Department of Mathematics **Uppsala University**
Stochastic processes **Spring 2014**

Assignment 1
5 Feb. 2014

1. Let $\mathscr{A}$ be the smallest σ-field of subsets of $\mathbb{R}^{[0,\infty)}$ containing cylinder sets. Show that the set $C[0,\infty)$ of continuous functions $f:[0,\infty)\to\mathbb{R}$ is not an element of $\mathscr{A}$.

2. If X, Y are random variables, defined on some probability space $(\Omega,\mathscr{F},\mathbb{P})$, and taking values in $C[0,1]$—the set of continuous functions $f:[0.1]\to\mathbb{R}$, equipped with the σ-algebra generated by the open sets in the topology of uniform convergence—show that equality of the finite-dimensional distributions of X and Y implies equality of their laws (i.e., that $\mathbb{P}\circ X^{-1}=\mathbb{P}\circ Y^{-1}$.)

3. Show that time-inversion of a standard Brownian motion gives a standard Brownian motion.

4. If W_t, $t\geq 0$, is a standard Brownian motion, show that the finite-dimensional distributions of $Z_t := e^{-t}W(e^{2t})$, $t\geq 0$, are invariant under time shifts. Also show that Z is Markov and Gaussian. (This Z is known as [the stationary version of an] Ornstein-Uhlenbeck process.)

5. Write down a complete proof of the fact that if $\mathscr{F}_t$, $t\geq 0$, is a right-continuous filtration, if A is a closed set, and if X_t, $t\geq 0$, has continuous paths, then $T_A := \inf\{t\geq 0:\ X_t\in A\}$ is a stopping time (with respect to $\mathscr{F}_t$).

6. Show that a family $\{X_t\}$ of random variables on $(\Omega,\mathscr{F},\mathbb{P})$ is uniformly integrable if and only if for all $\varepsilon>0$ there exists $\delta>0$ such that $\mathbb{E}(|X|;A)<\varepsilon$ whenever $\mathbb{P}(A)<\delta$. Use this to show that a martingale M_t, $t\geq 0$, is uniformly integrable when t is restricted on bounded sets.

7. Let W be a Brownian motion. Show that $M_t := W_t^3 - 3\int_0^t W_s\,ds$ and $N_t := W_t^3 - 3tW_t$ are both martingales. (The first martingale is, as we will see later, rather special because $A_t := 3\int_0^t W_s\,ds$ is differentiable in t and so has locally bounded variation on bounded intervals. We will later see that this A_t is, in some sense, unique.)

8. Exercise 3.16.

9. Exercise 3.17.