Asymptotics for the maximum of a random walk with negative increments

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Large Deviations, Lecture 1

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This lecture (SF):
One-dimensional random walk, asymptotics in two main cases (the "classical" and the "heavy-tailed") out of five.

Next lecture (TK):
The LD Principle in $\mathbb{R}^1$.

Lectures 3-5 (AP):
General LD Principle and Applications.

There will be 4 additional hours in LD's in the afternoons – for lecturing an additional matherial, discussions, and exercises.

1 Notation and definitions

Consider a sequence $\xi, \xi_1, \xi_2, \ldots$ of independent identically distributed (i.i.d.) random variables (r.v.'s) with a common distribution function $F(x) = \mathbf{P}(\xi \leq x)$. Let

$$\begin{aligned} S_0 &= 0, \quad S_n = \sum_1^n \xi_i, \quad n \geq 1, \\ M_n &= \max_{0 \leq i \leq n} S_i, \quad M = \sup_{n \geq 0} M_n \equiv M_\infty. \end{aligned}$$

We are interested in the asymptotics for $\mathbf{P}(M > x)$ when $x \to \infty$.

First, there is $0 - 1$ Law:
Either $\mathbf{P}(M < \infty) = 1$ (in other words, M is finite a.s.) or $\mathbf{P}(M = \infty) = 1$ (in other words, M is infinite a.s.).

Further, if
the mean (expectation, drift) of ξ exists (i.e. $\mathbf{E}|\xi| < \infty$)
then
$\mathbf{P}(M < \infty) = 1$ if and only if $\mathbf{E}\xi = -a < 0$ (se, e.g., Feller [5]).

If the mean does not exist ($\mathbf{E}|\xi| = \infty$), then M is finite a.s. if and only if

$$\int_0^\infty \frac{x dF(x)}{m(x)} < \infty \tag{1}$$

where $m(x) = \mathbf{E}(\min(x, -\xi)$ – see Erickson ([4]).

Let $\varphi(t) = \mathbf{E}e^{t\xi}$ be a moment generating function of ξ.. Clearly, $\varphi(0) = 1$. If $\varphi(t)$ is finite for some $t > 0$, then it is finite for all $0 < u < t$ and convex in this interval since $\varphi''(u) = \mathbf{E}\left(\xi^2 e^{u\xi}\right) \geq 0$.

There are 5 cases of interest:
(I) the mean does not exist, but condition (1) holds;
(II) $a > 0$ and $\varphi(t) = \infty$ for all $t > 0$ (the case of heavy-tailed distribution);
(III) $a > 0$ and there exists $\gamma > 0$ such that

$$\varphi(\gamma) = 1 \tag{2}$$

and $\varphi'(\gamma) < \infty$ (this is the "classical case", here $\varphi'(\gamma)$ means the left derivative);
(IV) $a > 0$, (2) holds, but $\varphi'(\gamma) = \infty$;
(V) $a > 0$, $\varphi(t) < \infty$ for some $t > 0$, but (2) is violated.

We consider today only cases (II) and (III). For the asymptotics in other cases, see recent papers [3], [8], [6], and futher references therein.

2 Applications

(a) Queueing theory. Stationary (limiting) waiting time W in a single server queue coincides in distribution with the supremum M of an associated random walk.

Consider a single-server queue with i.i.d. interarrival times t_n and independent of them i.i.d. service times σ_n. Assume that customer 1 arrives in an empty system. Then its waiting time (before service) is $W_1 = 0$. Customer 2 arrives t_1 units of time later, and its waiting time is $W_2 = \max(0, \sigma_1 - t_1)$. By induction,

$$W_{n+1} = \max(0, W_n + \sigma_n - t_n).$$

Let $\xi_n = \sigma_n - t_n$. Note that

$$\begin{aligned} W_{n+1} &= \max(0, W_n + \xi_n) \\ &= \max(0, \max(0, W_{n-1} + \xi_{n-1}) + \xi_n) \\ &= \max(0, \xi_n, \xi_n + \xi_{n-1} + W_{n-1}) \\ &= ... \\ &= \max(0, \xi_n, \xi_n + \xi_{n-1}, \ldots, \xi_n + \ldots + \xi_1) \end{aligned}$$

Since $\{\xi_n\}$ is an i.i.d. sequence, W_{n+1} has the same distribution as

$$M_n = \max(0, \xi_1, \xi_1 + \xi_2, \ldots, \xi_1 + \ldots + \xi_n).$$

Therefore, as $n \to \infty$, distributions of W_n converge weakly to that of M.

