap \{ \xi_n > x + R + n(a + \delta) \} \right)
= \sum_{n=1}^{\infty} P(B_{n-1}) \overline{F}(x + R + n(a + \delta))
\geq P(B) \sum_{n=1}^{\infty} \overline{F}(x + R + n(a + \delta))
\geq (1 - \epsilon) \frac{1}{a + \delta} \int_{x-R-a-\delta}^{\infty} \overline{F}(t) dt
\sim \frac{1 - \epsilon}{a + \delta} \int_{x}^{\infty} \overline{F}(t) dt.
\]

Since \( \epsilon \) and \( \delta \) are arbitrarily small, the coefficient \( \frac{1 - \epsilon}{a + \delta} \) may be made as close to \( 1/a \) as possible. Compare this lower bound with the result of Theorem 5!

Further comment: since \( F^a \) is long-tailed, \( \overline{F}(x) = o(F^b(x)) \), and any finite number of first jumps are (asymptotically) negligible.
6 Further Problems

**Problem 1.** Let $\xi_n$ have a light tail and let

$$v = \sup\{ t : \varphi(t) \leq 1 \}.$$ 

Show that

$$\lim_{x \to \infty} \frac{1}{x} \log P(M > x) = -v.$$ 

*Hint:* Use the following monotonicity property: for two random walks $S_n = \sum_1^n \xi_i$, $n = 1, 2, \ldots$ and $\hat{S}_n = \sum_1^n \hat{\xi}_i$, $n = 1, 2, \ldots$, if $\xi_i \leq \hat{\xi}_i$ a.s. for all $i$, then $M \leq \hat{M}$ a.s. and, in particular, for all $x$,

$$P(M > x) \leq P(\hat{M} > x).$$ 

You may consider the following choices for $\hat{\xi}_i$:
(a) $\hat{\xi}_i = \max(\xi_i, K)$ for some $K$ – for the upper bound, and
(b) $\hat{\xi}_i = \min(\xi_i, K)$ for some $K$ – for the lower bound.

**Problem 2.** Give a proof of Theorem 4.
References


