Clausen's problem and triangular lattices

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Abstract

Given an arbitrary triangle ABC, draw a line AA' meeting side BC at a point A' so that $BA' : BC = \alpha : (1-\alpha)$, for some $0 < \alpha < 1$. Repeat, symmetrically, with the other vertices and consider the triangle formed by the three lines AA', BB', CC'. Let ρ be the ratio of its area and the area of ABC. Then ρ depends on α only and not the shape of ABC. In particular, $\rho(1/3) = 1/7$. We explain this in 4 different ways, two of which are based on counting triangles or hexagons in lattices. We ask the question how we can relate the computation of $\rho(m/n)$ for integers m and n to lattices in the plane and find that, for each pair (m, n) of coprime integers, $m \le n$, the formula can be explained by constructing a sublattice of the standard triangular lattice.

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AMS 2000 *subject classifications.* Primary 51-01,05B45; secondary 52-XX,52C20

1 Introduction

In a wonderful little book by the famous probabilist Kai Lai Chung [2][endnote (20), page 91], the following apparently simple geometrical problem is mentioned.

Statement of the problem. *Consider an arbitrary triangle ABC in the plane. Let A', B', C' be points on the sides BC, CA, AB, respectively so that the distance of A' from B is one $1/3$ the distance of A' from C, etc. (Figure 1). Then the triangle $A''B''C''$ formed by the lines AA', BB', CC' has area $1/7$ the area of ABC.*

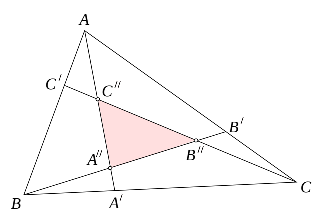

Figure 1: $BA' = \frac{1}{3}BC$, $CB' = \frac{1}{3}CA$, $AC' = \frac{1}{3}AB$

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Kai Lai Chung mentioned this to the famous physicist Richard Feynman. Feynman "found this unbelievable", and quickly worked out an approximation, concluding that "it must be wrong because his approximation did not support it". Chung also tells us that a friend of his, Carlo Riparbelli, an aeronautical engineer, later came up with two proofs of the statement. The earliest appearance of the problem is, to the best of our knowledge, in a note by Clausen [3]. This is why we refer to it as "Clausen's problem".

Elementary geometrical problems like the one above may have more to tell us than initially meets the eye. In fact, if not seen from the right point of view they may appear unbelievable. Chung does not tell us whether it was the 1/7 that Feynman found unbelievable or the fact that the ratio is a constant number regardless of the triangle. Regarding the last point, it is not hard at all to see that the ratio must be constant. Indeed, the fundamental theorem of Affine Geometry tells us that there is a unique affine transformation that maps the triangle ABC to any other triangle. Since affine transformations have the property that ratios of lengths of parallel segments are preserved and ratios of areas of any two sets are also preserved (see, e.g., Coxeter [4][13.32, Chapter 13]), it follows that the ratio of the area of $A''B''C''$ to ABC is constant. It so happens that the ratio is 1/7, that is, 1 over the number of non-overlapping triangles that the three lines AA', BB', CC' split the triangle ABC. Is that a coincidence?

We will show that it is not. We will give four methods for solving the problem. Each method tells its own story. As usual in mathematics, different methods shed different light to the problem. We shall then consider variants of the problem and ask what kind of numbers, instead of 1/7, we can obtain, that is, what kind of values can $\rho(m/n)$ obtain when m and n are coprime integers. We show that, for each such pair (m, n) there is a sublattice of the standard planar triangular lattice on which the computation becomes "visually obvious".