One can use also the "Loynes scheme". Assume that i.i.d.r.v.'s ξ_n are defined for all $-\infty < n < \infty$ (this extension may be done using Kolmogorov's theorem). Clearly, W_{n+1} has the same distribution as

$$\widetilde{M}_n = \max(0, \xi_{-1}, \xi_{-1} + \xi_{-2}, \ldots, \xi_{-1} + \ldots + \xi_{-n})$$

Here we do not reverse numbering – we just shift all indices by $(-n-1)$. Then $\widetilde{M}_{n+1} \geq \widetilde{M}_n$ a.s. for all n and $\widetilde{M}_n$ converge monotonically to the supremum

$$\widetilde{M} = \sup_{n \geq 0} \sum_{i=1}^{n} \xi_{-j}$$

of a random walk with increments $\{\xi_{-n}\}$.

Remark 1. You will meet Loynes scheme again tomorrow during the 2nd lecture on stability methods.

(b) Risk theory. In the risk theory, $\mathbf{P}(M > x)$ may be interpreted as a probability of ruin (in the infinite time horizon).

In the risk theory setting, an insurance company has an initial capital x. Consequent i.i.d. claims have sizes σ_n, and independent of them i.i.d. "interclaim times" are t_n. There is a surplus process with rate $c > 0$. A ruin occurs if

$$\inf_n \{x + \sum_{i=1}^{n} (ct_i - \sigma_i)\} < 0,$$

or, equivalently,

$$\sup_{n \geq 0} \sum_{i=1}^{n} (\sigma_i - ct_i) > x$$

where $\{\sigma_i - ct_i\}_{i \geq 1}$ is an i.i.d. sequence.

3 Elements of renewal theory

A r.v. ξ has a lattice (arithmetic) distribution with *span* $h > 0$ if

$$\sum_{n=-\infty}^{\infty} \mathbf{P}(\xi = nh) = 1$$

and if h is a maximal number with this property. Clearly, any r.v. has either a lattice distrubion (with uniquely determined span) or a *non-lattice (non-arithmetic)* distribution.

Let $\xi, \xi_1, \xi_2, \ldots$ be an i.i.d. sequence of random variables with a finite positive mean $b = \mathbf{E}\xi$. For any x, let

$$\tau(x) = \min\{n \ : \ S_n > x\}$$

be the firrst hitting time of (x, ∞), and

$$\chi(x) = S_{\tau(x)} - x$$

an *overshoot* over x.

Here are some basic facts:
(i) since $b > 0$, $\tau(x)$ is a.s. finite for any x and, moreover, $\mathbf{E}\tau(x) < \infty$.
(ii) If ξ has a non-arithmetic distribution, distributions of $\chi(x)$ converge (weakly), as $x \to \infty$, to a proper *continuous* distribution of, say, random variable $\chi(\infty)$. This means that

$$\mathbf{E}g(\chi(x)) \to \mathbf{E}g(\chi(\infty)), \quad x \to \infty$$

for any bounded continuous function, or, equivalently,

$$\mathbf{P}(\chi(x) \le t) \to \mathbf{P}(\chi(\infty) \le t), \quad x \to \infty$$

for any t.
(iii) If ξ has an arithmetic distribution with span h, distributions of $\{\chi(nh), n = 1, 2, \ldots\}$ converge weakly, as $n \to \infty$, to a proper *discrete* distribution, of, say, random variable $\widetilde{\chi}(\infty)$.

Remark 2. See, e.g., Asmussen ([1]), Feller ([5]), or/and Gut ([7]) for more detailed treatment of renewal theory and, in particular, for limiting distributions of .$\chi(\infty)$ and $\widetilde{\chi}(\infty)$.

4 The classical case

4.1 Exponential change of measure (Cramer transform)

Note that

$$\varphi(\gamma) \equiv \mathbf{E} e^{\gamma\xi} = \int_{-\infty}^{\infty} e^{\gamma x} dF(x) = 1.$$

Introduce a new distribution function

$$dF^*(x) = e^{\gamma x} dF(x).$$

Thus, F^* is a probability distribution function.

For any n and for any bounded function $g : \mathbb{R}^n \to \mathbb{R}$, let $\mathbf{E}^* g(\xi_1, \dots, \xi_n)$ be an expectation with respect to distribution F^*:

$$\mathbf{E}^* g(\xi_1, \dots, \xi_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \dots, x_n) dF^*(x_1) \dots dF^*(x_n)$$

and, in particular,

$$\mathbf{P}^*(g(\xi_1, \dots \xi_n) \in A) = \int \cdots \int_{g^{-1}(A)} dF^*(x_1) \dots dF^*(x_n).$$

