Department of Mathematics | **Uppsala University**
Stochastic processes | **Spring 2014**

Assignment 6

15 Mar. 2014

1. Show that 2-dimensional Brownian motion is neighborhood recurrent. Do this by using the harmonic function $h(x_1, x_2) = \log(x_1^2 + x_2^2)$ and immitate the computation in §21.1.

2. Exercise 21.7.

3. Let $T_r := \inf\{t \geq 0 : L_t^0 > r\}$, $r \geq 0$, where L_t^0, $t \geq 0$, is the local time of a standard Brownian morion at 0. Show that T_r, $r \geq 0$, is Markov by showing the stronger result that it has independent increments: if $r_0 < r_1 < \cdots < r_n$ then $T_{r_1} - T_{r_0}, \ldots, T_{r_n} - T_{r_{n-1}}$ are independent random variables. Moreover, show that the distribution of $T_{r+s} - T_s$ does not depend on s.

4. Consider the Markov chain X_n, $n = 0, 1, \ldots$, in $\mathbb{Z}_+$ with transition probabilities $P(x, y) = \mathbb{P}(X_{n+1} = y \mid X_n = x)$ given by $P(x, x+1) = p$, $P(x, x-1) = 1 - p$, for $x \geq 1$, and $P(0, 1) = 1$. We assume $p \in (0, 1)$. Let $A := \{\text{for all } n \geq 1\ X_n \neq 0\}$. Let $\mathbb{P}_x$ be the law of the Markov chain, on its canonical space, when $X_0 = x$. For all p, but espectially for $p < 1/2$, define the conditional probability $\mathbb{Q} = \mathbb{P}(\cdot \mid A)$ in such a way that, under $\mathbb{Q}$, the process is still Markov, and specify the transition probabilities under $\mathbb{Q}$. (Note: if $p < 1/2$, $\mathbb{P}(A) = 0$.)

5. Let W_t, $0 \leq t \leq 1$, be a standard Brownian motion, $W_0 = 0$ on its canonical space $\Omega = C[0, 1]$. Define the regular conditional probability $\mathbb{Q}(W(1), A) := \mathbb{P}(W \in A \mid W(1))$, $A \subset C[0, 1]$, Borel. Then let[1] $X_t := W_t - tW_1$, $0 \leq t \leq 1$. Show that the law of X under $\mathbb{P}$ is $\mathbb{Q}(0, \cdot)$.

6. Let A be a $d \times d$ real matrix and B a $d \times n$ real matrix. Let W_t, $t \geq 0$, be a standard Brownian motion in $\mathbb{R}^n$ and consider the stochastic equation

$$X_t = x_0 + \int_0^t AX_s ds + BW_t,$$

where x_0 is a given point in $\mathbb{R}^d$. Show that it has a unique pathwise solution and find the solution. Show that the solution is a Gaussian process X_t, $t \geq 0$, with values in $\mathbb{R}^d$, and compute the mean function $\mu(t) := \mathbb{E}X_t$ (the expectation taken componentwise) and the covariance function $c(s, t)$ defined as a $d \times d$ matrixed-valued function on $\mathbb{R}_+ \times \mathbb{R}_+$ with entries $c_{k\ell}(s, t) = \operatorname{cov}(X_k(s), X_\ell(t))$. Show that if all eigenvalues of A have strictly negative real part, then the distribution of X_t converges as $t \to \infty$. (Hint: The distribution of X_t is normal in $\mathbb{R}^d$ and so weak convergence is characterized by convergence of its parameters.)

7. To appreciate the use of the comparison theorem 24.5 in the uniqueness proof of a SDE under the Yamada-Watanabe condition, prove (or recall!) the following simple result for ordinary differential equations. Consider the ordinary differential equation

$$\frac{dx}{dt} = f(x),$$

[1]This X is called standard Brownian bridge. In a sense, Brownian motion is the limit of sampling from an urn with replacement. In the same sense, Brownian bridge is the limit of sampling from an urn with replacement.

where $f : \mathbb{R} \to \mathbb{R}$ satisfies:

$$|f(x_1) - f(x_2)| \leq \varphi(|x_1 - x_2|),$$

where φ is a strictly positive continuous function satisfying[2]

$$\int_0^\varepsilon \frac{du}{|\varphi(u)|} = \infty,$$

for some $\varepsilon > 0$. Show that if $x_1(t)$, $x_2(t)$ are two solutions of the ODE with $x_1(0) = x_2(0)$ then $x_1(t) = x_2(t)$ for all t.

8. Exercise 24.7. The point is this: if W_t, $t \geq 0$, is a d-dimensional standard BM starting from x_0, then $Y_t := |W_t|^2 = \sum_{j=1}^d W_{j,t}^2$ satisfies the SDE $dY_t = 2\sqrt{Y_t} dW_t + 2dt$, $Y_0 = |x_0|^2$.

9. Exercise 25.5. This is an extension of Lévy's forgery theorem. Roughly speaking, this theorem states that if you run a Brownian motion long enough then it will immitate your signature, a.s. The point of this exercise is to show that this theorem can be proved for other processes using the machinery of stochastic calculus.

10. Exercise 26.4. Let L_t^x, $t \geq 0$, $x \in \mathbb{R}$, be the local times for a standard BM W. Prove that, for fixed t_0, the process $(L_{t_0}^x,\ x \in \mathbb{R})$ does not have the Markov property. (What a contrast with Exercise 3 above!)

11. For a bare-hands proof of the Ray-Knight theorem, see the book of Peter Mörters and Yuval Peres, Brownian Motion, Cambridge University Press. And now that you finished this course, take a look at http://www2.math.uu.se/~takis/L/BMseminar/

[2] This is the classical Osgood condition. Can you interpret the condition physically?