Department of Mathematics **Uppsala University**
Stochastic processes **Spring 2014**

Reminder: In order to pass this course, you must successfully solve 50% of the assignments. I assign problems roughly weekly, rather than waiting until the end of the term for a big assignment.

Assignment 3

16 Feb. 2014

1. Exercise 7.1.
2. Why did we use three intervals in the proof of non-differentiability of Brownian motion? What could we do with more intervals? Do Exercise 7.4.
3. Exercise 8.4.
4. (The Brownian bridge[1]) Let W be a standard Brownian motion. Let $0 < t_1 < \cdots < t_n < 1$. Compute the law $Q_{W(1)}(t_1, \ldots, t_n)$ of $(W(t_1), \ldots, W(t_n))$ conditional on $W(1)$, and show that it is Gaussian in $\mathbb{R}^n$ with a certain mean and covariance matrix and compute them. Let $V = (V(t), 0 \le t \le 1)$ be a stochastic process with the finite-dimensional distributions $Q_0(t_1, \ldots, t_n)$. Show that V is a Gaussian process with continuous paths.
5. (The Brownian meander[2]) Let W be a standard Brownian motion. Define $M_t := \inf_{0 \le s \le t} W_s$. Use the reflection principle in order to show that, for $\varepsilon > 0$, and $x > 0$,

$$\mathbb{P}(W_t \ge x - \varepsilon | M_t \ge -\varepsilon) = \frac{\mathbb{P}(|x + W_t| \le \varepsilon)}{\mathbb{P}(|W_t| \le \varepsilon)}.$$

   Then let $\varepsilon \downarrow 0$, to prove that the conditional density of W_t given the event $\{W_s \ge 0 \text{ for all } 0 \le s \le t\}$ is

$$\frac{x}{t} e^{-x^2/2t}, \quad x > 0.$$

[1] The standard Brownian bridge on the time interval $[0, 1]$ is informally defined as standard Brownian motion on the time interval $[0, 1]$ conditioned to hit 0 at time 1.

[2] The standard Brownian meander is informally defined as Brownian motion conditioned to stay positive.