Department of Mathematics **Uppsala University**
Stochastic processes **Spring 2014**

50% of the assignments must be solved correctly.
Deadline: Precisely 1 week = 168 hours after the end of the last lecture.

Assignment 4
23 Feb. 2014

1.* Let M be a local martingale, that is, there is a sequence T_n of finite stopping times, such that $T_n \uparrow \infty$, a.s. and $M_{t\wedge T_n}$, $t \geq 0$, is a martingale for all n. (A sequence T_n of such stopping times is called a localizing sequence.) Suppose that M is a continuous local martingale. Show that we can take $T_n := \inf\{t \geq 0 : |M_t| > n\}$ as a localizing sequence.

2.* Show that if M_t, $t \geq 0$, is a uniformly integrable martingale and S, T are a.s. finite stopping times with $S \leq T$, a.s., then $\mathbb{E}(M_T|\mathscr{F}_S) = M_S$.

3.* Does Proposition 9.5 use the usual conditions for the filtration?

4.* For a uniformly integrable martingale M, show that the family of random variables $\{M_T : T \text{ finite stopping time}\}$ is uniformly integrable.

5.** Let M_t, $t \geq 0$, be a square integrable martingale with quadratic variation $\langle M \rangle$. Given predictable function $H : R_+ \times \Omega \to \mathbb{R}$ define, as in Ch. 10,

$$\|H\|_2 := \sqrt{\mathbb{E}\int_0^\infty H_t^2 d\langle M \rangle_t}.$$

If $\|H\|_1 < \infty$, show that there exists a sequence $H^{(N)}$, $N \in \mathbb{N}$, of elementary predictable functions[1] such that $\|H - H^{(N)}\|_2 \to 0$, as $N \to \infty$. (Hint: truncate space and time and use hint near the end of pg. 67.)

6.** (If the integrands are deterministic then we may allow the parameter space to be "general"–not necessarily time!) Let $\mathscr{B}^d$ denote the Borel σ-field on $\mathbb{R}^d$. Define a family $\{W(B), B \in \mathscr{B}^d\}$ of random variables by requiring that every finite subfamily is normal (Gaussian) and that $\mathbb{E}W(B) = 0$, $\mathbb{E}W(A)W(B) = \lambda(A \cap B)$, with λ being the d-dimensional Lebesgue measure. By using arguments similar do the ones of Ch. 10, give a meaning to the integral $\int_{\mathbb{R}^d} \varphi(x)\, dW(x)$ where φ is a square-integrable function: $\int_{\mathbb{R}^d} \varphi(x)^2\, \lambda(dx) < \infty$, and by requiring that $\int_{\mathbb{R}^d} \mathbf{1}_B\, dW(x) := W(B)$, whenever $B \in \mathscr{B}^d$, $\lambda(B) < \infty$.

7.*** [Optional problem] Construct a function $f : [0, 1] \to \mathbb{R}$ such that (i) f is càdlàg , (ii) $f(t) - f(t-) \geq 0$ for all t, (iii) f has unbounded variation. Can such a function be increasing?

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[1] An elementary predictable function is a linear combination of finitely many step predictable functions; a step predictable function is a function: $\mathbb{R}_+ \times \Omega \to \mathbb{R}$ of the form $K(\omega)\mathbf{1}_{(a,b]}(t)$ where $K : \Omega \to \mathbb{R}$ is $\mathscr{F}_a$-measurable and bounded.

The following need not be submitted, but should be done in connection with your study of the construction of the stochastic integral for which I only provided a (hopefully illuminated) road map in class.

1. Suppose that $A : \mathbb{R} \to \mathbb{R}$ is a continuous function with locally bounded variation. Show, from first principles, that if $f \in C^1$, then
$$f(A(t)) - f(A(s)) = \int_s^t f'(A(u))\, dA(u).$$
If A is the Cantor function on $[0, 1]$, then compute $\int_0^x \exp(A(u))\, dA(u)$, for $0 \le x \le 1$.