2 Four methods

2.1 Method 1: by similarity and trigonometry

As explained above, by the fundamental theorem of affine geometry, the ratio of the areas is constant. So we may consider any triangle. We choose ABC to be an equilateral triangle. Form the triangle $A''B''C''$ as described above. By symmetry (invariance with respect to a rotation by 120^o about the centroid of ABC) the triangle $A''B''C''$ is also equilateral. Since

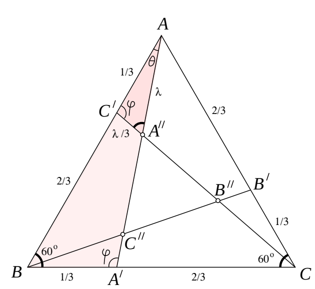

Figure 2: Triangles $AA''C'$ and ABA' are similar

$\theta := \angle A'AB = \angle B'BC$, and $\angle C'A''A = \angle A'B''B = 60^o$, the triangles $AA''C'$ and ABA'

are similar. Hence

$$\frac{AA''}{AB} = \frac{A''C'}{BA'} = \frac{C'A}{A'A} =: \lambda.$$

Taking, without loss of generality, $AB = 1$, we have

$$AA'' = \lambda, \quad A''C' = \lambda/3, \quad C'A = 1/3.$$

The law of sines for the triangle $AA''C$ gives

$$\frac{\lambda}{\sin\varphi} = \frac{\lambda/3}{\sin\theta} = \frac{1/3}{\sqrt{3}/2},$$

and so

$$\sin\varphi = \frac{3\sqrt{3}}{2}\lambda, \quad \sin\theta = \frac{\sqrt{3}}{2}\lambda.$$

Since $\varphi + \theta = 120^o$, we have

$$\sin\varphi\,\cos\theta + \sin\theta\,\cos\varphi = \sin(120^o) = \sqrt{3}/2.$$

We have $\cos\theta = \sqrt{1-\sin^2\theta}$ because θ is acute, but $\cos\varphi = -\sqrt{1-\sin^2\varphi}$ because φ is obtuse. So

$$3\lambda\sqrt{1-3\lambda^2/4} - \lambda\sqrt{1-27\lambda^2/4} = 1. \qquad (1)$$

(The minus sign is due to the fact that $\varphi > 90^o$.) It is easy to see that $\lambda = 1/\sqrt{7}$ satisfies the above. To show that this is the only solution, we observe see that the left-hand side is a strictly increasing function of λ on the interval $0 < \lambda < \lambda^*$, where λ^* being the largest value of λ for which both the terms inside the radicals are positive. [1] Thus, the ratio of the area of $AA''C''$ and that of ABA' is $\lambda^2 = 1/7$ and this implies that the ration of the areas of $A''B''C''$ and ABC is also $1/7$. [2]

2.2 Method 2: using coordinates

Suppose that the triangle ABC is isosceles with a right angle at A. Introduce Cartesian coordinates, placing A at the point $(0,0)$, B at $(1,0)$ and C at $(0,1)$. See Figure 3. The line ℓ_0 through A and A' has equation $y = x/2$. The line ℓ_1 through B and B' has equation $y = 1-3x$. The line ℓ_2 through C and C' has equation $y = 2(1-x)/3$. The point A'' is the intersection of ℓ_0 and ℓ_2. Solving $y = x/2 = 1-3x$ gives $x = 2/7$ and $y = 1/7$. The point B'' is the intersection of ℓ_0 and ℓ_1 and C'' is the intersection of ℓ_2 and ℓ_1. We find:

$$A'' = (2/7, 1/7), \quad B'' = (4/7, 2/7), \quad C'' = (1/7, 4/7).$$

The parallelogram with sides the vectors $\overrightarrow{A''B''} = B'' - A'' = (2/7, 1/7)$ and $\overrightarrow{A''C''} = C'' - A'' = (-1/7, 3/7)$ has area twice the area of the triangle $A''B''C''$. Hence

$$2\,\text{area}(A''B''C'') = \det\begin{pmatrix} 2/7, & 1/7 \\ -1/7, & 3/7 \end{pmatrix} = \frac{1}{7}.$$

Since the area of ABC is $1/2$, the result follows.

