On the extendibility of finitely exchangeable probability measures

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Abstract

We give necessary and sufficient conditions in order that a finite sequence $(X_1, \ldots, X_n)$ of exchangeable random elements in a fairly general space S be extendible to a longer finite or to an infinite exchangeable sequence. This is done by formulating the extendibility problem as the extension problem for certain bounded linear functionals on suitable normed spaces and by using the Hahn-Banach theorem and other functional and measure-theoretic techniques. We examine when such a finitely exchangeable random sequence is a mixture (with respect to a probability measure) of product measures and also study the preservation of the extendibility property under suitable limiting operations.

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1 Introduction and motivation

Exchangeability is one of the most important topics in probability theory with a wide range of applications. Most of the literature is concerned with exchangeability for an infinite sequence of random variables, in the sense that the distribution of the sequence does not change when we permute finitely many of the variables. David Aldous' survey [1] of the topic gives a very good overview. The basic theorem in the area is de Finetti's theorem stating that under suitable topological assumptions on the range of the variables (but see Dubins and Freedman [9] for a counterexample) an exchangeable sequence is a mixture of i.i.d. random variables [5, 11, 15, 17]. The mixing measure is a probability measure and is often referred to as the directing measure. Our paper is concerned with *finite exchangeability* of finite sequences of random elements of a fairly general space S, defined as invariance of its law under any of the $n!$ permutations of the variables. The following issues are

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well-known. First, finitely exchangeable sequences need not be mixtures of i.i.d. random variables. Second, finitely exchangeable sequences of length n may not be extendible to longer (finite or infinite) exchangeable sequences.

Regarding the first issue, there is the following, at first surprising, result: Let $(S, \mathscr{S})$ be an arbitrary measurable space and let $(X_1, \ldots, X_n)$ be an n-exchangeable random element of the n-fold product space $(S^n, \mathscr{S}^n)$, that is, for any permutation σ of $\{1, \ldots, n\}$, the law of $(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$ is the same as the law of $(X_1, \ldots, X_n)$. Then there is a *signed* measure ν on the space $\mathscr{P}(S)$ of probability measures on S such that

$$\mathbb{P}((X_1, \ldots, X_n) \in A) = \int_{\mathscr{P}(S)} p^n(A)\, \nu(dp), \quad A \in \mathscr{S}^n. \tag{1}$$

In this, we assume that $\mathscr{P}(S)$ is equipped with the standard σ-algebra generated by $p \mapsto p(A)$ for all $A \in \mathscr{S}$. This result was first proved by Jaynes [16] for the case where S is a 2-element set. See also the paper of Diaconis [7] for a clear discussion of the geometry behind this formula. The general case, i.e., for arbitrary measurable space $(S, \mathscr{S})$, was proved by Kerns and Székely [19]. We refer to (1) as the *finite exchangeability representation result*. In [20] we gave a short proof of the formula (also clarifying/correcting some subtle points) and established some notation which is also used in this current paper. We express the above result by saying that the law of an n-exchangeable random vector is a (signed) mixture of product measures. We note that such a signed measure must necessarily have $\nu(S) = 1$ and finite total variation. That is, if ν^+, ν^- are the positive and negative parts of ν, then $\nu^+(S) - \nu^-(S) = 1$, while $\|\nu\| := \nu^+(S) + \nu^-(S) < \infty$. Moreover, ν is not necessarily unique and, typically, it is not. (This is in sharp contrast to de Finetti's theorem for infinite sequences.) The signed measure ν is referred to as a directing signed measure. By "signed" we of course mean "not necessarily nonnegative". Indeed, it may very well be the case that ν is a nonnegative measure in which case ν is a probability measure; in such a case, we say that the law of $(X_1, \ldots, X_n)$ is a *true mixture* of product measures.

Regarding the second issue, i.e., that of extendibility, it is not difficult to see that a finitely exchangeable sequence may not be extendible. For instance, let P_n be the probability measure corresponding to sampling without replacement from an urn with $n \geq 2$ different items. Then there is no $N > n$ for which P_n is N-extendible. To be precise, if P_n is an n-exchangeable probability measure on S^n and $N \geq n$, we say that P_n is N-extendible if there is an N-exchangeable probability measure P_N on S^N such that $P_N(A \times S^{N-n}) = P_n(A)$, $A \in \mathscr{S}^n$. One of our goals in this paper is to give necessary and sufficient conditions for N-extendibility. If P_n is a true mixture of product measures then it is N-extendible for all $N \geq n$; P_n being a true mixture is only a sufficient condition. Regarding N-extendibility, no general criteria exist.

The extendibility problem has attracted much attention in the literature. When S is a finite set, then the finite extendibility problem reduces to the question of determining whether a point is located in a convex set in a multidimensional real vector space. Moreover, the extreme points of this convex set are known and the number of them is finite. This geometric point of view was initiated by de Finetti [6] and later followed-up by Crisma [2, 3], Spizzichino [26], and Wood [28]. The complexity of the problem increases when S is an infinite set. In this case, there is no general method characterizing finite extendibility, albeit a concept with useful applications. Following the aforementioned geometric point of view, Diaconis [7] and Diaconis and Freedman [8] show that, given a certain finite extendibility, one may bound the total variation distance between the given exchangeable sequence and

the closest (true) mixture of i.i.d. random variables, which helps establish easily de Finetti's theorem in the Hewitt-Savage setting [15].

Recall that an infinite random sequence $(X_1, X_2, \ldots)$ is called exchangeable if its law is invariant under permutations of finitely many variables. If P_n is an n-exchangeable probability measure, we say that it is infinitely extendible if there is an infinite exchangeable random sequence $(X_1, X_2, \ldots)$ such that $P_n(A) = \mathbb{P}((X_1, \ldots, X_n) \in A)$ for all $A \in \mathscr{S}^n$. The standard de Finetti's theorem says that if $(S, \mathscr{S})$ is a Borel space then the law of $(X_1, X_2, \ldots)$ is a true mixture of product measures, for a unique directing probability measure.

Regarding infinite extendibility, de Finetti [4] gives a condition for the binary case ($|S| = 2$) using characteristic functions. When S is a general measurable space, no criteria for finite or infinite extendibility exist. For either problem, only *necessary* conditions exist; see, e.g., Scarsini [24], von Plato [27], and Scarsini and Verdicchio [25]. For the case $S = \mathbb{R}$ and when variances exist, one simple necessary (but far from sufficient) condition for infinite extendibility is that any pair of variables have nonnegative covariance ([19, page 591]). This is certainly not sufficient (see Appendix A for a counterexample). As far as we know, the extendibility problem has been resolved in the literature by explicitly finding a longer exchangeable sequence which happens to be an extension. This is done, e.g., by Gnedin [14] for a class of exchangeable sequences and in [21] where Liggett, Steif and Tóth solve a particular problem of infinite extendibility within the context of statistical mechanical systems.

In this paper, we provide general and non-trivial criteria for the extendibility problem. First of all, we do not restrict ourselves to the finite S case. One of our concerns is to work with as general S as possible. There are topological restrictions to be imposed on S, arising from the methods of our proofs. We define certain linear operators via symmetrizing functionals on finite products of S and use functional analysis techniques to give necessary and sufficient conditions for extendibility. We also address the problem of infinite extendibility. The paper is organized as follows. In Section 2 we give some representation results on urn measures, some of which may be standard, some, perhaps, new (e.g., Proposition 1), and some are given in order to establish the basic elements of our paper. Urn measures are used in order to define certain symmetrizing operations (Section 3). These allow us to define, in Section 4, certain linear functionals and analyze their properties. Through these functionals we construct set functions which, under suitable conditions, turn out to be countably additive and nonnegative and which provide the required extensions. The first main result of this paper concerns a necessary and sufficient condition for finite extendibility and this is stated in Theorem 1 of Section 5. Regarding infinite extendibility, we also have a necessary and sufficient condition, stated in Theorem 2 of Section 6. Section 7 considers the problem of the finite exchangeability representation result (1) with true mixing, i.e., with ν being a probability measure. With a view towards applications, we establish some approximation results in Section 8. By means of a simple example, we show how the approximation results can be used. In Section 9 we take a closer look of what kind of signed directing measures are possible in representation (1) when S is a finite set. In particular, we prove that there is always one which has least total variation. A number of side issues are covered in the appendices.

2 Urn measures

Let n, N be positive integers, with $n \leq N$. We let $[n] := \{1, \ldots, n\}$ and denote by $\mathfrak{S}[n, N]$ the set of injective functions $\sigma : [n] \to [N]$. If $n = N$ then $\mathfrak{S}[n] := \mathfrak{S}[n, n]$ is the set of permutations of the first n positive integers. Let S be a set and S^n the n-fold Cartesian product of S by itself. Given $x = (x_1, \ldots, x_N) \in S^N$ and $\sigma \in \mathfrak{S}[n, N]$, with $n \leq N$, we let $\sigma x := (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. If $n = N$, then σx is obtained from x by permuting its coordinates according to σ. Similarly, the set $\mathfrak{S}[n, \mathbb{N}]$ is the set of injections of $[n]$ to the set $\mathbb{N}$ of positive integers. The set $\mathfrak{S}^{\uparrow}[n, \mathbb{N}]$ is the set of increasing injections from $[n]$ to $\mathbb{N}$.

Definition 1. *Let $\mathscr{S}$ be a σ-algebra on S, let $\mathscr{S}^n$ the product σ-algebra on S^n and $\mathscr{S}^{\mathbb{N}}$ the product σ-algebra on $S^{\mathbb{N}}$. A (probability) measure on $(S^n, \mathscr{S}^n)$ is said to be n-exchangeable if it is invariant under any $\sigma \in \mathfrak{S}[n]$. A probability measure on $(S^{\mathbb{N}}, \mathscr{S}^{\mathbb{N}})$ is said to be exchangeable (resp., contractable–see Kallenberg [18, Section 1.1]) if, for any n, it is invariant under any $\sigma \in \mathfrak{S}[n, \mathbb{N}]$ (resp., under any $\sigma \in \mathfrak{S}^{\uparrow}[n, \mathbb{N}]$).*

***Remark* 1.** The standard result for infinite sequences, is de Finetti's theorem (also known as de Finetti/Ryll-Nardzewski theorem) stating that, if S is a Borel space, then, given a contractable probability measure P on $(S^{\mathbb{N}}, \mathscr{S}^{\mathbb{N}})$, if η is the regular conditional probability of the first coordinate given the invariant σ-algebra $\mathscr{I}$, then the regular conditional probability $P(\cdot \mid \mathscr{I})$ is the infinite product η^{∞}, P-almost surely. This implies that P is a mixture of product measures and that P is exchangeable. Hence, for a Borel space, the generally stronger notion of contractability is equivalent to exchangeability. For the best proof of this result, see Kallenberg [18, Theorem 1.1]. Things are different for the finite-dimensional case not least because there is no obvious analog of $\mathscr{I}$. This is one of the reasons that a finite-dimensional analog of this result requires different machinery.

Note that

$$|\mathfrak{S}[n, N]| = (N)_n = N(N-1)\cdots(N-n+1).$$

For every $x \in S^N$, define the probability measure corresponding to having an urn with N items labelled $x_1, \ldots, x_N$ and picking $n \leq N$ of them at random without replacement. In other words, the probability measure

$$U^N_{n,x} := \frac{1}{(N)_n} \sum_{\sigma \in \mathfrak{S}[n,N]} \delta_{\sigma x}$$

is the distribution of the first n components $(Y_1, \ldots, Y_n)$ of a random element $(Y_1, \ldots, Y_N)$ of S^N obtained by permuting the components of $(x_1, \ldots, x_N)$ uniformly at random. Note that $U^N_{n,x}$ is a probability measure on S^n and is n-exchangeable. (One may also think of $K(x, B) \equiv U^N_{n,x}(B)$, $x \in S^N$, $B \subset S^n$, as a kernel [17, p. 20] from $x \in S^N$ to S^n. Later, see equation (7), we will integrate functions on S^n with respect to this kernel.) When sampling from an urn without replacement the only thing that matters is how many items with the same label we have in the urn. Thus, if

$$\varepsilon_x := \sum_{i=1}^{N} \delta_{x_i}, \quad x \in S^N,$$

then $U^N_{n,x} = U^N_{n,y}$ if $\varepsilon_x = \varepsilon_y$. We consider ε_x as a point measure (i.e., a measure with values in $\mathbb{Z}_+$) defined on some set T such that

$$\{x_1, \ldots, x_N\} \subset T \subset S.$$

We use the notation $\mathscr{N}(T)$ for the set of point measures on T, and $\mathscr{N}_k(T)$ those point measures having total mass k. If $\lambda, \nu \in \mathscr{N}(T)$, the inequality $\lambda \leq \nu$ is interpreted pointwise: $\lambda(B) \leq \nu(B)$ for all $B \subset T$, and the notation $(\nu)_\lambda$ stands for

$$(\nu)_\lambda := \prod_{a \in T} (\nu\{a\})_{\lambda\{a\}} = \prod_{a \in T} \nu\{a\}\,(\nu\{a\} - 1) \cdots (\nu\{a\} - \lambda\{a\} + 1),$$

in full analogy to the symbol $(N)_n$ for integers N, n. Given $\nu \in \mathscr{N}_N(T)$, we define the probability measure

$$u^N_{n,\nu} := \sum_{z \in T^n,\, \varepsilon_z \leq \nu} \frac{(\nu)_{\varepsilon_z}}{(N)_n}\, \delta_z. \tag{2}$$