Then

$$\begin{aligned}
\mathbf{E} g(\xi_1, \dots, \xi_n) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \dots, x_n) dF(x_1) \dots dF(x_n) \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \dots, x_n) e^{-\gamma x_1} dF^*(x_1) \dots e^{-\gamma x_n} dF^*(x_n) \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\gamma \sum_{i=1}^n x_i} g(x_1, \dots, x_n) dF^*(x_1) \dots dF^*(x_n) \\
&= \mathbf{E}^* \left(e^{-\gamma S_n} g(\xi_1, \dots, \xi_n) \right)
\end{aligned}$$

and, similarly,

$$\mathbf{E} \left(g(\xi_1, \dots, \xi_n) e^{\gamma S_n} \right) = \mathbf{E}^* g(\xi_1, \dots, \xi_n).$$

In particular,

$$\mathbf{E}^* \xi = \int_{-\infty}^{\infty} x dF^*(x) = \int_{-\infty}^{\infty} x e^{\gamma x} dF(x) = \mathbf{E} \left(\xi e^{\gamma \xi} \right),$$

and this number is positive (!) Therefore

$$\mathbf{P}^*(\tau(x) < \infty) = 1 \tag{3}$$

for all x.

Exercise 1. In the classical case, the following are equivalent:
(a) F has a lattice distribution with span h;
(b) F^* has a lattice distribution with span h.

Exercise 2. Assume that $t > 0$ is such that $\varphi(t) < \infty$. Let

$$dF_t(x) = \frac{e^{tx} dF(x)}{\varphi(t)}$$

and denote by $\mathbf{E}_t$ the corresponding extectation operator. Find an expression for $E_t g(\xi_1, \ldots \xi_n)$ in this case.

4.2 The asymptotics for $\mathbf{P}(M > x)$

Let $\mathbf{I}$ be an indicator function, i.e., for any (random) event B,

$$\mathbf{I}(B) = 1 \quad \text{if the event} \quad B \quad \text{occurs and} \quad \mathbf{I}(B) = 0 \quad \text{otherwise.}$$

In particular, for any $x \in \mathbb{R}^n$ and for any measurable set $A \subset \mathbb{R}^n$,

$$\mathbf{I}(x \in A) = 1 \quad \text{if} \quad x \in A \quad \text{and} \quad \mathbf{I}(x \in A) = 0 \quad \text{otherwise.}$$

Let $\tau(x) = \min\{n \geq 0 \ : \ S_n > x\}$ if there exists such n, and $\tau(x) = \infty$ otherwise. Then, for $x > 0$,

$$\begin{aligned}
\mathbf{P}(M > x) &= \mathbf{P}(\tau(x) < \infty) \\
&= \sum_{n=1}^{\infty} \mathbf{P}(\tau(x) = n) \\
&= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathbf{I}((x_1, \ldots, x_n) \in A_n(x)) dF(x_1) \ldots dF(x_n)
\end{aligned}$$

where

$$A_n(x) = \{(z_1, \ldots z_n) \ : \ z_1 \leq x, \ldots, z_1 + \ldots + z_{n-1} \leq x, z_1 + \ldots + z_n > x\}.$$

Further,

$$\begin{aligned}
\mathbf{P}(M > x) &= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\gamma \sum_{i=1}^{n} x_i} \mathbf{I}((x_1, \ldots, x_n) \in A_n(x)) dF^*(x_1) \ldots dF^*(x_n) \\
&= \sum_{n=1}^{\infty} \mathbf{E}^* \left(e^{-\gamma S_n} \mathbf{I}(\tau(x) = n) \right) \\
&= \mathbf{E}^* \left(e^{-\gamma S_{\tau(x)}} \mathbf{I}(\tau(x) < \infty) \right) \\
&= \mathbf{E}^* e^{-\gamma S_{\tau(x)}}.
\end{aligned}$$

The very last equality follows from (3).
Since

$$S_{\tau(x)} = x + \chi(x),$$

we get finally

Theorem 1. *(1) In the classical case,*

$$\mathbf{P}(M > x) = e^{-\gamma x} \mathbf{E}^* e^{-\gamma \chi(x)}.$$

(2) **Cramer upper bound**. *Since $\chi(x) \geq 0$, for any $x > 0$,*

$$\mathbf{P}(M > x) \leq e^{-\gamma x}.$$

(3) From the basic renewal theory,
(a) if F is non-lattice, then

$$\mathbf{P}(M > x) e^{\gamma x} \to \mathbf{E}^* e^{-\gamma \chi(\infty)} \in (0, \infty), \quad x \to \infty, \tag{4}$$

(b) If F is lattice with span h, then

$$\mathbf{P}(M > nh) e^{\gamma nh} \to \mathbf{E}^* e^{-\gamma \tilde{\chi}(\infty)} \in (0, \infty), \quad n \to \infty. \tag{5}$$

Exercise 3. Show that (in both lattice and non-lattice cases) $\sup_{x\geq 0} \mathbf{P}(M > x)e^{\gamma x} < \infty$ and $\inf_{x\geq 0} \mathbf{P}(M > x)e^{\gamma x} > 0$.