2. Let $(\Omega, \mathscr{F})$ be a measurable space with a filtration $\mathscr{F}_t$, $t \ge 0$. Consider the class (call it class A) of bounded functions $F : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that $t \mapsto F(t, \omega)$ is left-continuous for all ω and $\omega \mapsto F(t, \omega)$ is $\mathscr{F}_t$–measurable for all t. Let $\mathscr{P}$ be the σ-field of subsets of $\mathbb{R}_+ \times \Omega$ generated by class A, i.e., $\mathscr{P} := \sigma(\text{class A})$. Next, consider the class (call it class B) of functions $G : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ of the form $G(t, \omega) = K(\omega)\mathbf{1}_{(a,b]}(t)$ where $0 \le a \le b$, and where $K : \Omega \to \mathbb{R}$ is $\mathscr{F}_a$–measurable and bounded. Let $\mathscr{Q} := \sigma(\text{class B})$. Show that $\mathscr{P} = \mathscr{Q}$, i.e., that every set in $\mathscr{P}$ is in $\mathscr{Q}$ and vice versa.

3. An elementary predictable process is a linear combination of finitely many functions from class B. Let H be an elementary predictable process and M a square-integrable continuous martingale. We defined $N_t := \int_0^t H_s dM_s$ in the obvious way. Show that this definition makes sense (if H is represented in two ways as linear combination of functions from class B, then the result is the same.) Show that N_t, $t \ge 0$, is a square-integrable continuous martingale. Therefore, there is a random variable N_∞ such that $(N_t, 0 \le t \le \infty)$ is a martingale. Show that $\langle N \rangle_t = \int_0^t H_s^2 d\langle M \rangle_s$, for all t, a.s., and that $\langle N \rangle_\infty := \lim_{t\to\infty} \langle N \rangle_t = \int_0^\infty H_s^2 d\langle M \rangle_s < \infty$, a.s., and, moreover, $\mathbb{E}N_\infty^2 = \langle N \rangle_\infty$.

4. (Exercise 9.5) Let $\mathscr{H}$ be the collection of processes $Y : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ which are jointly measurable in (t, ω) and such that $\mathbb{E} \sup_{0\le t<\infty} Y_t^2 < \infty$. Define $d(Y, Z) := \sqrt{\mathbb{E} \sup_{0\le t<\infty} (Y_t - Z_t)^2}$ and show that it is a pseudo-metric. Identify processes Y, Y' such that $d(Y, Y') = 0$ and let $\widetilde{\mathscr{H}}$ be the collection of equivalence classes. Show that d naturally extends to $\widetilde{\mathscr{H}}$. (All that is "standard machinery"!) Moreover (and this is the important thing) show that $\widetilde{\mathscr{H}}$ is complete (i.e., if Y_n is a Cauchy sequence, meaning that $d(Y_n, Y_m) \to 0$ as $(m, n) \to (\infty, \infty)$, then there is Y such that $d(Y_n, Y) \to 0$.

5. Explain how the Hilbert-space isometry mentioned in class is an alternative way of seeing what's going on in "taking limits" on pages 67-68.

6. Let M be a continuous square-integrable martingale and let T be a stopping time. Let $L_t := M_{t\wedge T}$. Show that L_t is a continuous square-integrable martingale and that $\langle L \rangle_t = \langle M \rangle_{t\wedge T}$, $t \ge 0$, a.s.

7. We have carefully defined $\langle M \rangle$ for a continuous square-integrable martingale. Have we defined it for any continuous local martingale too?

8. If X is a continuous semimartingale with Doob-Meyer decomposition $X = M + A$ why does it make sense to define $\langle X \rangle = \langle M \rangle$? Also, assuming M is not constant, why is the variation of X infinite on any interval of positive length?

9. (Bonus problem!) Let $A : \mathbb{R} \to \mathbb{R}$ be a bounded increasing function (not necessarily continuous). Let $A(-\infty) := \lim_{t\downarrow-\infty} A(t)$, and $A(+\infty) := \lim_{t\downarrow+\infty} A(t)$, both of which exist and finite by monotonicity and boundedness, respectively. Let $B : \mathbb{R} \to [-\infty, +\infty]$ be the "inverse" of A (keeping in mind that $\inf \varnothing := +\infty$, $\sup \varnothing := -\infty$): $B(x) := \inf\{t \in \mathbb{R} : A(t) > x\}$, $x \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}_+$ be measurable and non-negative function. Show that
$$\int_{\mathbb{R}} \varphi(t)\, dA(t) = \int_{A(-\infty)}^{A(+\infty)} \varphi(x)\, dx.$$
(The left side is a Lebesgue-Stieltjes integral; the right is a Lebesgue integral.)