We then have

Lemma 1. *For any positive integers $n \leq N$, and any $\nu \in \mathscr{N}_N(T)$, the measure $u^N_{n,\nu}$ is a probability measure on T^n which is n-exchangeable. Moreover, if $x \in S^N$ and if T is a subset of S containing the range of x,*

$$U^N_{n,x} = u^N_{n,\varepsilon_x}.$$

Proof. We prove the latter equality. Fix $x \in S^N$. Take $T \supset \{x_1, \ldots, x_N\}$ and let $z \in T^n$. Let σ be a random element of $\mathfrak{S}[n, N]$ with uniform distribution. Then the probability of $\{x_{\sigma(1)} = z_1, \ldots, x_{\sigma(n)} = z_n\}$ is zero unless all the z_i belong to $\{x_1, \ldots, x_N\}$ and unless $\varepsilon_z \leq \varepsilon_x$. In this case, the probability equals $(\varepsilon_x)_{\varepsilon_z}/(N)_n$. This proves that $U^N_{n,x} = u^N_{n,\varepsilon_x}$. Since any $\nu \in \mathscr{N}_N(T)$ can be written as $\nu = \varepsilon_x$ for some $x \in T^N$ (take x to have components in the support of ν with each point a repeated $\nu\{a\}$ times), it follows that $u^N_{n,\nu} = U^N_{n,x}$ and so $u^N_{n,\nu}$ is a probability measure. The n-exchangeability of $u^N_{n,\nu}$ follows then from the n-exchangeability of $U^N_{n,x}$. □

We now look a bit more closely at the set of n-exchangeable probability measures on a set T. Let $\nu \in \mathscr{N}_n(T)$. Define

$$T^n(\nu) := \{z \in T^n : \varepsilon_z = \nu\}.$$

The sets $T^n(\nu)$ where ν ranges over $\mathscr{N}_n(T)$ form a partition of T^n. In other words,

$$T^n = \bigcup_{\nu \in \mathscr{N}_n(T)} T^n(\nu),$$

and the union is disjoint. The equality is clear from the fact that every $z \in T^n$ is an element of $T^n(\varepsilon_z)$, by definition. Note that, regardless of the cardinality of T, the set $T^n(\nu)$ is finite with cardinality

$$|T^n(\nu)| = \binom{n}{\nu} \equiv \frac{n!}{(\nu)!},$$

where

$$(\nu)! = (\nu)_\nu = \prod_{a \in T} \nu\{a\}!.$$

Convention: Throughout the paper, we shall assume that the σ-algebra $\mathscr{S}$ is rich enough to contain all singletons. Any subset of S will, by convention, be endowed with the σ-algebra induced by this $\mathscr{S}$.

Lemma 2. *For $\nu \in \mathscr{N}_n(T)$, the uniform measure on $T^n(\nu)$ is*

$$u^n_{n,\nu} = \binom{n}{\nu}^{-1} \sum_{z \in T^n(\nu)} \delta_z. \tag{3}$$

Moreover, $u^n_{n,\nu}$ is the only n-exchangeable probability measure supported on $T^n(\nu)$.

Proof. Since $T^n(\nu)$ has size $\binom{n}{\nu}$, the uniform measure assigns mass $\binom{n}{\nu}^{-1}$ to each point in $T^n(\nu)$. So the right-hand side of (3) is the uniform measure on $T^n(\nu)$. To see that this is $u^n_{n,\nu}$, given by (2) with $N = n$, notice that if ν is a point measure on T with total mass n and if $z \in T^n$ then $\varepsilon_z \le \nu$ is equivalent to $\varepsilon_z = \nu$. The coefficient in front of δ_z in (3) equals $(\nu)_{\varepsilon_z}/(N)_n = (\nu)_\nu/(n)_n = \binom{n}{\nu}^{-1}$. Hence equality holds. □

An immediate consequence of this is that if T is countable and Q is an n-exchangeable probability measure on T^n then $Q(\cdot \cap T^n(\nu))$ is a multiple of $u^n_{n,\nu}$:

Corollary 1. *Let T be a countable set. If Q is a probability measure on T^n which is n-exchangeable then*

$$Q = \sum_{\nu \in \mathscr{N}_n(T)} Q(T^n(\nu))\, u^n_{n,\nu}.$$

A further consequence is that if T is countable then the probability measure $u^N_{n,\lambda}$ on T^n, where $N \ge n$, and $\lambda \in \mathscr{N}_N(T)$, can be expressed as

$$u^N_{n,\lambda} = \sum_{\nu \in \mathscr{N}_n(T)} a(\lambda, \nu)\, u^n_{n,\nu}, \tag{4}$$

in a unique way. The coefficients $a(\lambda, \nu)$ are nonnegative and $\sum_{\nu \in \mathscr{N}_n(T)} a(\lambda, \nu) = 1$, for all λ. Let $\mathrm{supp}(\lambda)$ be the set of points x with $\lambda\{x\} \ne 0$. The following says that (4) can be inverted.

Proposition 1. *Let $\lambda \in \mathscr{N}_n(T)$. If T is countable, then there exist (not necessarily unique) real numbers $c(\lambda, \nu)$, $\nu \in \mathscr{N}_N(T)$, such that $c_\lambda := [c(\lambda, \nu),\, \nu \in \mathscr{N}_N(T)]$ has only finite number of non-zero elements and*

$$u^n_{n,\lambda} = \sum_{\nu \in \mathscr{N}_N(T)} c(\lambda, \nu)\, u^N_{n,\nu}. \tag{5}$$

Moreover, c_λ can be chosen in a way that it depends only on the cardinality of $|\mathrm{supp}(\lambda)|$ and the multiplicities of the points in this support.

Proof. Let $T_\lambda = \mathrm{supp}(\lambda) = \{a_1, a_2, \ldots, a_k\}$ with $1 \le k \le n$. Put an order on T_λ such that $a_i < a_j$ if $1 \le i < j \le k$. This order induces another order on $\mathscr{N}_n(T_\lambda)$ such that any two points measures ν_1, ν_2 in $\mathscr{N}_n(T_\lambda)$ satisfy $\nu_1 > \nu_2$ if and only if there exists $1 \le r \le k$ such that

$$\nu_1(a_i) = \nu_2(a_i), \text{ for } i \le r-1; \quad \nu_1(a_r) > \nu_2(a_r).$$

For any $\nu \in \mathscr{N}_n(T_\lambda)$, let $\nu^N = \nu + (N-n)\delta_{a_k}$. Then $\nu^N \in \mathscr{N}_N(T_\lambda)$ and

$$\mathscr{M}_N(T_\lambda) := \{\nu^N : \nu \in \mathscr{N}_n(T_\lambda)\} \subset \mathscr{N}_N(T_\lambda).$$

We call ν the associated point measure in $\mathscr{N}_n(T_\lambda)$ of ν^N. It is obvious that each element in $\mathscr{M}_N(T_\lambda)$ has only one associated point measure in $\mathscr{N}_n(T_\lambda)$.
Note that $\mathscr{M}_N(T_\lambda)$ and $\mathscr{N}_n(T_\lambda)$ have the same finite cardinality and for any $\nu^N \in \mathscr{M}_N(T_\lambda)$, using (4), we have

$$u^N_{n,\nu^N} = \sum_{\mu \in \mathscr{N}_n(T_\lambda)} a(\nu^N, \mu)\, u^n_{n,\mu}. \tag{6}$$

The aim is to show that any $u^n_{n,\mu}$ with $\mu \in \mathscr{N}_n(T_\lambda)$ can be written as a linear combination of those elements u^N_{n,ν^N} with $\nu^N \in \mathscr{M}_N(T_\lambda)$. Consider the point measures $u^n_{n,\mu}$, $\mu \in \mathscr{N}_n(T_\lambda)$, and arrange them in decreasing order of μ, thus forming the row vector $\boldsymbol{V}_1$. Consider next u^N_{n,ν^N}, $\nu^N \in \mathscr{M}_N(T_\lambda)$, and let $\boldsymbol{V}_2$ be the row vector formed by arranging them in decreasing order of ν, the associated point measure in $\mathscr{N}_n(T_\lambda)$ of ν^N. Then (6) implies that there exists a square matrix A such that

$$\boldsymbol{V}_2 = \boldsymbol{V}_1 A.$$

If $\nu_1 > \nu_2$, for $\nu_1, \nu_2 \in \mathscr{N}_n(T_\lambda)$, we have $u^N_{n,\nu_2^N}(T^n_\lambda(\nu_1)) = 0$ and $u^N_{n,\nu_2^N}(T^n_\lambda(\nu_2)) > 0$. This implies that A is an upper triangular matrix with diagonal elements positive. Therefore A is invertible and $\boldsymbol{V}_1 = \boldsymbol{V}_2 A^{-1}$. This means that any probability measure $u^n_{n,\mu}$ with $\mu \in \mathscr{N}_n(T_\lambda)$ is a linear combination of elements u^n_{n,ν^N} with $\nu^N \in \mathscr{M}_N(T_\lambda)$. It is clear that A^{-1} does not depend on the location of the elements of T_λ. In particular, letting $\lambda = \mu$, we obtain (5). The linear representation is not unique, since $\{u^N_{n,\nu^N} : \nu \in \mathscr{N}_n(T_\lambda)\}$ was just a particular choice of linearly independent vectors. □

3 Symmetrizing operations

Definition 2 (symmetric functions and operators)**.** *A function $g : S^k \to \mathbb{R}$ is called symmetric if $g(x) = (g \circ \sigma)(x) := g(\sigma x)$, $x \in S^k$, for all permutations $\sigma \in \mathfrak{S}[k]$. Let $\Phi(S^k)$ be a linear space of real-valued functions closed under symmetrization of functions. A linear functional $\mathcal{L} : \Phi(S^k) \to \mathbb{R}$ is called symmetric if $\mathcal{L}g = \mathcal{L}(g \circ \sigma)$ for all $g \in \Phi(S^k)$ and all $\sigma \in \mathfrak{S}[k]$.*

Consider a function $g : S^n \to \mathbb{R}$ and let $N \geq n$. We can obtain a symmetric function of a larger number of variables in the following way:

$$(U^N_n g)(x) := U^N_{n,x} g \equiv \int_{S^n} g(y)\, U^N_{n,x}(dy) = \frac{1}{(N)_n} \sum_{\sigma \in \mathfrak{S}[n,N]} g(\sigma x), \quad x \in S^N. \tag{7}$$

There is a slight abuse of notation here, in that $U^N_{n,x} g$ denotes integration with respect to the probability measure $U^N_{n,x}$, but we also view U^N_n as a linear operator from functions on S^n into functions on S^N. It is clear that for any g on S^n, the function $U^N_n g$ is a symmetric function on S^N. We may refer to $U^N_n g$ as the N-symmetrized version of g. The case $N = n$ corresponds to the standard symmetrization operation: $(U^n_n g)(x) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}[n]} g(\sigma x)$, so $U^n_n g = g$ if and only if g is symmetric.

Let Φ_n be the linear space of all bounded and measurable functions $g : S^n \to \mathbb{R}$, equipped with the sup norm

$$\|g\| = \sup_{x \in S^n} |g(x)|,$$

and let $\overline{\Phi}_n$ be the subset of Φ_n consisting of symmetric functions. Notice that Φ_n is a Banach space and $\overline{\Phi}_n$ is a Banach subspace of Φ_n.

Lemma 3. *(i) U_n^N maps Φ_n into $\overline{\Phi}_N$ and $\|U_n^N g\| \leq \|g\|$.*
(ii) (projectivity property) For any $n \leq N \leq M$,

$$U_n^M = U_N^M \circ U_n^N.$$

(iii) The norm $\|U_n^N g\|$ decreases as N increases.

Proof. (i) is immediate from the definition of U_n^N. (iii) follows from (ii). So we just prove (ii). Consider an urn with M items labelled from some alphabet $T = \{x_1, \ldots, x_M\}$. Sample n items at random without replacement. This can be done in two stages: first sample N items at random without replacement and place them in a second urn, and then sample n items from the second urn. The two-stage procedure is exactly equivalent to the first one. □

Next comes a measurability observation which basically says that all (bounded measurable) symmetric functions of N variables are "obtained" as measurable functions of N-symmetrized functions of $n \leq N$ variables.

Proposition 2. *Let $U_n^N \Phi_n$ be the image of Φ_n under U_n^N and $\sigma(U_n^N \Phi_n)$ the σ-algebra generated by all functions in $U_n^N \Phi_n$. Then the set $\overline{\Phi}_N$ of all symmetric bounded measurable functions on $(S^N, \mathscr{S}^N)$ coincides with the set of all bounded measurable real-valued functions on $(S^N, \sigma(U_n^N \Phi_n))$.*

Proof. Let $\mathfrak{F}_N$ be the set of all bounded measurable functions on the space $(S^N, \sigma(\widetilde{\Phi}_N))$. On one hand, $U_n^N \Phi_n \subset \overline{\Phi}_N$, so $\mathfrak{F}_N \subset \overline{\Phi}_N$. On the other hand, let Π_N be the set of symmetric rectangles in S^N, i.e., sets of the form B^N, where $B \in \mathscr{S}$. Since $\overline{\Phi}_N$ is the set of all bounded $\sigma(\Pi_N)$-measurable functions on S^N, we need show that every $\sigma(\Pi_N)$-measurable function is contained in $\mathfrak{F}_N$. To this end, it suffices to show that $G := \mathbf{1}_{B^N} \in \mathfrak{F}_N$ for any $B \in \mathscr{S}$. Let $F = U_n^N \mathbf{1}_{B^n}$. This follows from the observation that G is a function of F because $F(x_1, \ldots, x_N) = 1$ if and only if $G(x_1, \ldots, x_N) = 1$. □

4 Extending functionals

We start addressing the extension problem (extending an exchangeable probability measure on S^n to an exchangeable probability measure on S^N for fixed $N \geq n$) by introducing and constructing "extending functionals" (see Definitions 4 and 5 below). In this section, we are concerned with the properties of these functionals, whereas in Section 5 we use them to derive a necessary and sufficient condition for N-extendibility.

Definition 3. *Let $n \leq N$ be positive integers. If $X = (X_1, \ldots, X_n)$ is a random element of S^n with exchangeable law, we say that it is N-extendible if there exists a random element $(Y_1, \ldots, Y_N)$ of S^N with exchangeable law such that $(X_1, \ldots, X_n) \overset{(d)}{=} (Y_1, \ldots, Y_n)$.*

Remark 2. Strictly speaking, we should refer to (n, N)-extendibility because the property depends on both indices. But we keep n fixed throughout the paper.