Corollary 1. *In the classical case,*

$$\lim_{x\to\infty} \frac{1}{x} \log \mathbf{P}(M > x) = -\gamma.$$

Remark 3. This a a particular example of LDP (to be introduced in Lecture 2.)

4.3 "Typical" sample paths

Since $\mathbf{E}^*\xi = b > 0$, the SLLN says that

$$\mathbf{P}^*\left(\frac{S_n}{n} \to b\right) = 1, \quad \text{or} \quad \frac{S_n}{n} \to b \quad \mathbf{P}^* - \text{a.s.} \tag{6}$$

This also means that, for any $\varepsilon > 0$,

$$\mathbf{P}^*\left(\sup_{m\geq n}\left|\frac{S_n}{n} - b\right| > \varepsilon\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Exercise 4. Show that (6) is equivalent to the following:
for any $\varepsilon \in (0,1)$ and any $\delta \in (0,1)$, there exists $R > 0$ such that

$$\mathbf{P}^*(B) \equiv \mathbf{P}^*\left(-R + n(b-\delta) \leq S_n \leq R + n(b+\delta) \quad \text{for all} \quad n\right) \geq 1 - \varepsilon, \tag{7}$$

and also to the following:
for any $\varepsilon \in (0,1)$, there exist $R > 0$ and a sequence $\delta_n \to 0$ such that

$$\mathbf{P}^*(B) \equiv \mathbf{P}^*\left(-R + n(b-\delta_n) \leq S_n \leq R + n(b+\delta_n) \quad \text{for all} \quad n\right) \geq 1 - \varepsilon. \tag{8}$$

Let

$$B(x) = \{-R + n(b-\delta) \leq S_n \leq R + n(b+\delta) \quad \text{for all} \quad n \leq \tau(x)\}.$$

Theorem 2. *For any $\varepsilon, \delta \in (0,1)$, one can choose $R > 0$ such that, for any $x > 0$,*

$$\mathbf{P}(B(x) \mid M > x) \geq 1 - \varepsilon/C$$

where $C = E^* e^{-\gamma\chi(x)}$ or $C = \inf_{x\geq 0} E^* e^{-\gamma\chi(x)}$.

Corollary 2. *For any $\varepsilon > 0$, as $x \to \infty$,*

$$\mathbf{P}\left(\left|\frac{\tau(x)}{x} - \frac{1}{b}\right| > \varepsilon \mid M > x\right) \to 0 \quad \textit{and} \quad \mathbf{P}\left(\sup_{k\leq\tau(x)}\left|\frac{S_k}{x} - \frac{kb}{x}\right| > \varepsilon \mid M > x\right) \to 0.$$

Proof of Theorem 2. From Bayes formula,

$$\mathbf{P}(B(x) \mid M > x) = \frac{\mathbf{P}(B(x), M > x)}{\mathbf{P}(M > x)}.$$

Here

$$\mathbf{P}(M > x) = e^{-\gamma x} E^* e^{-\gamma\chi(x)}$$

and, similarly,

$$\begin{aligned}
\mathbf{P}(B(x), M > x) &= \dots \\
&= \mathbf{E}^*\left(e^{-\gamma S_{\tau(x)}} \cdot \mathbf{I}(B(x))\mathbf{I}(\tau(x) < \infty)\right) \\
&= \mathbf{E}^*\left(e^{-\gamma S_{\tau(x)}} \cdot \mathbf{I}(B(x))\right) \\
&\geq \mathbf{E}^*\left(e^{-\gamma S_{\tau(x)}} \cdot \mathbf{I}(B)\right) \\
&= e^{-\gamma x}\left(\mathbf{E}^* e^{-\gamma\chi(x)} - \mathbf{E}^*\left(e^{-\gamma\chi(x)}\mathbf{I}(\overline{B})\right)\right).
\end{aligned}$$

Here

$$0 \leq \mathbf{E}^*\left(e^{-\gamma\chi(x)}\mathbf{I}(\overline{B})\right) \leq \mathbf{E}^* I(\overline{B}) = \mathbf{P}^*(\overline{B}) \leq \varepsilon,$$

and the result follows.

Exercise 5. Prove Corollary 2.

