50% of the assignments must be solved correctly.
Deadline: Precisely 1 week = 168 hours after the end of the last lecture.
I reserve the right to ask you to explain to me how you solved a problem.
Almost all the exercises below are easy. Exercise 3 (11.2) asks you to write down a *careful derivation* of the formula.

Assignment 5
3 Mar. 2014

1. Let S_n, $n \in \mathbb{Z}$, is a simple symmetric random walk in $\mathbb{Z}$. That is, $S_0 = 0$, $S_n = \xi_1 + \cdots + \xi_n$, sum of i.i.d. Bernoulli random variables, $\mathbb{P}(\xi_1 = \pm 1) = 1/2$. Let $M_n := \max(S_0, S_1, \ldots, S_n)$. (i) Show that $M_n - S_n$, $n \geq 0$, is Markov. This is (a partial) discrete-time analog of Theorem 14.2. (ii) What else can you squeeze out of these processes? Is $\sum_{k=1}^{n} (\operatorname{sgn} S_k)\xi_k$, $n \geq 0$, a simple symmetric random walk?

2. Let W_t be standard Brownian motion, $M_t = \sup_{s \leq t} W_s$. Use the fact that $|W|$ has the same law as $Q := M - W$ to show that $|W|$ is Markov. (Hint: Show that $Q_t = [Q_s + (W_t - W_s)] \vee \sup_{s \leq r \leq t}(W_r - W_s)$, for $s < t$.)

3. Exercise 11.2. Very important if you really want to understand Tanaka's formula and, therefore, the concept of local time as defined *through* Tanaka's formula.

4. Exercise 13.5. This is the more general version of Girsanov's theorem for continuous martingales.

5. Let $\mathbb{P}$ be a probability measure on $\mathbb{R}^n$ and let $X : \mathbb{R}^n \to \mathbb{R}^n$ be the identity function (regarded as a random variable with law $\mathbb{P}$). Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. (i) Show that there is a probability measure $\mathbb{Q}$ on $\mathbb{R}^d$ such that $Y := g(X)$ has law $\mathbb{P}$ under $\mathbb{Q}$. (Just take $\mathbb{Q}$ to be the law of $g^{-1}(X)$ under $\mathbb{P}$, and show that it works.) (ii) Consider a "discrete-time" Brownian motion $X := (X_1, X_2, \ldots, X_n)$, where $X_k := \xi_1 + \cdots + \xi_k$, where $\xi_1, \xi_2, \ldots$ are i.i.d. N(0,1). Let $\mathbb{P}$ be the law of X on $\mathbb{R}^n$. Let $Y = g(X) := (X_1 - c_1, X_2 - (c_1 + c_2), \ldots, X_n - \sum_{j=1}^{n} c_j)$, where $c_1, c_2, \ldots$ are real numbers. Let $\mathbb{Q}$ be defined via
$$\frac{d\mathbb{Q}}{d\mathbb{P}}(x_1, \ldots, x_n) = \exp\left(\sum_{j=1}^{n} c_j (x_j - x_{j-1}) - \frac{1}{2} \sum_{j=1}^{n} c_j^2 \right)$$
($x_0 := 0$), and show that Y has law $\mathbb{P}$ under $\mathbb{Q}$.

6. Exercise 10.5.

7. Let M_t, $t \geq 0$, be a positive martingale on $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$, with $\mathbb{E}M_t = 1$, and where $\mathscr{F} := \bigvee_{t \geq 0} \mathscr{F}_t$. Define, for each t, a probability measure $\mathbb{Q}_t$ on $(\Omega, \mathscr{F}_t, \mathbb{P})$ by $d\mathbb{Q}_t/d\mathbb{P} = M_t$, i.e., by $\mathbb{Q}_t(A) = \mathbb{E}[M_t; A]$, when $A \in \mathscr{F}_t$. Show that, for $s < t$, $\mathbb{Q}_s$ is the restriction of $\mathbb{Q}_t$ on $\mathscr{F}_s \subset \mathscr{F}_t$. Show that there is a probability measure $\mathbb{Q}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ such that $\mathbb{Q}$ restricted on $\mathscr{F}_t$ equals $\mathbb{Q}_t$.

8. In the proof of Girsanov's theorem, we showed that $U_t := W_t - \int_0^t H_r dr$ is martingale under $\mathbb{Q}$, where $\mathbb{Q}$ is the probability measure defined by (13.3)-(13.4). Complete the proof by showing that $U_t^2 - t$ is also a martingale with respect to $\mathbb{Q}$.

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9. Let W_t, $t \geq 0$, be a standard Brownian motion on its canonical probability space $\Omega = C[0,\infty)$ of continuous function $\omega : [0,\infty) \to \mathbb{R}$, and $W_t(\omega) = \omega(t)$ and let $\mathbb{P}$ be its law. Let $\mathscr{F}_t := \sigma(W_s, s \leq t)$. Let

$$Z_t^{\sigma,c} := \sigma W_t + ct, \quad t \geq 0,$$

for $\sigma \neq 0$, $c \in \mathbb{R}$. We think of $Z^{\sigma,c}$ as a random variable on Ω with values in $C[0,\infty)$. Let $\mathbb{P}^{\sigma,c}$ be the law of $Z^{\sigma,c}$. (i) Explain why Girsanov's theorem says that

$$\frac{d\mathbb{P}^{1,c}}{d\mathbb{P}}(\omega) = e^{-c\omega(t) - \frac{1}{2}c^2 t}, \quad \text{on } \mathscr{F}_t, \qquad \text{for all } t \in [0,\infty)$$

(part of the statement is that $\mathbb{P}^{1,c}$ is absolutely continuous with respect to $\mathbb{P}$ on $(\Omega, \mathscr{F}_t)$, but makes no claim about the absolute continuity of the two measures on $\mathscr{F}_\infty := \bigvee_{t\in[0,\infty)} \mathscr{F}_t$). (ii) Explain why, for any $\sigma \neq 1$,

$$\frac{d\mathbb{P}^{\sigma,c}}{d\mathbb{P}} \text{ does not exist on any } \mathscr{F}_t.$$

(Hint: Compute the quadratic variation of $Z^{\sigma,c}$.)