Note first the following characterization of N-extendibility, which is a rewriting of Proposition 1.4 in Spizzichino [26].

Proposition 3. *Let $X = (X_1, \dots, X_n)$ be a random element of S^n with exchangeable law. Then X is N-extendible if and only if there exists an exchangeable probability measure μ on $(S^N, \mathscr{S}^N)$ such that, for any $g \in \Phi_n$,*

$$\mathbb{E}g(X) = \int_{S^N} (U_n^N g)\, d\mu. \tag{8}$$

Proof. Suppose first that $X = (X_1, \dots, X_n)$ is N-extendible and let $Y = (Y_1, \dots, Y_N)$ have exchangeable law and is such that $(X_1, \dots, X_n) \overset{(d)}{=} (Y_1, \dots, Y_n)$. Then, for any $\sigma \in \mathfrak{S}[n, N]$, we have the equivalence in distribution: $(X_1, \dots, X_n) \overset{(d)}{=} (Y_{\sigma(1)}, \dots, Y_{\sigma(n)})$. Then (8) holds with μ being the law of Y. For the converse, assume (8) holds for any $g \in \Phi_n$. Then

$$\mathbb{E}g(X) = \frac{1}{(N)_n} \sum_{\sigma \in \mathfrak{S}[n,N]} \int_{S^N} g(\sigma x) \mu(dx),$$

and since μ is N-exchangeable all terms in the sum are equal. So

$$\mathbb{E}g(X) = \int_{S^N} g(x_1, \dots, x_n)\, \mu(dx_1 \cdots dx_n \cdots dx_N), \quad g \in \Phi_n.$$

Let $Y = (Y_1, \dots, Y_N)$ be a random element of S^N with law μ. Then the last display is equivalent to $(X_1, \dots, X_n) \overset{(d)}{=} (Y_1, \dots, Y_n)$. □

The following observation is crucial.

Proposition 4. *Let $X = (X_1, \dots, X_n)$ be a random element of S^n with n-exchangeable law. Then for every bounded measurable function $g : S^n \to \mathbb{R}$ such that $U_n^N g = 0_{S^N}$ (the identically zero function on S^N) we have $\mathbb{E}g(X) = 0$.*

Proof. Let $g : S^n \to \mathbb{R}$ be bounded and measurable. By exchangeability of the law of X, we have $g(X) \overset{(d)}{=} (U_n^n g)(X)$ and so

$$\mathbb{E}g(X) = \mathbb{E}(U_n^n g)(X).$$

From Lemma 1 and Proposition 1, we have, with $T \equiv T_X := \{X_1, \dots, X_n\}$,

$$(U_n^n g)(X) = U_{n,X}^n g = u_{n,\varepsilon_X}^N g = \sum_{\nu \in \mathscr{N}_N(T_X)} c(\varepsilon_X, \nu)\, u_{n,\nu}^N g. \tag{9}$$

Assume $U_n^N g = 0_{S^N}$. Then $(U_n^N g)(z) = U_{n,z}^N g = 0$ for all $z = (z_1, \dots, z_N) \in T_X^N$. But for every $\nu \in \mathscr{N}_N(T_X)$ we can find a $z = (z_1, \dots, z_N) \in T_X^N$ with $u_{n,\nu}^N = U_{n,z}^N$ (arrange the atoms of ν counting their multiplicities). Hence $u_{n,\nu}^N g = 0$ for all $\nu \in \mathscr{N}_N(T_X)$. From the last display, we get $(U_n^n g)(X) = 0$ and so $\mathbb{E}g(X) = 0$. □

This proposition allows us to give the following definition.

Definition 4 (primitive N-extending functional of an exchangeable measure)**.** *For any exchangeable $X = (X_1, \dots, X_n)$ we define its "primitive N-extending functional" $\mathcal{E}_n^N : U_n^N \Phi_n \to \mathbb{R}$ by*

$$\mathcal{E}_n^N : U_n^N g \mapsto \mathbb{E} g(X).$$

***Remark* 3.** That this definition makes sense follows from the following simple argument: if $U_n^N g = U_n^N h$ then $U_n^N (g - h) = 0$ and, by Proposition 4, $\mathbb{E} g(X) = \mathbb{E} h(X)$.

Now consider $U_n^N \Phi_n$ as a Banach subspace of Φ_N.

Lemma 4. *The primitive N-extending functional is linear and bounded.*

Proof. Linearity is immediate. To show boundedness we will show that there is a finite constant K such that $|\mathcal{E}_n^N (U_n^N g)| \le K \|U_n^N g\|$, for all $g \in \Phi_n$, i.e., that $|\mathbb{E} g(X)| \le K \|U_n^N g\|$. To this end, write $\mathbb{E} g(X) = \mathbb{E}(U_n^n g)(X)$ and then use (9) with $T_X = \{X_1, \dots, X_n\}$.

$$\begin{aligned} |\mathbb{E} g(X)| &\le \mathbb{E} |(U_n^n g)(X)| \\ &\le \mathbb{E} \sum_{\nu \in \mathscr{N}_N(T_X)} |c(\varepsilon_X, \nu)| \, \|u_{n,\nu}^N g\| \end{aligned}$$

But notice that

$$\|U_n^N g\| = \sup_{z \in S^N} |U_{n,z}^N g| = \sup_{\nu \in \mathscr{N}_N(T_X)} |u_{n,\nu}^N g|.$$

Hence $|\mathbb{E} g(X)| \le K \|U_n^N g\|$ with $K = \mathbb{E} \sum_{\nu \in \mathscr{N}_N(T_X)} |c(\varepsilon_X, \nu)|$. By Proposition 1, K is a finite number. □

Proposition 5. *If $X = (X_1, \dots, X_n)$ is n-exchangeable and N-extendible then $\|\mathcal{E}_n^N\| = 1$.*

Proof. Let μ be a probability measure on $(S^N, \mathscr{S}^N)$ forming an extension of the law of X. By Proposition 3,

$$|\mathbb{E} g(X)| = \left| \int_{S^N} (U_n^N g) \, d\mu \right| \le \|U_n^N g\|, \qquad \text{for all } g \in \Phi_n,$$

implying that $|\mathcal{E}_n^N f| \le \|f\|$ for all f in the domain of $\mathcal{E}_n^N$. Since $\mathcal{E}_n^N (U_n^N \mathbf{1}_{S^n}) = \mathcal{E}_n^N \mathbf{1}_{S^N} = 1$, it follows that $\|\mathcal{E}_n^N\| = 1$. □

Definition 5 (N-extending functional and N-extending set function)**.** *Given positive integers n, N, with $n \le N$, and an n-exchangeable $(X_1, \dots, X_n)$, an "N-extending functional" $\mathcal{L}_{n,N}$ is a symmetric bounded linear functional on Φ_N such that $\mathcal{L}_{n,N}$ and $\mathcal{E}_n^N$ agree on $U_n^N \Phi_n$ and $\|\mathcal{L}_{n,N}\| = \|\mathcal{E}_n^N\|$. The "$N$-extending set function" $\mu_{n,N} : \mathscr{S}^N \to \mathbb{R}$ corresponding to the N-extending functional $\mathcal{L}_{n,N}$ is defined by*

$$\mu_{n,N}(A) := \mathcal{L}_{n,N} \mathbf{1}_A, \quad A \in \mathscr{S}^N,$$

where $\mathbf{1}_A$ is the indicator function of A.

The Hahn-Banach theorem [10, Theorem 6.1.4] guarantees that an N-extending functional always exists.

Proposition 6. *For any exchangeable $X = (X_1, \dots, X_n)$, with values in an arbitrary measurable space, an N-extending functional always exists.*

Proof. First, consider $U_n^N \Phi_n$ as a Banach subspace of $\overline{\Phi}_N$, the space of symmetric bounded measurable functions on S^N. By the Hahn-Banach theorem, we can define a bounded linear functional $\mathcal{L}_n'^N$ on $\overline{\Phi}_N$ which agrees with $\mathcal{E}_n^N$ on $U_n^N \Phi_n$ and which has the same norm. Then let $\mathcal{L}_{n,N} : S^N \to \mathbb{R}$ be defined by

$$\mathcal{L}_{n,N} := \mathcal{L}_n'^N \circ U_N^N. \tag{10}$$

□

Therefore an N-extending set function $\mu_{n,N}(A)$, $A \in \mathscr{S}^N$, always exists. But, *a priori*, it is not clear that $\mu_{n,N}$ is a nonnegative set function, neither that it is countably additive. (In fact, the latter may not be true: see Theorem 1, Remark 5 and Appendix B.) Without additional assumptions, we can only guarantee the following:

Lemma 5 (basic properties of N-extending set functions)**.** *An N-extending set function $\mu_{n,N}$ is finitely additive with total mass 1 and is an exchangeable set function, in the sense that*

$$\mu_{n,N}(\sigma A) = \mu_{n,N}(A), \quad \sigma \in \mathfrak{S}[N],\ A \in \mathscr{S}^N.$$

Proof. Let $g := \mathbf{1}_{S^n}$ (the function that is identically equal to 1 on S^n.) Then $U_n^N g = \mathbf{1}_{S^N}$. By the definition of $\mathcal{E}_n^N$, we have $\mathcal{E}_n^N$ maps $U_n^N g$ to $\mathbb{E}g(X) = 1$. Since $\mathcal{L}_{n,N}$ agrees with $\mathcal{E}_n^N$ on $U_n^N \Phi_N$ we also have $\mathcal{L}_{n,N}(U_n^N g) = 1$. Therefore

$$\mu_{n,N}(S^N) = \mathcal{L}_{n,N} \mathbf{1}_{S^N} = \mathcal{L}_{n,N}(U_n^N \mathbf{1}_{S^n}) = 1.$$

Finite additivity follows from the linearity of $\mathcal{L}_{n,N}$. Exchangeability follows from the requirement that an N-extending functional is symmetric. □

The following simple (and standard) observations will be used several times in the sequel.

Lemma 6. *Let $\mathcal{L}$ be a linear operator from Φ_N to $\mathbb{R}$ such that $\mathcal{L}(\mathbf{1}_{S^N}) = 1$ and $\|\mathcal{L}\| = 1$. Then*
(i) $\mathcal{L}$ is monotone and $\mu(A) = \mathcal{L}(\mathbf{1}_A) \geq 0$ for any $A \in \mathscr{S}^N$.
(ii) For any function $f \in \Phi_N$ with $f \leq 1$ we have

$$\mathcal{L}\mathbf{1}_{f \leq t} \leq \frac{1 - \mathcal{L}f}{1 - t}, \quad 0 < t < 1.$$

Proof. (i) It is enough to prove that $\mathcal{L}$ maps nonnegative functions into nonnegative real numbers. Assume not, i.e., assume there is $g \in \Phi_N$, $g \geq 0$, such that $\mathcal{L}g < 0$. Let $h := \|g\| \mathbf{1}_{S^N} - g$. Then h is nonnegative and not identically equal to zero. Since $\mathcal{L}\mathbf{1}_{S^N} = 1$, we have $\mathcal{L}h = \|g\| - \mathcal{L}g > \|g\|$. On the other hand, since $h \geq 0$, we have $\|h\| = \sup_{x \in S^N}\{\|g\| - g(x)\} = \|g\| - \inf_{x \in S^N} g(x) \leq \|g\|$. Hence $\|h\| \leq \|g\| < \mathcal{L}h \leq \|\mathcal{L}\|\,\|h\|$, and, since h is not identically zero, this implies that $\|\mathcal{L}\| > 1$.
(ii) For any $0 < t < 1$, we have $f + (1-t)\mathbf{1}_{f \leq t} \leq \mathbf{1}_{S^N}$. Since $\mathcal{L}$ is monotone, $\mathcal{L}(f + (1 - t)\mathbf{1}_{f \leq t}) \leq \mathcal{L}\mathbf{1}_{S^N} = 1$. □

5 Properties of the N-extending set functions and a criterion for N-extendibility

So far, we have defined the notion of an N-extending functional and its corresponding N-extending set function under no special assumptions on S (other than $\mathscr{S}$ contains all singletons) or the probability distribution of $(X_1, \ldots, X_n)$. Some restrictions will be used in this section. We will assume that S is a locally compact Hausdorff space and let $\mathscr{S}$ be its Borel σ-algebra, i.e., the smallest σ-algebra containing all open sets. Note that this Borel σ-algebra contains all singletons. For any positive integer k, S^k is equipped with the product topology and product σ-algebra $\mathscr{S}^k$. We let $C_c(S^k)$ be the set of continuous functions $f : S^k \to \mathbb{R}$ vanishing outside some compact set. We will also assume that the law of X_1 is regular, meaning inner and outer regular. Inner regular means that

$$\mathbb{P}(X_1 \in B) = \sup\{\mathbb{P}(X_1 \in K) : K \subset B,\, K \text{ compact} \subset S\},$$

for all $B \in \mathscr{S}$. Outer regular means that

$$\mathbb{P}(X_1 \in B) = \inf\{\mathbb{P}(X_1 \in G) : G \supset B,\, G \text{ open} \subset S\},$$

for all $B \in \mathscr{S}$.

Our first result is that restricting an N-extending functional $\mathcal{L}_{n,N}$ on $C_c(S^N)$ will not reduce its norm, if $\|\mathcal{L}_{n,N}\| = 1$.