4.4 Distributional asymptotics for the cycle maxima

Let $\theta = \min\{n \geq 1 \ : \ S_n \leq 0\}$ be the first hitting time of the negative half-line. Note that $\theta < \infty$ a.s. Let

$$M_\theta = \max_{n \leq \theta} S_n$$

Theorem 3. *In the classical case, as $x \to \infty$,*

$$\mathbf{P}(M_\theta > x)e^{\gamma x} \to \left(1 - \mathbf{E}e^{\gamma S_\theta}\right)\mathbf{E}^* e^{-\gamma \chi(\infty)}.$$

Proof of Theorem 3. Let

$$\widetilde{M} = \sup_{n \geq \theta}(S_n - S_\theta).$$

Then $\widetilde{M}$ does not depend on $\theta, S_1, \ldots, S_\theta$ and, in particular, on M_θ and S_θ. Further,

$$\begin{aligned}
\mathbf{P}(M > x) &= \mathbf{P}(M_\theta > x) + \mathbf{P}(M_\theta \leq x, S_\theta + \widetilde{M} > x) \\
&= \mathbf{P}(M_\theta > x) + \mathbf{P}(S_\theta + \widetilde{M} > x) - \mathbf{P}(M_\theta > x, S_\theta + \widetilde{M} > x).
\end{aligned}$$

Here

$$\begin{aligned}
0 &\leq \mathbf{P}(M_\theta > x, S_\theta + \widetilde{M} > x) \leq \mathbf{P}(M_\theta > x, \widetilde{M} > x) \\
&= \mathbf{P}(M_\theta > x)\mathbf{P}(\widetilde{M} > x) = o(\mathbf{P}(M_\theta > x)).
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbf{P}(M_\theta > x) &\sim \mathbf{P}(M > x) - \mathbf{P}(M > x - \widetilde{S}_{\widetilde{\theta}}) \\
&= \int_0^\infty \mathbf{P}(-S_\theta \in dt)\mathbf{P}(M \in (x, x+t])
\end{aligned}$$

where $\widetilde{S}_{\widetilde{\theta}}$ conicides in distribution with S_θ and does not depend on M. Here $f(x) \sim g(x)$ means that $f(x)/g(x) \to 1$ as $x \to \infty$.

From Theorem 2,

$$\begin{aligned}
\mathbf{P}(M \in (x, x+t]) &= e^{-\gamma x}\mathbf{E}^* e^{-\gamma\chi(x)} - e^{-\gamma(x+t)}\mathbf{E}^* e^{-\gamma\chi(x+t)} \\
&= e^{-\gamma x}\left(\mathbf{E}^* e^{-\gamma\chi(x)} - e^{-\gamma t}\mathbf{E}^* e^{-\gamma\chi(x+t)}\right).
\end{aligned}$$

Thus, by dominate convergence theorem,

$$\begin{aligned}
\mathbf{P}(M_\theta > x) &\sim e^{-\gamma x}\mathbf{E}^* e^{-\gamma\chi(\infty)} \int_0^\infty \mathbf{P}(-S_\theta \in dt)\left(1 - e^{-\gamma t}\right) \\
&= e^{-\gamma x}\mathbf{E}^* e^{-\gamma\chi(\infty)}\left(1 - \mathbf{E}e^{-\gamma S_\theta}\right).
\end{aligned}$$

5 The heavy tail case

In the heavy tail case, we need some spicific assumptinos to derive the asymptotics for $\mathbf{P}(M > x)$.

5.1 Definition and basic properties of long-tailed and subexponential distributions

5.1.1 Long-tailed distributions

Definition. A distribution F with is long-tailed (belong to the class $\mathcal{L}$) if $\overline{F}(x) > 0$ for all x and $\overline{F}(x+1) \sim \overline{F}(x)$, i.e.

$$\frac{\overline{F}(x+1)}{\overline{F}(x)} \to 1 \quad \text{as} \quad x \to \infty.$$

Exercise 6. Show that if $F \in \mathcal{L}$, then, for any fixed $-\infty < y < \infty$,

$$\frac{\overline{F}(x+y)}{\overline{F}(x)} \to 1 \quad \text{as} \quad x \to \infty.$$

Exercise 7. Show that if $F \in \mathcal{L}$, then there exists a function $h(x) \geq 0$, $h(x) \to \infty$ as $x \to \infty$ such that

$$\frac{\overline{F}(x+h(x))}{\overline{F}(x)} \to 1 \quad \text{as} \quad x \to \infty.$$

Exercise 8. Show that if $F \in \mathcal{L}$ and if $\overline{F}(x) \sim \overline{G}(x)$, then $G \in \mathcal{L}$.