[1] Another way to solve equation (1) is by taking squares twice and, by algebra, show that it yields

$$91\lambda^4 - 20\lambda^2 + 1 = 0.$$

This equation has four solutions $\pm 1/\sqrt{7}, \pm 1/\sqrt{13}$. Only the ones with positive sign should be kept. Of those, $1/\sqrt{13}$ does not satisfy (1). Hence $1/\sqrt{7}$ is the only solution.

[2] Alternatively, we have that $A''C'' = AA' - \lambda - C''A' = \frac{1}{3\lambda} - \lambda - \frac{\lambda}{3} = \frac{1}{\sqrt{7}}$.

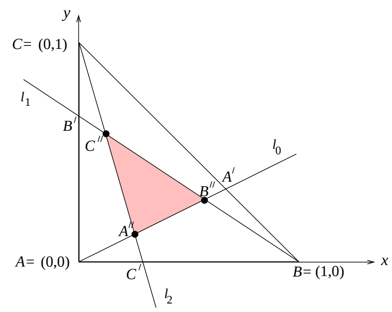

Figure 3: Assume one angle is right

2.3 Method 3: counting areas

Consider again an equilateral triangle ABC. Mark the 1/3, 2/3 points of side BC by A', A_1', and do the same for the other sides. Let O be the centroid (that is, the intersection of the medians) of the triangle ABC. Clearly, O is also the centroid of $A''B''C''$. Draw a line through O parallel to BB'. This line cuts $A''C''$ at a point P and AC at a point Q; see Figure 4.

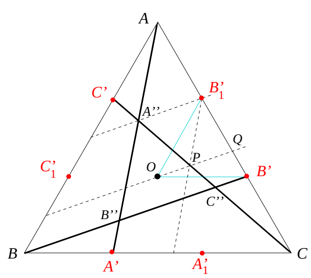

Figure 4

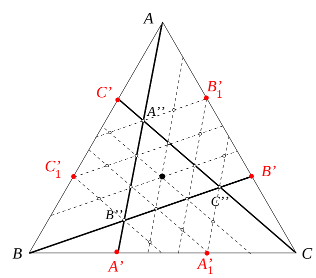

Figure 5

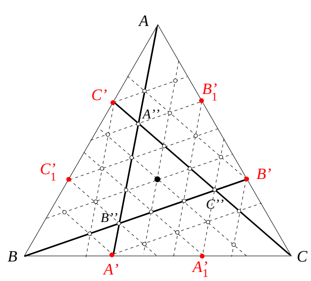

Figure 6

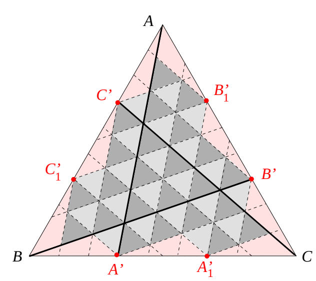

Figure 7

Since O is the centroid of $A''B''C''$, we have $PC'' = \frac{1}{3}A''C''$. Since the triangle $B_1'OB'$ is similar to ABC, we have that $QB' = \frac{1}{3}B'C = \frac{1}{3}B_1'B'$. Therefore, $QB'/B_1'B' = PC''/A''C''$ and so, by Thales' theorem, the line $B_1'A''$ is parallel to OQ. Similarly, $B_1'P$ is parallel to $A''B''$.

Repeat the same process two more times. That is, draw lines through O parallel to the sides of $A''B''$ and $A''C''$, then lines through A_1' and lines through C_1'. We arrive at Figure

5. In other words, we rotated Figure 4 by 60^o and 120^0 and superimposed the lines.

We next draw lines through A' parallel to $B''C''$ and to $A''C''$ and, cyclically, through B' and C'. We obtain Figure 6. We can easily see that several triplets of lines pass through common points.