Proposition 7. *Fix positive integer n and assume that $X = (X_1, \ldots, X_n)$ is exchangeable and that the law of X_1 is regular. Let $N \geq n$ and let $\mathcal{L}_{n,N}$ be an N-extending functional in the sense of Definition 5. If S is a locally compact Hausdorff space and if the norm of $\mathcal{L}_{n,N}$ is 1 then its restriction on $C_c(S^N)$ has norm 1 also.*

Proof. Let $0 < \varepsilon < 1/n$. By the inner regularity of the law of X_1, we can find compact $K \subset S$ such that $\mathbb{P}(X_1 \in K) \geq 1 - \varepsilon$, and hence $\mathbb{P}(X \in K^n) \geq 1 - n\varepsilon > 0$. Let

$$g(x_1, \ldots, x_n) := \mathbf{1}_{K^n}(x_1, \ldots, x_n)$$

and consider its N-symmetrized version

$$(U_n^N g)(x_1, \ldots, x_N) = \frac{1}{(N)_n} \sum_{\sigma \in \mathfrak{S}[n,N]} \mathbf{1}_{K^n}(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

If $x = (x_1, \ldots, x_N) \in K^N$ then $\sigma x = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in K^n$ and so $(U_n^N g)(x) = 1$. If $x \notin K^N$ then there is integer $1 \leq i \leq N$ such that $x_i \notin K$. Hence the number of injections $\sigma \in \mathfrak{S}[n, N]$ such that $(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in K^n$ is a number $I < (N)_n$. So, for $x \notin K^N$, we have $(U_n^N g)(x) \leq I/(N)_n =: t < 1$. Since we assume that $\|\mathcal{L}_{n,N}\| = 1$, Lemma 6(ii) applied to the function $f = U_n^N g$ gives

$$\mathcal{L}_{n,N} \mathbf{1}_{f \leq t} \leq \frac{1 - \mathcal{L}_{n,N} f}{1 - t}.$$

But $f(x) \leq t \Leftrightarrow x \notin K^N$, so $\mathcal{L}_{n,N} \mathbf{1}_{f \leq t} = \mathcal{L}_{n,N}(1 - \mathbf{1}_{K^N})$. On the other hand, $\mathcal{L}_{n,N} f = \mathcal{E}_n^N(U_n^N g) = \mathbb{E} g(X) = \mathbb{P}(X \in K^n) \geq 1 - n\varepsilon$, and so

$$\mathcal{L}_{n,N} \mathbf{1}_{K^N} \geq 1 - \frac{n\varepsilon}{1 - t}.$$

By Urysohn's lemma [23, p. 39], there exists $F \in C_c(S^N)$ with $0 \le F \le 1$ and $F = 1$ on K^N. Using monotonicity of $\mathcal{L}_{n,N}$ again, we have

$$\sup_{h \in C_c(S^N),\, \|h\| \le 1} |\mathcal{L}_{n,N} h| \ge \mathcal{L}_{n,N} F \ge \mathcal{L}_{n,N} \mathbf{1}_{K^N} \ge 1 - \frac{n\varepsilon}{1-t}.$$

Letting $\varepsilon \downarrow 0$, we obtain that the norm of the restriction of $\mathcal{L}_{n,N}$ to $C_c(S^N)$ is at least 1. Since the norm of the restriction is always less than the norm of the operator, it follows that the norm of the restriction is actually equal to 1. □

For any positive integer k, let $\mathscr{R}_k$ be the collection of rectangles $B_1 \times \cdots \times B_k$, where $B_1, \ldots, B_k \in \mathscr{S}$, and let $\alpha(\mathscr{R}_k)$ be the algebra of subsets of S^k generated (=smallest class of sets closed under finite unions and complementations) by $\mathscr{R}_k$. The following result is used in the proof of the main theorem.

Proposition 8. *Assume that $X = (X_1, \ldots, X_n)$ is exchangeable. Let $\mathcal{L}_{n,N}$ be an N-extending functional and $\mu_{n,N}$ its corresponding N-extending set function (Definition 5). Suppose that $\|\mathcal{L}_{n,N}\| = 1$ and that $\mu_{n,N}$ is countably additive on $\alpha(\mathscr{R}_N)$. Then there exists a unique probability measure $\nu_{n,N}$ on $\mathscr{S}^N$ such that*
(i) $\nu_{n,N}$ agrees with $\mu_{n,N}$ on $\alpha(\mathscr{R}_N)$,
(ii) $\nu_{n,N}$ is an N-extension of the law of X.

Proof. By Lemma 5, $\mu_{n,N}$ is finitely additive with total mass 1. Since $\mathcal{L}_{n,N}$ has norm 1, by Lemma 6(i), $\mu_{n,N}$ is nonnegative. Since we assume that $\mu_{n,N}$ is also countably additive on $\alpha(\mathscr{R}_N)$, by the Carathéodory extension theorem, there exists a unique probability measure $\nu_{n,N}$ on $\mathscr{S}^N$ such that $\nu_{n,N} = \mu_{n,N}$ on $\alpha(\mathscr{R}_N)$. By Lemma 5, $\mu_{n,N}$ is exchangeable and therefore so is $\nu_{n,N}$. Define

$$\nu(A) := \int_{S^N} (U_n^N \mathbf{1}_A)\, d\nu_{n,N} = \frac{1}{(N)_n} \sum_{\sigma \in \mathfrak{S}[n,N]} \nu_{n,N}(\sigma^{-1} A), \quad A \in \mathscr{S}^n,$$

where $\sigma^{-1} A = \{x \in S^N : (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in A\}$. Then ν is a probability measure on $(S^n, \mathscr{S}^n)$. Suppose $A \in \alpha(\mathscr{R}_n)$. Then $\sigma^{-1} A \in \alpha(\mathscr{R}_N)$. So $\nu_{n,N}(\sigma^{-1} A) = \mu_{n,N}(\sigma^{-1} A)$. Therefore,

$$\begin{aligned}
\nu(A) &= \frac{1}{(N)_n} \sum_{\sigma \in \mathfrak{S}[n,N]} \mu_{n,N}(\sigma^{-1} A) \\
&= \frac{1}{(N)_n} \sum_{\sigma \in \mathfrak{S}[n,N]} \mathcal{L}_{n,N} \mathbf{1}_{\sigma^{-1} A} \\
&= \mathcal{L}_{n,N} \left(\frac{1}{(N)_n} \sum_{\sigma \in \mathfrak{S}[n,N]} \mathbf{1}_{\sigma^{-1} A} \right) = \mathcal{E}_n^N (U_n^N \mathbf{1}_A) = \mathbb{P}(X \in A).
\end{aligned}$$

Since ν agrees with $\mathbb{P}(X \in \cdot)$ on $\alpha(\mathscr{R}_n)$, it follows, again by the Carathéodory extension theorem, that $\nu = \mathbb{P}(X \in \cdot)$ on $\mathscr{S}^n$. Hence

$$\mathbb{P}(X \in A) = \int_{S^N} (U_n^N \mathbf{1}_A)\, d\nu_{n,N}, \quad A \in \mathscr{S}^n. \tag{11}$$

By Proposition 3, we have that X is N-extendible and one of the extensions has law $\nu_{n,N}$. As for uniqueness, if $\nu'_{n,N}$ is another N-exchangeable probability measure on $(S^N, \mathscr{S}^N)$ extending X and agreeing with $\mu_{n,N}$ on $\alpha(\mathscr{R}_N)$, then, necessarily, $\nu'_{n,N} = \nu_{n,N}$ on $\mathscr{S}^N$. □

***Remark* 4.** Proposition 8 does not say that the law of an exchangeable random element $X = (X_1, \dots, X_n)$ is uniquely N-extendible (this is not true, in general), but that the set of extensions which are constructed from the N-extending set functional $\mu_{n,N}$ is a singleton if the assumptions of this lemma are satisfied.

Our first main result is proved next.

Theorem 1. *Suppose that S is a locally compact Hausdorff space and $\mathscr{S}$ its Borel σ-algebra. Let $X = (X_1, \dots, X_n)$ be an exchangeable random element of S^n such that the law of X_1 is regular. Then $(X_1, \dots, X_n)$ is N-extendible if and only if $\|\mathcal{E}_n^N\| = 1$.*

Proof. The necessity is given by Proposition 5. For the sufficiency part, assume that $\mathcal{L}_{n,N}$ is any N-extending functional with norm $\|\mathcal{L}_{n,N}\| = 1$. (By Proposition 6 such an N-extending functional always exists.) To apply Proposition 8, we need to show that the corresponding $\mu_{n,N}$ is countably additive on $\alpha(\mathscr{R}_N)$. Since S is locally compact and Hausdorff, the Riesz representation theorem [23, p. 40] applied to the bounded linear functional $\mathcal{L}_{n,N}$ on $C_c(S^N)$, guarantees the existence of a unique regular measure λ on $(S^N, \mathscr{S}^N)$ such that

$$\mathcal{L}_{n,N} f = \int_{S^N} f \, d\lambda, \quad f \in C_c(S^N),$$

and

$$\lambda(G) = \sup\{\mathcal{L}_{n,N} f : f \in C_c(S^N),\, 0 \le f \le 1,\, \mathrm{supp}(f) \subset G\}, \qquad \text{open } G \subset S^N, \tag{12}$$

$$\lambda(E) = \inf\{\lambda(G) : G \text{ open},\, E \subset G\}, \qquad E \in \mathscr{S}^N, \tag{13}$$

where $\mathrm{supp}(f) = \{x : f(x) \neq 0\}$. By Proposition 7, λ is a probability measure. We shall prove that λ provides the announced N-extension. Fix $R = A_1 \times \cdots \times A_N \in \mathscr{R}_N$, and $0 < \varepsilon < 1/N^2$. By (13), there exists open $R_\varepsilon \subset S^N$ such that

$$R \subset R_\varepsilon, \quad \lambda(R) \le \lambda(R_\varepsilon) \le \lambda(R) + \varepsilon. \tag{14}$$

On the other hand, by the outer regularity of X_1, for each $i \le N$, pick open set $B_{i,\varepsilon} \subset S$ such that

$$A_i \subset B_{i,\varepsilon}, \qquad \mathbb{P}(X_1 \in A_i) \le \mathbb{P}(X_1 \in B_{i,\varepsilon}) \le \mathbb{P}(X_1 \in A_i) + \varepsilon, \quad 1 \le i \le N.$$

Consider now the set

$$F_{n,\varepsilon} := \{x \in S^n : \exists j \le n\ \exists i \le N\ x_j \in B_{i,\varepsilon} - A_i\} \in \mathscr{S}^n.$$

We then have

$$\mathbb{P}(X \in F_{n,\varepsilon}) \le \sum_{j \le n} \sum_{i \le N} \mathbb{P}(X_j \in B_{i,\varepsilon} - A_i) = \sum_{j \le n} \sum_{i \le N} \mathbb{P}(X_1 \in B_{i,\varepsilon} - A_i) \le nN\varepsilon. \tag{15}$$

Next, let

$$O_\varepsilon := R_\varepsilon \cap (B_{1,\varepsilon} \times \cdots \times B_{N,\varepsilon}).$$

Since $R \subset R_\varepsilon$, and $R = A_1 \times \cdots \times A_N \subset B_{1,\varepsilon} \times \cdots \times B_{N,\varepsilon}$, it follows that

$$R \subset O_\varepsilon.$$

Now let

$$G_{N,\varepsilon} := \{x \in S^N : \exists j \le N \ \exists i \le N \ x_j \in B_{i,\varepsilon} - A_i\} \in \mathscr{S}^N.$$

Then

$$O_\varepsilon - R \subset G_{N,\varepsilon}. \tag{16}$$

We aim to apply Lemma 6(ii) to the function

$$f = U_n^N \mathbf{1}_{S^n - F_{n,\varepsilon}}.$$

This is a function on S^N with $0 \le f \le 1$. We check that $\|f\| = 1$. To see this, notice that $\mathbb{P}(X_1 \in \bigcup_{i\le N}(B_{i,\varepsilon} - A_i)) \le N\varepsilon < 1$, which implies that $\mathbb{P}(X_1 \in S - \bigcup_{i\le N}(B_{i,\varepsilon} - A_i)) > 0$, and so

$$S - \bigcup_{i\le N}(B_{i,\varepsilon} - A_i) \ne \varnothing.$$

Therefore we can select an element $a \in S - \bigcup_{i\le N}(B_{i,\varepsilon} - A_i)$. Let $\boldsymbol{a} = (a, \dots, a) \in S^N$ and notice that $f(\boldsymbol{a}) = 1$. Hence $\|f\| = 1$. Thus Lemma 3.1 applies to f:

$$\mathcal{L}_{n,N}\mathbf{1}_{f\le t} \le \frac{1 - \mathcal{L}_{n,N} f}{1-t}, \text{ for any } 0 < t < 1.$$

If $x \in G_{N,\varepsilon}$, then choosing uniformly n items without replacement from N coordinates of x where at least one coordinate belongs to $\bigcup_{i\le N}(B_{i,\varepsilon} - A_i)$, the probability for that none of the chosen n-coordinates is located in $F_{n,\varepsilon}$ is not larger than $\binom{N-1}{n}/\binom{N}{n} = \frac{N-n}{N}$, that is, $f(x) \le \frac{N-n}{N}$. Hence

$$G_{N,\varepsilon} \subset \{x \in S^N : f(x) \le \frac{N-n}{N}\}.$$

Since $\mathcal{L}_{n,N}$ is monotone (Lemma 6(i)),

$$\mathcal{L}_{n,N}\mathbf{1}_{G_{N,\varepsilon}} \le \mathcal{L}_{n,N}\mathbf{1}_{f\le \frac{N-n}{N}} \le \frac{1 - \mathcal{L}_{n,N} f}{1 - \frac{N-n}{N}}.$$

By Definition 4 and (15)

$$\mathcal{L}_{n,N} f = \mathbb{P}(X \in S^n - F_{n,\varepsilon}) = 1 - \mathbb{P}(X \in F_{n,\varepsilon}) \ge 1 - nN\varepsilon.$$

Combining the last two displays and using (16) we get

$$\mathcal{L}_{n,N}\mathbf{1}_{O_\varepsilon - R} \le \mathcal{L}_{n,N}\mathbf{1}_{G_{N,\varepsilon}} \le N^2\varepsilon.$$

Hence

$$\mathcal{L}_{n,N}\mathbf{1}_R \le \mathcal{L}_{n,N}\mathbf{1}_{O_\varepsilon} \le \mathcal{L}_{n,N}\mathbf{1}_R + N^2\varepsilon.$$