Examples of long-tailed distributions:
(1) Pareto distribution:

$$\overline{F}(x) = x^{-\alpha} \quad \text{for} \quad x \geq 1.$$

(2) Weibull distribution with parameters $\beta \in (0,1)$ and $c > 0$:

$$\overline{F}(x) = e^{-cx^{\beta}} \quad \text{for} \quad x \geq 0.$$

(3) Log-normal distribution:
a r.v. X has a log-normal distribution with parameters (a, σ^2) if $X = e^Y$ where Y has a distribution $N(a, \sigma^2)$. In other words, X has a density function

$$f(x) = \frac{1}{\sqrt{2\pi}x} e^{-\frac{\log x - a)^2}{2\sigma^2}}.$$

5.1.2 Subexponential distributions

For any distribution F on $[0, \infty)$ with unbounded support,

$$\liminf_{x\to\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} \quad \geq \quad 2.$$

Indeed, for i.i.d.r.v.'s ξ_1 and ξ_2 with common distribution F,

$$\begin{aligned} \overline{F * F}(x) &= Pr(\xi_1 + \xi_2 > x) \geq \mathbf{P}\left(\{\xi_1 > x\} \bigcup \{\xi_2 > x\}\right) \\ &= 2\mathbf{P}(\xi_1 > x) - \mathbf{P}(\xi_1 > x)^2 = \overline{F}(x)(2 - \overline{F}(x)) \sim 2\overline{F}(x). \end{aligned}$$

Now we introduce a special class of distributions for which a limit of the latter ratio exists and equals 2.

Definition. Let F be a distribution on $[0, \infty)$ with unbounded support. We say that F is *subexponential* and write $F \in \mathcal{S}$ if

$$\overline{F * F}(x) \quad \sim \quad 2\overline{F}(x) \quad \text{as } x \to \infty.$$

Equivalently, a non-negative random variable ξ has a subexponential distribution if, for two independent copies ξ_1 and ξ_2 of ξ,

$$\mathbf{P}\{\xi_1 + \xi_2 > x\} \quad \sim \quad 2\mathbf{P}\{\xi > x\} \quad \text{as } x \to \infty.$$

Since the equivalence

$$\mathbf{P}\{\max(\xi_1, \xi_2) > x\} = 1 - (1 - \mathbf{P}\{\xi > x\})^2 \quad \sim \quad 2\mathbf{P}\{\xi > x\}$$

always holds as $x \to \infty$, we can say that F is a subexponential distribution iff

$$\mathbf{P}\{\xi_1 + \xi_2 > x\} \quad \sim \quad \mathbf{P}\{\max(\xi_1, \xi_2) > x\} \quad \text{as } x \to \infty.$$

Moreover, since the event $\{\max(\xi_1, \xi_2) > x\}$ implies the event $\{\xi_1 + \xi_2 > x\}$, in subexponential case we have the following relation:

$$\mathbf{P}\{\xi_1 + \xi_2 > x, \max(\xi_1, \xi_2) \le x\} \quad = \quad o(\mathbf{P}\{\xi > x\}) \quad \text{as } x \to \infty. \tag{9}$$

Examples of SE distributions: Pareto, Weibull, Log-normal.

Exercise 9. Show (by induction) that if $F \in \mathcal{S}$, then, for any $n = 3, 4, \ldots$,

$$\overline{F^{*n}}(x) \sim n\overline{F}(x).$$

Exercise 10. Show that $\mathcal{S} \subset \mathcal{L}$, i.e. any subexponential distribution on the positive half-line is long-tailed.

Proposition 1. *Let F be a distribution on $[0, \infty)$ and ξ_1, ξ_2 be two independent random variables with distribution F. Then the following assertions are equivalent:*

(i) *F is subexponential;*

(ii) *F is long-tailed and, for every function $h(x) \to \infty$,*

$$\mathbf{P}\{\xi_1 + \xi_2 > x,\ \xi_1 > h(x),\ \xi_2 > h(x)\} \quad = \quad o(\overline{F}(x)) \text{ as } x \to \infty; \tag{10}$$

(iii) *there exists a function $h(x) < x/2$ such that $h(x) \to \infty$, $\overline{F}(x - h(x)) \sim \overline{F}(x)$ as $x \to \infty$, and* (10) *holds.*

Proof of (i)$\Rightarrow$(ii). F is long-tailed by Exercise 10. Note that if (10) is valid for some $h(x)$, then it follows for any $h_1 \ge h$. So without loss of generality we assume that $h(x) < x/2$. For $h(x) < x/2$, the probability of the event $B = \{\xi_1 + \xi_2 > x\}$ is equal to

$$\mathbf{P}\{B,\ \xi_1 \le h(x)\} + \mathbf{P}\{B,\ \xi_2 \le h(x)\} + \mathbf{P}\{B,\ \xi_1 > h(x),\ \xi_2 > h(x)\}.$$

Since

$$\begin{aligned} \mathbf{P}\{B,\ \xi_1 \le h(x)\} &= \mathbf{P}\{B,\ \xi_2 \le h(x)\} \\ &= \int_0^{h(x)} \overline{F}(x-y)F(dy) \sim \overline{F}(x) \int_0^{h(x)} F(dy) \sim \overline{F}(x), \end{aligned} \tag{11}$$

the conclusion follows from the relation $\mathbf{P}\{B\} \sim 2\overline{F}(x)$.