Since O_ε is open, (12) gives

$$\lambda(O_\varepsilon) \le \mathcal{L}_{n,N}\mathbf{1}_{O_\varepsilon}.$$

Combining the last inequalities with the way that O_ε was selected in (14), we get

$$\lambda(R) \le \lambda(O_\varepsilon) \le \mathcal{L}_{n,N}\mathbf{1}_{O_\varepsilon} \le \mathcal{L}_{n,N}\mathbf{1}_R + N^2\varepsilon,$$

and, letting $\varepsilon \downarrow 0$,

$$\lambda(R) \le \mathcal{L}_{n,N}\mathbf{1}_R.$$

But (by definition) $\mu_{n,N}(R) = \mathcal{L}_{n,N}\mathbf{1}_R$. So $\lambda(R) \le \mu_{n,N}(R)$ for all rectangles $R \in \mathscr{R}_N$. Since $\lambda(S^N) = 1 = \mu_{n,N}(S^N)$, the last inequality is actually an equality. Thus $\lambda = \mu_{n,N}$ on $\mathscr{R}_N$. Since λ is countably additive on $\alpha(\mathscr{R}_N)$, so is $\mu_{n,N}$. By Proposition 8, the countably additive set function $\mu_{n,N}|_{\alpha(\mathscr{R}_N)}$ extends to a unique probability measure $\nu_{n,N}$ on $(S^N, \mathscr{S}^N)$ which is also an N-extension of the law of $(X_1, \dots, X_n)$. Necessarily, $\nu_{n,N} = \lambda$. □

***Remark* 5.** We point out a subtlety in the proof of of Theorem 1. We started with an n-exchangeable $(X_1, \dots, X_n)$ and assumed that its primitive N-extending functional $\mathcal{E}_n^N$ has norm 1. We took $\mathcal{L}_{n,N}$ to be any N-extending functional and defined the probability measure $\nu_{n,N} = \lambda$ on $(S^N, \mathscr{S}^N)$ via the Riesz representation of this functional restricted to $C_c(S^N)$. On the other hand, we have the N-extending set function $\mu_{n,N}$ defined by $\mu_{n,N}(A) = \mathcal{L}_{n,N}\mathbf{1}_A$, $A \in \mathscr{S}^N$. The bulk of the proof was devoted to proving that $\mu_{n,N} = \nu_{n,N}$ on $\mathscr{R}_N$. However, it is not necessarily the case that $\mu_{n,N} = \nu_{n,N}$ on the whole of $\mathscr{S}^N$. In fact, $\mu_{n,N}$ need not be a probability measure at all. See Appendix B for an example.

***Remark* 6.** Note that the proof of the theorem provides a (not unique in general) regular exchangeable probability measure on $(S^N, \mathscr{S}^N)$ which is an extension of $(X_1, \dots, X_n)$. This is not a specialty of this particular extension, because any N-extension of $(X_1, \dots, X_n)$ is regular (see Proposition 10 in Appendix C.)

Corollary 2. *Given an n-exchangeable probability measure P_n on S^n and $N > n$, we can formulate the criterion for N-extendibility as follows:*

$$\forall \varepsilon > 0 \ \forall g \in \Phi_n \ \exists a_1, \dots, a_N \in S \text{ such that}$$
$$\left| \int_{S^n} g(x)\, P_n(dx) \right| \le \frac{1+\varepsilon}{(N)_n} \left| \sum g(a_{\sigma(1)}, \dots, a_{\sigma(n)}) \right|, \tag{17}$$

where the sum is taken over all injections $\sigma : [n] \to [N]$.

The "$\forall g \in \Phi_n$" assumption in (17) can be weakened. One way to do this is as in the following proposition (which holds without any topological assumptions).

Proposition 9. *Let $\mathscr{A}$ denote an algebra of subsets of S^n generating $\mathscr{S}^n$. Let $\mathfrak{F}$ be the set of linear span of indicator functions with supports being elements of $\mathscr{A}$. Let $\widetilde{\mathfrak{F}} = \{U_n^N g : g \in \mathfrak{F}\}$. Then $\|\mathcal{E}_n^N\| = 1$ if and only if $\mathcal{E}_n^N$ restricted to $\widetilde{\mathfrak{F}}$ has norm 1.*

Proof. Note that the whole set S^n belongs to any algebra generating $\mathscr{S}^n$ and $\|U_n^N \mathbf{1}_{S^n}\| = \mathcal{E}_n^N(U_n^N \mathbf{1}_{S^n}) = 1$. So if $\|\mathcal{E}_n^N\| = 1$ then necessarily the norm of $\mathcal{E}_n^N$ restricted to $\widetilde{\mathfrak{F}}$ is 1. Assume next that the restriction of $\mathcal{E}_n^N$ on $\widetilde{\mathfrak{F}}$ has norm 1. We need to prove that $|\mathbb{E}[g(X)]| \le \|U_n^N g\|$ for any $g \in \Phi_n$. First of all, we consider $g = \mathbf{1}_A$ for $A \in \mathscr{S}^n$ and $A \notin \mathscr{A}$. For any $0 < \varepsilon < \|U_n^N \mathbf{1}_A\|$, there must exist $A_i \in \mathscr{A}, i = 1, 2, \dots$, such that $A \subset \bigcup_{i \ge 1} A_i$ and $\mathbb{P}(X \in A) \ge \mathbb{P}(X \in \bigcup_{i \ge 1} A_i) - \varepsilon$. Similarly, there exist $B_i \in \mathscr{A}$, $i = 1, 2 \dots$, such that $\bigcap_{i \ge 1} B_i \subset A$ and $\mathbb{P}(X \in A) \le \mathbb{P}(X \in \bigcap_{i \ge 1} B_i) + \varepsilon$. Note that, as $m \to \infty$,

$$U_n^N \mathbf{1}_{\cup_{i=1}^m A_i} \uparrow U_n^N \mathbf{1}_{\cup_{i \ge 1} A_i}, \quad U_n^N \mathbf{1}_{\cap_{i=1}^m B_i} \downarrow U_n^N \mathbf{1}_{\cap_{i \ge 1} B_i},$$

and also

$$\mathcal{E}_n^N(U_n^N \mathbf{1}_{\bigcup_{i=1}^m A_i}) = \mathbb{P}(X \in \cup_{i=1}^m A_i) \uparrow_{m \to \infty} \mathbb{P}(X \in \cup_{i \ge 1} A_i) = \mathcal{E}_n^N(U_n^N \mathbf{1}_{\cup_{i \ge 1} A_i}),$$
$$\mathcal{E}_n^N(U_n^N \mathbf{1}_{\cap_{i=1}^m A_i}) = \mathbb{P}(X \in \cap_{i=1}^m A_i) \downarrow_{m \to \infty} \mathbb{P}(X \in \cap_{i \ge 1} A_i) = \mathcal{E}_n^N(U_n^N \mathbf{1}_{\cap_{i \ge 1} A_i}).$$

Letting m tend to infinity we obtain

$$\frac{|\mathcal{E}_n^N(U_n^N \mathbf{1}_{\cup_{i \ge 1} A_i})|}{\|U_n^N \mathbf{1}_{\cup_{i \ge 1} A_i}\|} \le 1, \quad \frac{|\mathcal{E}_n^N(U_n^N \mathbf{1}_{\cap_{i \ge 1} B_i})|}{\|U_n^N \mathbf{1}_{\cap_{i \ge 1} B_i}\|} \le 1.$$

Note that

$$\|U_n^N \mathbf{1}_{\cap_{i\geq 1} B_i}\| \leq \|U_n^N \mathbf{1}_A\| \leq \|U_n^N \mathbf{1}_{\cup_{i\geq 1} A_i}\|,$$

$$|\mathcal{E}_n^N(U_n^N \mathbf{1}_{\cup_{i\geq 1} A_i})| - \varepsilon \leq |\mathcal{E}_n^N(U_n^N \mathbf{1}_A)| \leq |\mathcal{E}_n^N(U_n^N \mathbf{1}_{\cap_{i\geq 1} B_i})| + \varepsilon.$$

Since ε can be arbitrarily small, we get

$$\frac{|\mathcal{E}_n^N(U_n^N \mathbf{1}_A)|}{\|U_n^N \mathbf{1}_A\|} = \frac{|\mathbb{P}(X \in A)|}{\|U_n^N \mathbf{1}_A\|} \leq 1, \quad A \in \mathscr{S}^n.$$

To prove $\frac{|\mathbb{E}[g(X)]|}{\|U_n^N g\|} \leq 1$ for any $g \in \Phi_n$, we approximate g by simple functions which can also be bounded from above and from below by pointwise limits of functions in $\widetilde{\mathfrak{F}}$. The procedure is exactly the same as for $\mathbf{1}_A$. □

6 A criterion for infinite extendibility

We are seeking conditions that enable us to extend an n-exchangeable probability measure to an exchangeable probability measure on $S^{\mathbb{N}}$ in the standard sense (see Definition 1.) It seems natural to posit that X is N-extendible for all $N \geq n$ if and only if X is extendible to $S^{\mathbb{N}}$. One direction is clear: If X is an exchangeable random element of $S^{\mathbb{N}}$ then $(X_1, \ldots, X_n)$ is N-exchangeable for all $N \in \mathbb{N}$. But the other direction is not *a priori* obvious since we may have an N'-extension and an N''-extension of X, for some $n < N' < N''$, but the N''-extension may not be an extension of the N'-extension. Even worse, the N'-extension could be not extendible any more.

One may attempt to use Prohorov's theorem (since any finite extension is tight, see Remark 6) to prove the infinite extendibility by obtaining an appropriate infinite exchangeable sequence. This is possible in a metric space. But we work with a locally compact Hausdorff space (not necessarily metrizable). For a locally compact Hausdorff space there is a version of Prohorov's theorem [12] whose conclusion is stated in terms of continuous functions with compact support. This class of functions is not big enough for our purposes. Indeed, as in Proposition (3), the set of test functions required for N-extendibility is $\{U_n^N g : g \in \Phi_n\}$ and these functions are merely bounded. To bypass this difficulty, we will rely on a functional analytic approach (and use the Hahn-Banach theorem again) in the theorem below.

Theorem 2. *Assume that S is locally compact Hausdorff space and $X = (X_1, \ldots, X_n)$ an exchangeable random element of S^n such that X_1 has a regular law. The following are equivalent:*
(a) X is N-extendible for all $N \geq n$.
(b) There is a random element $Y = (Y_1, Y_2, \ldots)$ of $S^{\mathbb{N}}$ with exchangeable law such that $(X_1, \ldots, X_n) \overset{(d)}{=} (Y_1, \ldots, Y_n)$.

Proof. Fix n and assume (a) holds. Let Φ^* be the set of all bounded measurable real-valued functions $f(x_1, \ldots, x_N)$ on S^N for some $N \geq n$:

$$\Phi^* := \bigcup_{N \geq n} \Phi_N.$$

If $k < \ell$ then Φ_k is naturally embedded into Φ_ℓ: if $f \in \Phi_k$ then we can define $\widetilde{f} \in \Phi_\ell$ by $\widetilde{f}(x_1, \ldots, x_k, \ldots, x_\ell) := f(x_1, \ldots, x_k)$. We shall write $\Phi_k \subset \Phi_\ell$ for this embedding; this

should be read in the sense that the image of Φ_k under $f \mapsto \widetilde{f}$ is contained in Φ_ℓ. If $f \in \Phi^*$ then there is an $k \geq n$ such that $f \in \Phi_k$. The N-symmetrized version of f is $U_k^N f$ as in (7). Since $\Phi_k \subset \Phi_\ell$ for $k \leq \ell$, we can also consider $U_\ell^N f$ for $k \leq \ell \leq N$. We can easily see $U_\ell^N f = U_k^N f$.

If $f \in \Phi^*$ then there is $N \geq n$ such that $f \in \Phi_N$. We let i_f be the minimum such N. We next define a relation $\sim$ on Φ^* by

$$f \sim g \iff \exists N \; U_{i_f}^N f = U_{i_g}^N g, \quad f, g \in \Phi^*, N \geq \max\{i_f, i_g\}.$$

We see that $\sim$ is an equivalence relation. To check transitivity, suppose $f \sim g$ and $g \sim h$. Then $U_{i_f}^N f = U_{i_g}^N g$ and $U_{i_g}^M g = U_{i_h}^M h$ for some M and N. Letting $L := \max(M, N)$ and using the projectivity property (Lemma 3(ii)) we have $U_{i_f}^L f = U_{i_g}^L g$ and $U_{i_g}^L g = U_{i_h}^L h$, implying that $f \sim h$. In particular, notice that $f \sim U_k^N f$ for all $N \geq k \geq i_f$. From the discussion above and the projectivity property, we also see that

$$f \sim g \iff \exists k_0 \text{ so that if } k_0 \leq k \leq N \text{ then } U_k^N f = U_k^N g. \tag{18}$$

Let $[f]$ be the equivalence class of f:

$$[f] := \{g \in \Phi^* : \; g \sim f\},$$

and let $[\Phi^*]$ be the collection of equivalence classes:

$$[\Phi^*] := \{[f] : \; f \in \Phi^*\}.$$

We can easily check using (18) that if $f \sim f'$ and $g \sim g'$ then, for all $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g \sim \alpha f' + \beta g'$. Hence we can define

$$\alpha[f] + \beta[g] := [\alpha f + \beta g],$$

which means that $[\Phi^*]$ is a linear space with origin $[0]$, the set of functions equivalent to the identically zero function.

By Lemma 3(iii), the norm $\|U_{i_g}^N g\|$ decreases as N increases, so we attempt to define a norm on $[\Phi^*]$ by

$$\|[g]\| := \lim_{N \to \infty} \|U_{i_g}^N g\| = \inf_{N \geq i_g} \|U_{i_g}^N g\|.$$

First, it is clear that if $g \sim h$ then $\|[g]\| = \|[h]\|$; so $[g] \mapsto \|[g]\|$ is a well-defined function. To see that the triangle inequality holds, use the fact that $f \sim U_k^N f$ for all $N \geq k \geq i_f$ and (18). Let $g_1, g_2 \in \Phi^*$. Then we can choose k so that for $g_1 \sim U_k^N g_1$ and $g_2 \sim U_k^N g_2$, for all large N. Then $g_1 + g_2 \sim U_k^N g_1 + U_k^N g_2 = U_k^N (g_1 + g_2)$ and so

$$\begin{aligned}
\|[g_1 + g_2]\| &= \inf_{N \geq k} \|U_k^N (g_1 + g_2)\| \\
&\leq \inf_{N \geq k} (\|U_k^N g_1\| + \|U_k^N g_1\|) \\
&= \lim_{N \to \infty} (\|U_k^N g_1\| + \|U_k^N g_1\|) = \|[g_1]\| + \|[g_2]\|.
\end{aligned}$$

To check positive definiteness we prove the following:

Lemma 7. *If $g \in \Phi^*$ has $\|[g]\| = 0$ and if $f \in \Phi_N$ is a symmetric function such that $f \sim g$ then f is identically zero.*

Proof. Let $f \in \Phi_N$ be symmetric. Let p be a probability measure on $(S, \mathscr{S})$. Then

$$p^N(f) := \int_{S^N} f \, dp^N = \int_{S^{N+M}} (U_N^{N+M} f) \, dp^{N+M}.$$

So then $\|p^N(f)\| \leq \|U_N^{N+M} f\|$. Note that $\lim_{M\to\infty} \|U_N^{N+M} f\| = \|[f]\|$. Let $g \in \Phi^*$ have $\|[g]\| = 0$ and assume $f \sim g$. Then $\|[f]\| = \|[g]\| = 0$. Hence

$$p^N(f) = 0, \qquad \text{for any probability measure } p.$$

By (1),

$$\mathbb{E}[f(Y)] = 0, \qquad \text{for any } N\text{−exchangeable } Y = (Y_1, \ldots, Y_N).$$

Together with the symmetry of f, this implies that f is identically 0. □

Suppose now that $\|[g]\| = 0$. Then $g \sim U_k^N g$ for some k and N. By Lemma 7, $U_k^N g$ is identically zero. Thus $[g] = [0]$. We have thus shown that $[\Phi^*]$ is a normed linear space. Consider now

$$[\Phi_n] := \{[f] : f \in \Phi_n\}.$$

Clearly, $[\Phi_n]$ is a linear subspace of $[\Phi^*]$. It is normed by the same norm. We now attempt to define a linear functional

$$\mathcal{L}^0 : [\Phi_n] \to \mathbb{R}$$

based on the following observation. If $f, g \in \Phi_n$ have $\mathbb{E}f(X_1, \ldots, X_n) \neq \mathbb{E}g(X_1, \ldots, X_n)$ Then, by Proposition 4, $U_n^N f \neq U_n^N g$ for all $N \geq n$. So then $f \not\sim g$ and so $[f] \neq [g]$. Therefore,

$$\mathcal{L}^0 : [g] \mapsto \mathbb{E}g(X_1, \ldots, X_n) \tag{19}$$

is a function; in fact, a linear function from $[\Phi_n]$ into $\mathbb{R}$. Consider the norm of $\mathcal{L}^0$:

$$\|\mathcal{L}^0\| = \sup_{g\in\Phi_n} \frac{|\mathcal{L}^0([g])|}{\|[g]\|} = \sup_{g\in\Phi_n} \sup_N \frac{|\mathbb{E}g(X_1, \ldots, X_n)|}{\|U_n^N g\|} = \sup_N \|\mathcal{E}_n^N\|,$$

where the last equality is due to the definition of the primitive N-extending functional $\mathcal{E}_n^N$; see Definition 4. Since, by assumption, $(X_1, \ldots, X_n)$ is N-exchangeable for all $N \geq n$, we have (Theorem 1) that $\|\mathcal{E}_n^N\| = 1$ for all $N \geq n$ and so

$$\|\mathcal{L}^0\| = 1.$$

Using the Hahn-Banach theorem we extend $\mathcal{L}^0$ from $[\Phi_n]$ to a linear functional $\mathcal{L}^*$ on $[\Phi^*]$, that is,

$$\mathcal{L}^*\big|_{[\Phi_n]} = \mathcal{L}^0, \tag{20}$$

such that

$$\|\mathcal{L}^*\| = \|\mathcal{L}^0\| = 1.$$

We then define

$$\mathcal{L} : \Phi^* \to \mathbb{R}; \quad \mathcal{L}g := \mathcal{L}^*([g]). \tag{21}$$

Note that $\mathcal{L}$ is a linear functional which is moreover symmetric in the sense of Definition 2, that is, $\mathcal{L}g = \mathcal{L}g'$ is g' is obtained from g by permuting its arguments. Since $\|\mathcal{L}^*\| = 1$, we have, for all $g \in \Phi^*$,

$$|\mathcal{L}g| = |\mathcal{L}^*([g])| \leq \|[g]\| = \inf_N \|U_{i_g}^N g\| \leq \|g\|,$$

and so

$$\|\mathcal{L}\| = 1.$$

In particular, let

$$\mathcal{L}_{n,N} := \mathcal{L}\big|_{\Phi_N} \tag{22}$$

be the restriction of $\mathcal{L}$ onto Φ_N. Then

$$\|\mathcal{L}_{n,N}\| = 1. \tag{23}$$

Recalling the definition of the primitive N-extending functional $\mathcal{E}_n^N$ (Definition 4), we claim that

$$\mathcal{L}_{n,N} = \mathcal{E}_n^N \quad \text{on } U_n^N \Phi_n. \tag{24}$$

To see this, let $f \in U_n^N \Phi_n$. Then $f = U_n^N g$ for some $g \in \Phi_n$. By Definition 4, $\mathcal{E}_n^N f = \mathbb{E} g(X_1, \ldots, X_n)$. On the other hand,

$$\mathcal{L}_{n,N} f \overset{(22)}{=} \mathcal{L}(U_n^N g) \overset{(21)}{=} \mathcal{L}^*([U_n^N g]) = \mathcal{L}^*([g]) \overset{(20)}{=} \mathcal{L}^0([g]) \overset{(19)}{=} \mathbb{E} g(X_1, \ldots, X_n).$$

Note that the symmetry of $\mathcal{L}_{n,N}$ is inherited from $\mathcal{L}$. Therefore $\mathcal{L}_{n,N}$ is an N-extending functional of $(X_1, \cdots, X_n)$. As in Definition 5, the corresponding N-extending set function $\mu_{n,N}$ is given by

$$\mu_{n,N}(A) := \mathcal{L}_{n,N} \mathbf{1}_A, \quad A \in \mathscr{S}^N.$$

By Remark 5, we have that $\mu_{n,N}$ is countably additive on $\alpha(\mathscr{R}_N)$. Since $\mathcal{L}_{n,N}$ was constructed via the operator $\mathcal{L}$, we have the consistency property:

$$\mu_{n,N}(A) = \mu_{n,N'}(A \times S^{N'-N}), \quad A \in \alpha(\mathscr{R}_N), \quad n \le N \le N'. \tag{25}$$

By Proposition 8, for each $N \ge n$, there exists a unique N-exchangeable probability measure $\nu_{n,N}$ agreeing with the set function $\mu_{n,N}$ on $\alpha(\mathscr{R}_N)$ and such that $\mathbb{P}((X_1, \ldots, X_n) \in B) = \nu_{n,N}(B \times S^{N-n})$ for all $B \in \mathscr{S}^n$. From the consistency (25), we have consistency of the family $\{\nu_{n,N},\, N \ge n\}$. By Kolmogorov's extension theorem [22, p. 82], there exists a probability measure ν on $(S^{\mathbb{N}}, \mathscr{S}^{\mathbb{N}})$ such that $\nu(A \times S^\infty) = \nu_{n,N}(A)$ if $A \in \mathscr{S}^N$, for all $N \ge n$. By the N-exchangeability of $\nu_{n,N}$ for all N, we have that ν is an exchangeable probability measure on $(S^{\mathbb{N}}, \mathscr{S}^{\mathbb{N}})$ (see Definition 1). Let $Y = (Y_1, Y_2, \ldots)$ be a random element of $S^{\mathbb{N}}$ with law ν. Then $(X_1, \ldots, X_n) \overset{(d)}{=} (Y_1, \ldots, Y_n)$. This completes the proof. □

7 True mixing in finite exchangeability representation results

In this section we study the problem of when an exchangeable $X = (X_1, \ldots, X_n)$ is a true mixture of i.i.d. random variables. In other words, we ask when (1) holds with ν being a probability measure.

For any $g \in \Phi_n$ define the function

$$I_n g : \mathscr{P}(S) \to \mathbb{R}$$

as the expectation of $g(Y_1, \ldots, Y_n)$ when the Y_i are i.i.d. with common law p:

$$I_n g(p) := \int_{S^n} g(y_1, \ldots, y_n)\, p(dy_1) \cdots p(dy_n). \tag{26}$$

Notice that the map $g \mapsto I_n g$ is linear and so

$$\Psi_n := \{I_n g : g \in \Phi_n\}$$

is a linear space which is equipped with the natural norm

$$\|I_n g\| := \sup_{p \in \mathscr{P}(S)} |I_n g(p)|.$$

Define next the linear functional

$$\mathcal{T}_n : \Psi_n \to \mathbb{R}$$

by the formula

$$\mathcal{T}_n(I_n g) := \mathbb{E} g(X_1, \ldots, X_n),$$

where $(X_1, \ldots, X_n)$ is the given exchangeable random element of S^n. By (1),

$$|\mathcal{T}_n(I_n g)| = |\mathbb{E} g(X)| \leq \|I_n g\| \, \|\nu\|,$$

where $\|\nu\|$ is the total variation of the signed measure ν. Hence $\mathcal{T}_n$ is a bounded linear functional and so

$$\|\mathcal{T}_n\| = \sup_{g \in \Phi_n} \frac{|\mathbb{E} g(X_1, \ldots, X_n)|}{\|I_n g\|} \leq \|\nu\| < \infty, \tag{27}$$

the inequality being true for any signed measure ν satisfying (1).

Theorem 3. *Let $X = (X_1, \ldots, X_n)$ be an exchangeable random element of S^n. Suppose that the hypotheses of Theorem 1 hold. In addition, assume that $\mathscr{S}$ is generated by $C_c(S)$ (i.e., $\mathscr{S}$ is the Baire σ-algebra of S). Then the following three assertions are equivalent:*

1. *X is N-extendible for all $N \geq n$ (or, equivalently, by Theorem 2, X is infinitely extendible)*
2. *$\|\mathcal{T}_n\| = 1$;*
3. *there exists a probability measure ν on $\mathscr{P}(S)$ satisfying (1).*

Proof. $1 \Rightarrow 3$: If X is N-extendible for all $N \geq n$, Theorem 2 tells that X can be embedded into an infinite exchangeable random sequence $Y = (Y_1, Y_2, \ldots)$. Since $\mathscr{S}$ is the Baire σ-algebra, Theorem 7.4 of Hewitt and Savage [15] applies: there exists a probability measure ν on $\mathscr{P}(S)$ such that:

$$\mathbb{P}(Y \in A) = \int_{\mathscr{P}(S)} p^\infty(A) \, \nu(dp), \quad A \in \mathscr{S}^\infty,$$

and hence

$$\mathbb{P}(X \in B) = \int_{\mathscr{P}(S)} p^n(B) \, \nu(dp), \quad B \in \mathscr{S}^n.$$

$3 \Rightarrow 2$: The last equality implies

$$|\mathbb{E} g(X)| \leq \sup_{p \in \mathscr{P}(S)} |p^n(g)| = \|I_n g\|, \quad g \in \Phi_n.$$

But $I_n \mathbf{1}_{S^n}(p) = 1$ for all $p \in \mathscr{P}(S)$. Hence $\|I_n \mathbf{1}_{S^n}\| = 1$. Also, $|\mathbb{E}\mathbf{1}_{S^n}(X)| = 1$. Thus $\|\mathcal{T}_n\| = 1$.
$2 \Rightarrow 1$: Let $N \geq n$. By symmetry, we can write (26) as

$$I_n g(p) = \int_{S^N} (U_n^N g)(x)\, p^N(dx)$$

Hence

$$\|I_n g\| = \sup_{p \in \mathscr{P}(S)} |I_n g(p)| \leq \sup_{x \in S^N} |(U_n^N g)(x)| = \|U_n^N g\|.$$

Therefore, the norm of $\mathcal{T}_n$, given by (27), satisfies

$$\|\mathcal{T}_n\| \geq \sup_{g \in \Phi_n} \frac{|\mathbb{E}g(X)|}{\|U_n^N g\|} = \|\mathcal{E}_n^N\|,$$

the latter being equal to the norm of the primitive N-extending functional (see Definition 4). By assumption $\|\mathcal{T}_n\| = 1$. Hence $\|\mathcal{E}_n^N\| \leq 1$ and, *a fortiori*, $\|\mathcal{E}_n^N\| = 1$. By Theorem 1 we conclude that $(X_1, \ldots, X_n)$ is N-extendible. □

8 Preservation of exchangeability under limits; applications

Throughout this section, we assume that the hypothesis of Theorem 2 are satisfied. We establish results concerning the preservation of extendibility property via limits. They can serve as practical criteria for extendibility, as illustrated by means of an example.

Theorem 4. *For each $i \in \mathbb{N}$, let $X^i := (X_1^i, \ldots, X_n^i)$ be an n-exchangeable random element of S^n which is N-extendible. Assume that one of the following two conditions hold:*

1. *X^i converges to X in total variation: $\lim_{i\to\infty} \sup_{B \in \mathscr{S}^n} |\mathbb{P}(X^i \in B) - \mathbb{P}(X \in B)| = 0$.*
2. *X^i converges to X vaguely, that is, $\lim_{i\to\infty} \mathbb{E}f(X^i) = \mathbb{E}f(X)$, for all $f \in C_c(S^n)$, and the Borel σ-algebra $\mathscr{S}$ is also the Baire σ-algebra of S.*

In either case, X is also N-extendible.
Moreover, the theorem remains true if "N-extendible" is replaced by "infinitely extendible".