(ii)⇒(iii). By Exercise 7, if F is long-tailed then there exists a function h such that $h(x) \to \infty$ and $\overline{F}(x - h(x)) \sim \overline{F}(x)$ as $x \to \infty$.

(iii)⇒(i). Substituting (11) and (10) into decomposition of the probability of the event B, we get the desired equivalence $\mathbf{P}\{B\} \sim 2\overline{F}(x)$. The proof is complete.

Let $\{\xi_n\}$ be a sequence of i.i.d. non-negative random variables with common distribution $F(B) = \mathbf{P}\{\xi_1 \in B\}$. Put $S_n = \xi_1 + \ldots + \xi_n$.

Proposition 2. *Assume that $F \in \mathcal{S}$. Then, for any $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ such that, for any $x \geq 0$ and $n \geq 1$,*

$$\overline{F^{*n}}(x) \quad \leq \quad c(\varepsilon)(1+\varepsilon)^n \overline{F}(x).$$

Proof. For $x_0 > 0$ and $k \geq 1$, put

$$A_k \equiv A_k(x_0) \quad = \quad \sup_{x > x_0} \frac{\overline{F^{*k}}(x)}{\overline{F}(x)}.$$

Take any $\varepsilon > 0$. It follows from (9) that there exists x_0 such that, for any $x > x_0$,

$$\mathbf{P}\{\xi_1 + \xi_2 > x,\ \xi_2 \leq x\} \quad \leq \quad (1+\varepsilon/2)\overline{F}(x).$$

We have the following decomposition

$$\begin{aligned} \mathbf{P}\{S_n > x\} &= \mathbf{P}\{S_n > x,\ \xi_n \leq x - x_0\} + \mathbf{P}\{S_n > x,\ \xi_n > x - x_0\} \\ &\equiv P_1(x) + P_2(x). \end{aligned}$$

By the definition of A_{n-1} and x_0, for any $x > x_0$,

$$\begin{aligned} P_1(x) &= \int_0^{x-x_0} \mathbf{P}\{S_{n-1} > x - y\}\mathbf{P}\{\xi_n \in dy\} \\ &\leq A_{n-1} \int_0^{x-x_0} \overline{F}(x-y)\mathbf{P}\{\xi_n \in dy\} \\ &= A_{n-1}\mathbf{P}\{\xi_1 + \xi_n > x,\ \xi_n \leq x - x_0\} \leq A_{n-1}(1+\varepsilon/2)\overline{F}(x). \end{aligned} \tag{12}$$

Further,

$$P_2(x) \quad \leq \quad \mathbf{P}\{\xi_n > x - x_0\} \leq L\overline{F}(x),$$

where

$$L \quad = \quad \sup_y \frac{\overline{F}(y - x_0)}{\overline{F}(y)}.$$

Since $F \in \mathcal{L}$, L is finite. Then, for any $x > x_0$,

$$P_2(x) \quad \leq \quad L\overline{F}(x). \tag{13}$$

It follows from (12) and (13) that $A_n \leq A_{n-1}(1+\varepsilon/2) + L$ for $n > 1$. Therefore, an induction argument yields:

$$A_n \quad \leq \quad A_1(1+\varepsilon/2)^{n-1} + L\sum_{l=0}^{n-2}(1+\varepsilon/2)^l \leq Ln(1+\varepsilon/2)^{n-1}.$$

This implies the conclusion of the proposition.

Let us consider now some random time τ with distribution $p_n = \mathbf{P}\{\tau = n\}$, $n \geq 0$ which is independent of $\{\xi_n\}$. Then the distribution of the randomly stopped sum S_τ is equal to

$$\mathbf{P}\{S_\tau \in B\} = \sum_{n \geq 0} p_n F^{*n}(B).$$

Theorem 4. *Assume* $F[0,\infty) = 1$ *and* $\mathbf{E}\tau < \infty$.
If $F \in \mathcal{S}$ *and* $\mathbf{E}(1+\delta)^\tau < \infty$ *for some* $\delta > 0$*, then*

$$\frac{\mathbf{P}\{S_\tau > x\}}{\overline{F}(x)} \quad \to \quad \mathbf{E}\tau \quad \textit{as } x \to \infty. \tag{14}$$

A proof of Theorem 4 follows from Proposition 2 and the dominated convergence theorem.

5.2 Asymptotics for $\mathbf{P}(M > x)$ in the heavy tail case

Definition. A distribution F on the whole line is subexponential if a distribution $F\mathbf{I}(x > 0$ is subexponential.