Proof. In either case, it suffices to prove the case of N-extendibility, since N-extendibility for all N implies infinite extendibility. In either case, since X^i is N-extendible, by Proposition 5,

$$\frac{|\mathbb{E}g(X^i)|}{\|U_n^N g\|} \leq 1, \quad g \in \Phi_n. \tag{28}$$

Assume the first condition holds. By the total variation convergence assumption we obtain

$$\frac{|\mathbb{E}g(X)|}{\|U_n^N g\|} \leq 1, \tag{29}$$

for all $g \in \Phi_n$. By Theorem 1 this implies that X is N-extendible.
Assume next that the second condition holds. Then (29) holds for any $g \in \operatorname{span}\{C_c(S^n), \mathbf{1}_{S^n}\}$, Given $g \in C_c(S^n)$ such that $0 < g \leq 1$, the set $A_{g,c} := \{g \geq c\}$ with $0 < c \leq 1$ is a

compact G_δ (i.e., intersection of countably many open sets) subset of S^n. Let $\mathscr{A}$ be the algebra generated by the collection $\{A_{g,c} : g \in \text{span}\{C_c(S^n), \mathbf{1}_{S^n}\}, 0 < c \leq 1\}$. Then $\mathscr{A}$ generates $\mathscr{S}^n$. Let $\mathfrak{F}, \widetilde{\mathfrak{F}}$ be the same as that in Proposition 9. We prove that every function in $\mathfrak{F}$ can be approached pointwise by a uniformly bounded sequence of functions in $\text{span}\{C_c(S^n), \mathbf{1}_{S^n}\}$. To this end, we only need to verify the case of $\mathbf{1}_{g\geq c}$. For any $0 < c \leq 1$, let $f := (g/c) \wedge 1 \in C_c(S^n)$. Then the m-th power f^m of f converges decreasingly and pointwise to $\mathbf{1}_{g\geq c}$. In view of (29), we deduce that the primitive N-extending functional $\mathcal{E}_n^N$ of X, restricted to $\widetilde{\mathfrak{F}}$, has norm less than or equal to 1. On the other hand, $\mathbf{1}_{S^n} \in \mathfrak{F}$ and $\frac{\mathbb{E}[\mathbf{1}_{S^n}(X)]}{\|U_n^N \mathbf{1}_{S^n}\|} = 1$. So the norm on $\widetilde{\mathfrak{F}}$ is exactly equal 1. We then apply Proposition 9 to finish the proof. □

Remark 7. This theorem is not trivial, since we are dealing with locally compact Hausdorff spaces. One may think of a sequence of probability measures $\{Y^i\}_{i\in\mathbb{N}}$ such that each Y^i is an N-extension of X^i and then by Prohorov's theorem [12], there exists a limit probability measure Y of vague convergence of a subsequence of $\{Y^i\}_{i\in\mathbb{N}}$. A natural idea is to prove that Y is an N-extension of X for both cases in the theorem. However it is difficult to proceed especially in case 1. We refer to the discussion above Theorem 2 for more details on the difficulty.

Theorem 5. *Let $\{\mathscr{S}_j\}_{j\geq 1}$ be a filtration (increasing sequence of σ-algebras) on S. Consider the algebra $\mathscr{A} := \bigcup_{j\geq 1} \mathscr{S}_j^n$ of subsets of S^n. Assume that $\sigma(\mathscr{A}) = \mathscr{S}^n$. For each $j \geq 1$, let P_j be an n-exchangeable probability measure on $(S^n, \mathscr{S}_j^n)$. Let P be the law of $X = (X_1, \ldots, X_n)$ and assume that*

$$\lim_{j\to\infty} \sup_{A\in\mathscr{S}_i^n} |P_j(A) - P(A)| = 0, \qquad \textit{for all } i \geq 1. \tag{30}$$

If every P_j is N-extendible (resp., infinitely extendible) then P is N-extendible (resp., infinitely extendible).

Proof. As in Theorem 4, one just needs to deal with the case of N-extendibility. Let $\mathfrak{F}, \widetilde{\mathfrak{F}}$ be the same as that in Proposition 9 with the algebra $\mathscr{A}$ defined as in the statement of the theorem. Using Proposition 9, it suffices to prove that

$$\sup_{g\in\mathfrak{F}} \frac{|\mathbb{E}[g(X)]|}{\|U_n^N g\|} \leq 1. \tag{31}$$

Since $g \in \mathfrak{F}$, where $\mathfrak{F}$ is the collection of linear combinations of indicator functions of sets from $\mathscr{A}$, it follows that there is an $i \geq 1$ such that g is $\mathscr{S}_i^n$-measurable. Hence g is also $\mathscr{S}_j^n$-measurable for any $j \geq i$. By (30) it follows that

$$\int_{S^n} g\, dP_j \to \int_{S^n} g\, dP, \qquad \text{as } j \to \infty.$$

Hence

$$\frac{\left|\int_{S^n} g\, dP_j\right|}{\|U_n^N g\|} \to \frac{\left|\int_{S^n} g\, dP\right|}{\|U_n^N g\|}, \qquad \text{as } j \to \infty.$$

Since g is $\mathscr{S}_j^n$-measurable for all $j \geq i$, by Proposition 5, the left-hand side of the last display is ≤ 1". Therefore, so is the right-hand side. By Proposition (9), it follows that (31) holds and so P is N-extendible. □

***Remark* 8.** Extendibility questions for particular exchangeable measures have been studied in the literature. In case S is a finite set with a small number of elements then the problem is relatively easy (see [13]). Otherwise, to the best of our knowledge, extendibility problems are resolved by constructing explicit extensions; see, for example, Gnedin [14], and Liggett *et al.* [21]. Explicit constructions can be difficult. In some cases, it might be still interesting to know that extensions exist without explicitly constructing them. It is with this in mind that the results of this section have been proved. In particular, Theorem 5 allows us to use coarse, discrete, approximations. For instance, we can let Π_j be a finite partition of S (finite collection of pairwise disjoint sets whose union is S) and $\mathscr{S}_j$ the algebra (and hence σ-algebra) generated by Π_j. Moreover, we let Π_j become finer as j increases ($\mathscr{S}_j \subset \mathscr{S}_{j+1}$ for all j) and assume that they generate the natural σ-algebra $\mathscr{S}$ of S, that is, $\mathscr{S} = \sigma(\bigcup_j \Pi_j)$. If Π_j^n is the collection of all sets of the form $A_1 \times \cdots \times A_n$ with $A_1, \ldots, A_n \in \Pi_j$, then Π_j^n, $j = 1, 2, \ldots$, is a sequence of nested partitions of S^n. Suppose that we wish to prove the N-extendibility of an exchangeable random element $X = (X_1, \ldots, X_n)$ of $(S^n, \mathscr{S}^n)$ whose law is denoted by P. Then it suffices to construct probability measures P_j on $(S^n, \sigma(\Pi_j^n))$, that approximate P, as $j \to \infty$, in the sense that $\max_{B \in \Pi_i^n} |P_j(B) - P(B)| \to 0$, as $j \to \infty$, for all i, and prove that each P_j is N-extendible. Since $(S^n, \sigma(\Pi_j^n), P_j)$ is a probability space with finite $\sigma(\Pi_j^n)$, the problem reduces to an extendibility problem on a finite space. This reduction may be easier to handle in some situations as the following example shows.

Example. This is borrowed from Theorem 2 of Gnedin [14]. Let $X = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}_+^n$ with nonnegative entries with density f_n of the form

$$f_n(x_1, \ldots, x_n) = g_n(x_1 \vee \cdots \vee x_n)$$

for some measurable function $g_n : \mathbb{R}_+ \to \mathbb{R}_+$. Clearly, X is n-exchangeable. Theorem 2 of [14] says that if g_n is nonincreasing, then X is representable as a true mixture of i.i.d. random variables (and this is equivalent to infinite extendibility). Although the representation (or infinite extension) can be explicitly given, we still use this simple example to show how Theorem 5 works.

Without loss of generality, let $n = 2$. Our assumption is that the law P of $X = (X_1, X_2)$ has density $f_2(x_1, x_2) = g(x_1 \vee x_2)$, $x_1, x_2 \geq 0$, for some nonincreasing function g. With $S = \mathbb{R}_+ = [0, \infty)$, we attempt to define (S^n, Π_j^n, P_j), for $j = 1, 2, \ldots$, "by discretization". Let $\mathbb{D}_j$ be the set of positive binary rationals $k/2^j$ in the interval $[0, j]$:

$$\mathbb{D}_j := \Big\{ \frac{k}{2^j} : 1 \leq k \leq j2^j \Big\},$$

and let Π_j be the partition of $\mathbb{R}_+$ corresponding to $\mathbb{D}_j$:

$$\Pi_j := \left\{ I_k(j) := \left[\frac{k-1}{2^j}, \frac{k}{2^j} \right) : 1 \leq k \leq j2^j \right\} \cup \big\{ [j, \infty) \big\}.$$

We define P_j on $\sigma(\Pi_j^2)$ by letting, for each $B \in \Pi_j^2$,

$$P_j(B) := \begin{cases} c_j \, g(\frac{k \vee \ell}{2^j}), & \text{if } B = I_k(j) \times I_\ell(j) \\ 0, & \text{otherwise} \end{cases}$$

and extending to each element of $\sigma(\Pi_j^2)$ by additivity. The positive constant c_j is chosen so that $P_j(\mathbb{R}_+^2) = 1$. We can easily see that

$$\max_{1 \leq k, \ell \leq i2^i} \left| P_j(I_k(i) \times I_\ell(i)) - \int_{I_k(i) \times I_\ell(i)} g(x \vee y) \, dxdy \right| \to 0, \quad \text{as } j \to \infty, \text{ for all } i.$$

Fix j. Clearly, P_j is 2-exchangeable on $(\mathbb{R}_+^2, \sigma(\Pi_j^2))$. Let $N \geq 2$. We now prove that P_j is infinitely extendible, or, equivalently (by Theorem 3), that it is a true mixture of product measures. Let Q_r, $1 \leq r \leq j2^j$, be probability measures on $(\mathbb{R}_+^2, \sigma(\Pi_j^2))$ defined by defining them on the each set $B \in \Pi_j^2$:

$$Q_r(B) := \begin{cases} 1/r^2, & \text{if } B = I_k(j) \times I_\ell(j),\ 1 \leq k, \ell \leq r \\ 0, & \text{otherwise .} \end{cases}$$

Clearly each Q_r is a product measure. Let $a_1 < a_2 < \cdots < a_m$ be the listing of the elements of $\mathbb{D}_j$ in increasing order. Thus, $m = j2^j$ and $a_k = k/2^j$. We claim that

$$P_j = \sum_{r=1}^{m-1} [c_j g(a_r) - c_j g(a_{r+1})] r^2 Q_r + c_j g(a_m) m^2 Q_m.$$

If the claim is true, then P_j is a true mixture of product measures with positive coefficients. To prove the claim, let $B = I_k(j) \times I_\ell(j)$, assuming $\ell \leq k \leq mj2^j$. But $Q_r(I_k(j) \times I_\ell(j)) = 0$ unless $r \geq k$. For $k \leq r \leq m$, we have $Q_r(I_k(j) \times I_\ell(j)) = 1/r^2$. Evaluating the right-hand side of the last display at $B = I_k(j) \times I_\ell(j)$ we obtain

$$\sum_{r=k}^{m-1} [c_j g(a_r) - c_j g(a_{r+1})] r^2 \frac{1}{r^2} + c_j g(a_m) m^2 \frac{1}{m^2} = c_j g(a_k) = P_j(I_k(j) \times I_\ell(j)).$$

□

The case studied in Theorem 1 of [14] can also be treated in this way. Here, the explicit infinite extension is not trivial.

9 A closer look at the finite S case

Theorem 3 shows that if $\|\mathcal{T}_n\| = 1$, then X is representable as a true mixture of i.i.d. random variables, under some good topological conditions. If $\|\mathcal{T}_n\| > 1$ then (1) holds for a signed measure ν with $\nu^- \neq 0$. The question we address in this section is what is the "least" signed measure ν which can be used in (1). The adjective "least" is with respect to the total variation $\|\nu\| = \nu^+(S) + \nu^-(S)$. If $\|\mathcal{T}_n\| = 1$ then the problem is solved: ν is a probability measure and so $\|\nu\| = \|\mathcal{T}_n\| = 1$. In general, $\|\nu\| \geq \|\mathcal{T}_n\|$; see (27). When S is a finite set then we can always find a signed measure ν with $\|\nu\|$ equal to $\|\mathcal{T}_n\|$:

Theorem 6. *If S is a finite set then there exists a signed measure ν satisfying (1) with $\|\nu\| = \|\mathcal{T}_n\|$.*

Proof. Suppose S has cardinality $c < \infty$, say, $S = \{1, 2, \ldots, c\}$. Then the space $\mathscr{P}(S)$ of probability measures on S is the unit simplex in $\mathbb{R}^c$:

$$\mathscr{P}(S) = \{(p_1, \ldots, p_c) \in \mathbb{R}^c : p_1, \ldots, p_c \geq 0; \sum_{i=1}^{c} p_i = 1\}.$$

Thus, $\mathscr{P}(S)$ is a compact subset of $\mathbb{R}^c$, whereas $\mathscr{P}(S^n)$ is a compact subset of $\mathbb{R}^{c^n}$. The set Φ_n (of bounded measurable functions from S^n into $\mathbb{R}$) is naturally identified with $\mathbb{R}^{c^n}$.

Recall the definition of $I_n g$ from (26), when $g \in \Phi_n$: it is the function $I_n g : \mathscr{P}(S) \to \mathbb{R}$ given by

$$I_n g(p) = \sum_{y_1 \in S} \cdots \sum_{y_n \in S} g(y_1, \ldots, y_n) p_{y_1} \cdots p_{y_n}.$$

Hence, for each $g \in \Phi_n$, $p \mapsto I_n(p)$ is a continuous function on the compact set $\mathscr{P}(S)$. Hence $\Psi_n = \{I_n g : g \in \Phi_n\}$ is a linear subspace of $C(\mathscr{P}(S), \mathbb{R})$, the set of continuous real-valued functions on $\mathscr{P}(S)$ (considered as a Banach space under the sup norm). Finally, $\mathcal{T}_n : \Psi_n \to \mathbb{R}$, defined by $\mathcal{T}_n(I_n g) = \mathbb{E} g(X_1, \ldots, X_n)$ is a bounded linear functional on Ψ_n. Using the Hahn-Banach theorem, we extend $\mathcal{T}_n$ to a bounded linear functional $\mathcal{T} : C(\mathscr{P}(S), \mathbb{R}) \to \mathbb{R}$ without increasing the norm: $\|\mathcal{T}\| = \|\mathcal{T}_n\|$. Using the Riesz-Markov representation theorem, we can find a unique regular signed measure ν on $\mathscr{P}(S)$ such that

$$\mathcal{T} f = \int_{\mathscr{P}(S)} f(\pi)\, \nu(d\pi), \quad f \in C(\mathscr{P}(S), \mathbb{R}), \tag{32}$$

and $\|\nu\| = \|\mathcal{T}\| = \|\mathcal{T}_n\|$. But then, for any $g \in \Phi_n$,

$$E g(X_1, \ldots, X_n) = \mathcal{T}_n(I_n g) = \mathcal{T}(I_n g) = \int_{\mathscr{P}(S)} (I_n g)(p) \nu(dp) = \int_{\mathscr{P}(S)} p^n(g) \nu(dp).$$

The first equality is due to the definition of $\mathcal{T}_n$ and the third equality follows from (32). So equality (1) holds with the signed measure ν of least total variation, i.e., $\|\nu\| = \|\mathcal{T}_n\|$. □

***Remark* 9.** If $\|\mathcal{T}_n\| > 1$, then $\nu^- \neq 0$. In this case, there is a largest finite $N \geq n$ such that $(X_1, \ldots, X_n)$ is N-extendible but not $(N+1)$-extendible.

A Extendibility and covariance

If $(X_1, \ldots, X_n)$ is an n-exchangeable sequence of real random variables with $\mathbb{E} X_1^2 < \infty$ then $\operatorname{cov}(X_1, X_2) \geq -\operatorname{var} X_1/(n-1)$. This follows from the expansion of $\mathbb{E}(X_1 + \cdots + X_n)^2$. If $Y = (Y_1, Y_2, \ldots)$ is an infinite exchangeable sequence of real random variables with $\mathbb{E} Y_1^2 < \infty$ then $\operatorname{cov}(Y_1, Y_2) \geq 0$ and, by de Finetti's theorem, the law of Y is a true mixture of product measures.

It might, at first sight, appear that nonnegativity of the covariance for the n-exchangeable sequence $X = (X_1, \ldots, X_n)$ might imply that the law of X is a true mixture of product measures or that it is N-exchangeable for some $N > n$ (or both). To dispel this, we give a simple example.

Take $n = 2$ and $S = \{1, 3/2, 2, 5/2\}$. Let $X = (X_1, X_2)$ be such that

$$\begin{aligned} \mathbb{P}\big(X_1 = 1, X_2 = \frac{3}{2}\big) &= \mathbb{P}\big(X_1 = \frac{3}{2}, X_2 = 1\big) = \mathbb{P}\big(X_1 = 2, X_2 = \frac{5}{2}\big) \\ &= \mathbb{P}\big(X_1 = \frac{5}{2}, X_2 = 2\big) = \frac{1}{4}. \end{aligned}$$

Then $\operatorname{cov}(X_1, X_2) = \frac{3}{16} > 0$, but it is easy to see that the law of X is not a true mixture. Indeed, if it were, we would have

$$\mathbb{P}(X_1 = X_2) = \int_{\mathscr{P}(S)} p^2\{(x_1, x_2) : x_1 = x_2\}\, \nu(dp),$$

for some probability measure ν on $\mathscr{P}(S)$. Since $p^2(\{(x_1,x_2) : x_1 = x_2\}) > 0$, under any probability measure p on S, we would have $\mathbb{P}(X_1 = X_2) > 0$. But, by definition, $\mathbb{P}(X_1 = X_2) = 0$.

B Extending set functions need not be probability measures

This is a continuation of Remark 5. We give an example of a space S and an extending functional $\mathcal{L}_{n,N}$ such that $\mu_{n,N}(A) = \mathcal{L}_{n,N}(1_A)$, $A \in \mathscr{S}^N$, is not a probability measure. Take S to be the unit interval $[0,1]$, and let $n = 1$, $N = 2$. Let X be uniformly distributed in S. We have $(U_1^2 g)(x_1,x_2) = \frac{1}{2}(g(x_1) + g(x_2))$. For $g \in \Phi_1$, the functional $\mathcal{E}_1^2$ maps $U_1^2 g$ to $\mathbb{E}g(X) = \int_0^1 g(t)dt$.

We now construct a particular 2-extending functional $\mathcal{L}_{1,2}$. We let $\mathfrak{F}$ consist of functions of the form

$$F = \sum_{i=0}^{m} c_i \mathbf{1}_{R_i} \tag{33}$$

where $c_i \in \mathbb{R}$, $R_i = A_i \times B_i$, with $A_i, B_i \in \mathscr{B}$ (the Borel σ-field of $[0,1]$). Let $D := \{(t,t) : 0 \le t \le 1\}$ and let $\mathfrak{D}$ consist of functions of the form

$$G = F + c1_D, \quad F \in \mathfrak{F}, \quad c \in \mathbb{R}.$$

Define the symmetric functional $\mathcal{L} : \mathfrak{D} \to \mathbb{R}$ by

$$\mathcal{L}(F + c1_D) := \mathbb{E}F(X,X) = \int_0^1 F(t,t)\,dt. \tag{34}$$

We claim that $\|\mathcal{L}\| = 1$. See below for the proof of this claim. Since $\mathfrak{D}$ is a normed linear subspace of Φ_2, there exists (by the Hahn-Banach theorem) a symmetric extension $\mathcal{L}_{1,2} : \Phi_2 \to \mathbb{R}$ of $\mathcal{L}$ with $\|\mathcal{L}_{1,2}\| = \|\mathcal{L}\| = 1$. We show that $\mathcal{L}_{1,2}$ is a 2-extending functional, i.e., that $\mathcal{L}_{1,2}(U_1^2 g) = \int_0^1 g(t)\,dt$, for all $g \in \Phi_1$. Let f_n (respectively, h_n) be an increasing (respectively, decreasing) sequence of simple functions on $[0,1]$ (i.e., linear combinations of finitely many indicator functions of Borel subsets of $[0,1]$) such that $f_n \uparrow g$ (respectively, $h_n \downarrow g$). We have $f_n \le g \le h_n$ for all n, and so $U_1^2 f_n \le U_1^2 g \le U_1^2 h_n$. Since $\|\mathcal{L}_{1,2}\| = 1$, by Lemma 6(i), $\mathcal{L}_{1,2}$ is a monotone operator. Hence $\mathcal{L}_{1,2}(U_1^2 f_n) \le \mathcal{L}_{1,2}(U_1^2 g) \le \mathcal{L}_{1,2}(U_1^2 h_n)$ for all n. Since $U_1^2 f_n \in \mathfrak{F}$, we have $\mathcal{L}_{1,2}(U_1^2 f_n) = \mathcal{L}(U_1^2 f_n) = \int_0^1 f_n(t)\,dt$. Similarly, $\mathcal{L}_{1,2}(U_1^2 h_n) = \int_0^1 h_n(t)\,dt$. By monotone convergence, $\lim_{n\to\infty} \int_0^1 f_n(t)\,dt = \lim_{n\to\infty} \int_0^1 h_n(t)\,dt = \int_0^1 g(t)\,dt$. Therefore, $\mathcal{L}_{1,2}(U_1^2 g) = \int_0^1 g(t)\,dt = \mathcal{E}_1^2(U_1^2 g)$, showing that $\mathcal{L}_{1,2}$ agrees with $\mathcal{E}_1^2$ on $U_1^2\Phi_1$.

As in the proof of Theorem 1, the functional $\mathcal{L}_{1,2}$ restricted on the space $C([0,1]\times[0,1])$ of continuous functions on $[0,1]\times[0,1]$ admits the Riesz representation

$$\mathcal{L}_{1,2}F = \int_{[0,1]\times[0,1]} F(x,y)\,\nu_{1,2}(dx,dy), \quad F \in C([0,1]\times[0,1]),$$

for some probability measure $\nu_{1,2}$ on $[0,1]\times[0,1]$ and this $\nu_{1,2}$ is a 2-extension of the law of X. It is easy to see that $\nu_{1,2}$ is the law of (X,X). Recall that $\mu_{1,2}$ is defined by $\mu_{1,2}(A) = \mathcal{L}_{1,2}(\mathbf{1}_A)$ for all Borel $A \subset [0,1]\times[0,1]$. We have (as in the proof of Theorem 1) $\mu_{1,2}(R) = \nu_{1,2}(R)$ for all rectangles R. Assume that $\mu_{1,2}$ is a probability measure on the Borel sets of $[0,1]\times[0,1]$. Then, necessarily, $\mu_{1,2} = \nu_{1,2}$. But $\mu_{1,2}(D) = \mathcal{L}_{1,2}(\mathbf{1}_D) = \mathcal{L}(\mathbf{1}_D) = 0$, and this contradicts $\nu_{1,2}(D) = \mathbb{P}((X,X) \in D) = 1$.

Proof of the claim that $\mathcal{L}$ has norm 1. We need to show that $|\mathcal{L}(F + c\mathbf{1}_D)| \le \|F + c\mathbf{1}_D\|$, for all $F \in \mathfrak{F}$ and all $c \in \mathbb{R}$. If $c = 0$, the inequality holds. If $c \neq 0$, divide by c and use (34) to reduce the claim to the proof of the inequality

$$\Big| \int_0^1 F(t,t)\,dt \Big| \le \max_{x\neq y} |F(x,y)| \vee \max_t |F(t,t)+1|, \quad F \in \mathfrak{F}.$$

We consider two cases. If $\max_{x\neq y} |F(x,y)| = \max_{x,y} |F(x,y)|$ then

$$\big| \int_0^1 F(t,t)\,dt \big| \le \max_{x,y} |F(x,y)| = \max_{x\neq y} |F(x,y)| \le \max_{x\neq y} |F(x,y)| \vee \max_t |F(t,t)+1|.$$

If not, there is $t_0 \in [0,1]$ such that $|F(t_0,t_0)| > \max_{x\neq y} |F(x,y)|$. Since F can be written as in (33) with pairwise disjoint R_i, it follows that one of the R_i must be a singleton, say, $R_0 = \{t_0\} \times \{t_0\}$. Without loss of generality, assume that this is the only singleton among the R_i's. Then $F = c_0 \mathbf{1}_{R_0} + H$, where $H = \sum_{i=1}^m c_i \mathbf{1}_{R_i}$ has the property of case 1, i.e., $\max_{x\neq y} |H(x,y)| = \max_{x,y} |H(x,y)|$. Then

$$\begin{aligned} \big| \int_0^1 F(t,t)\,dt \big| = \big| \int_0^1 H(t,t)\,dt \big| &\le \max_{x,y} |H(x,y)| = \max_{x\neq y} |H(x,y)| = \max_{x\neq y} |F(x,y)| \\ &\le \max_{x\neq y} |F(x,y)| \vee \max_t |F(t,t)+1|. \end{aligned}$$

□

C On regularity

Proposition 10. *Let S, T be two topological spaces with Borel σ-algebras $\mathscr{S}, \mathscr{T}$, respectively. Let μ and ν be probability measures on $(S, \mathscr{S})$, $(T, \mathscr{T})$, respectively. If μ and ν are regular (meaning: inner and outer regular) then any probability measure ρ on $(S \times T, \mathscr{S} \otimes \mathscr{T})$ with marginals μ, ν is regular.*

Proof. Say that $M \in \mathscr{S} \otimes \mathscr{T}$ is ρ-regular if $\rho(M)$ is simultaneously the supremum of compact subsets of M and the infimum of open supersets of M. Let $\mathscr{M}$ be the collection of all ρ-regular sets. We will show that $\mathscr{M} = \mathscr{S} \otimes \mathscr{T}$. It is easy to see that every rectangle, i.e., set of the form $A \times B$ with $A \in \mathscr{S}$ and $B \in \mathscr{T}$, is ρ-regular. Suppose that M_i, $i \in \mathbb{N}$, is an increasing sequence of elements of $\mathscr{M}$ with $\bigcup_i M_i = M$. Fix $\varepsilon > 0$. For each i, pick compact $K \subset M_i$ with $\rho(K_i) \ge \rho(M_i) - \varepsilon$. Also, pick j such that $\rho(M - M_j) \le \varepsilon$. Then $\rho(M) \le \rho(K_j) + 2\varepsilon$. Hence M is inner regular. Next, pick, for each i, open $O_i \supset M_i$ with $\rho(O_i) \le \rho(M_i) + \varepsilon 2^{-i}$ and let $O^j = \bigcup_{i\le j} O_i$. Then $\rho(O^j) - \rho(M_j) = \rho(O^j - M_j) \le \sum_{i\le j} \rho(O_i - M_i) \le \varepsilon$. Letting $j \to \infty$, and $O := \bigcup_i O_i$ (an open set containing M) we obtain $\rho(O) - \rho(M) \le \varepsilon$. Hence M is outer regular. For similar reasons, the intersection of a decreasing sequence of ρ-regular sets is ρ-regular. This shows that $\mathscr{M}$ is a monotone class of sets containing the algebra of rectangles. By the monotone class theorem $\mathscr{M}$ contains the σ-algebra $\mathscr{S} \otimes \mathscr{T}$. □

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