Theorem 5. *Let $S_n = \sum_1^n \xi_i$ be a random walk with negative mean $-a$. Assume that a distribution F^s with the tail*

$$\overline{F}^s(x) = \min\left(1, \int_x^\infty \overline{F}(t)dt\right)$$

is sunexponential. Then, as $x \to \infty$,

$$\mathbf{P}(M > x) \sim \frac{1}{a}\int_x^\infty \overline{F}(t)dt.$$

Sketch of Proof.

Step 1. Let

$$\eta = \min\{n \geq 1 \ : \ S_n > 0\} \leq \infty$$

and let a random variable ψ have a distribution

$$\mathbf{P}(\psi \leq x) = \mathbf{P}(S_\eta \leq x \mid \eta < \infty).$$

Direct probabilistic calculations show that if $F^s \in \mathcal{L}$, then

$$\overline{G}(x) \sim \frac{p}{qa}\overline{F}^s(x)$$

where $p = \mathbf{P}(M = 0)$ and $q = 1 - p$.
Further, if F^s subexponential distribution, then G is subexponential too.

Step 2. We use the following fact (see, e.g., Feller).
A random variable M has the same distribution with

$$\sum_1^\nu \psi_i$$

where ψ's are i.i.d. and have the same distribution with ψ and ν does not depend on them and has a distribution

$$\mathbf{P}(\nu = k) = pq^k \quad \text{for} \quad k = 0, 1, 2, \ldots$$

Then we can apply Theorem 4:

$$\mathbf{P}(M > x) \sim \mathbf{E}\nu\overline{G}(x)$$

where

$$\mathbf{E}\nu = \sum_{k\geq 1}\mathbf{P}(\nu \geq k) = q + q^2 + \ldots = \frac{q}{p}.$$

5.3 "Typical" sample paths

Use again the SLLN:

$$\mathbf{P}\left(\frac{S_n}{n} \to -a\right) = 1,$$

or for any $\varepsilon \in (0,1)$ and any $\delta \in (0,1)$, there exists $R > 0$ such that

$$\mathbf{P}(B) \equiv \mathbf{P}\left(-R - n(a+\delta) \le S_n \le R - n(a-\delta) \quad \text{for all} \quad n\right) \ge 1 - \varepsilon. \tag{15}$$

For $m = 0, 1, \ldots$, introduce events

$$B_m = \{R - n(a+\delta) \le S_n \le R - n(a-\delta) \quad \text{for all} \quad n \le m\}.$$

Clearly, $B \subset B_m$ for all m. Then

$$\begin{aligned}
\mathbf{P}(M > x) &\ge \sum_{n=1}^{\infty} \mathbf{P}\left(B_{n-1} \bigcap \{S_n > x\}\right) \\
&\ge \sum_{n=1}^{\infty} \mathbf{P}\left(B_{n-1} \bigcap \{\xi_n > x + R + n(a+\delta)\}\right) \\
&= \sum_{n=1}^{\infty} \mathbf{P}(B_{n-1}) \overline{F}(x + R + n(a+\delta)) \\
&\ge \mathbf{P}(B) \sum_{n=1}^{\infty} \overline{F}(x + R + n(a+\delta)) \\
&\ge (1-\varepsilon) \frac{1}{a+\delta} \int_{x-R-a-\delta}^{\infty} \overline{F}(t) dt \\
&\sim \frac{1-\varepsilon}{a+\delta} \int_{x}^{\infty} \overline{F}(t) dt.
\end{aligned}$$

Since ε and δ are arbitrarily small, the coefficient $\frac{1-\varepsilon}{a+\delta}$ may be made as close to $1/a$ as possible. Compare this lower bound with the result of Theorem 5 !

Further comment: since F^s is long-tailed, $\overline{F}(x) = o(\overline{F}^s(x)$, and any finite number of first jumps are (asymptotically) negligible.

6 Further Problems

Problem 1. Let ξ_n have a light tail and let

$$v = \sup\{t \ : \ \varphi(t) \le 1\}.$$

Show that

$$\lim_{x\to\infty} \frac{1}{x} \log \mathbf{P}(M > x) = -v.$$

Hint: Use the following monotonicity property: for two random walks $S_n = \sum_1^n \xi_i,\ n = 1, 2, \ldots$ and $\hat{S}_n = \sum_1^n \hat{\xi}_i,\ n = 1, 2, \ldots$, if $\xi_i \le \hat{\xi}_i$ a.s. for all i, then $M \le \hat{M}$ a.s. and, in particular, for all x,

$$\mathbf{P}(M > x) \le \mathbf{P}(\hat{M} > x).$$

You may consider the following choices for $\hat{\xi}_i$:
(a) $\hat{\xi}_i = \max(\xi_i, K)$ for some K – for the upper bound, and
(b) $\hat{\xi}_i = \min(\xi_i, K)$ for some K – for the lower bound.

Problem 2. Give a proof of Theorem 4.

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