Finally, Figure 7 shows that we have thus managed to fit several congruent equilateral triangles (indicated by gray color) inside ABC. The remaining portion of ABC, not covered by the gray triangles, consists of 9 congruent triangles (indicated by pink color). If we let m be the area of each of the equilateral triangles we can see (see Figure 8) that each of the pink triangles has area $2m$ because the parallelogram formed by one of the pink triangles and the union by its image obtained by a half-turn through the mid-point of its longest side consists of 4 equilateral triangles. There are $9 + 36 = 45$ gray equilateral triangles inside

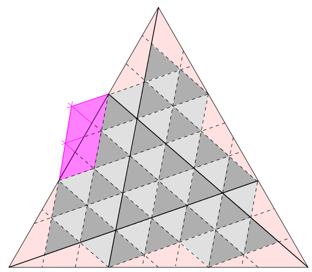

Figure 8: Each pink triangle has area equal to the area of two gray triangles

ABC and 9 pink triangles. Hence

$$\text{area}(ABC) = 45m + 9 \times 2m = 63m.$$

On the other hand,

$$\text{area}(A''B''C'') = 9m.$$

Therefore the ratio of the areas of $A''B''C''$ and ABC is $1/7$.

2.4 Interlude: a visual proof

Since affine mappings do preserve collinearity and ratios of lengths of parallel segments it follows that the splitting of the triangle into smaller congruent triangles works for an arbitrary triangle. Figure 9 can thus be classified as a visual proof of the problem statement.

2.5 Method 4: by tiling

Consider a regular hexagon centered at the point O. It is the red-shaded hexagon in Figure 10 below. Let R be its area. Draw 6 identical regular hexagons around it to obtain the red hexagonal flower indicated in the figure below.

The hexagonal flower is a non-convex polygon with 18 vertices. Label one of them A and then, moving clockwise, consider every third vertex, marking them by the letters B through F. Consider now the convex hexagon $ABCDEF$, indicated as green in the figure.

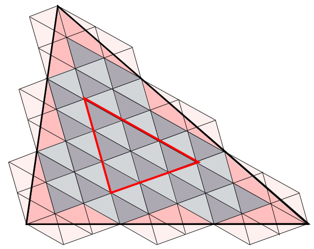

Figure 9: A visual proof of the problem statement for an arbitrary triangle

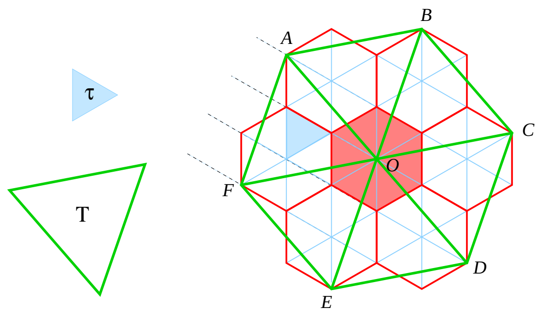

Figure 10: Hexagonal flower; area of red hexagon =: R; areas of $ABCDEF$ =: G

By symmetry, it is a regular hexagon. Let G be its area. By inspection, we immediately get that

$$G = 7R.$$

Draw the segments $OA, OB, \ldots, OF$, thus splitting the green hexagon into 6 equilateral (green) triangles, each of area, say, T:

$$G = 6T.$$

Draw the (blue) diameters of each small red hexagon, splitting it into 6 (blue) equilateral triangles of area, say, τ:

$$R = 6\tau. \tag{2}$$

Combining the above displays we obtain

$$T = 7\tau. \tag{3}$$

It remains to show that the small blue triangle is situated inside a green triangle as the triangle $A''B''C''$ is situated inside ABC in Method 1. But this is clear from the fact that the three doted lines split the edge AF into thirds.

2.6 Another visual proof

Finally, it is worthwhile observing that the 1/7 factor can be explained visually by the following figure showing two ways to tile the plane by regular hexagons.

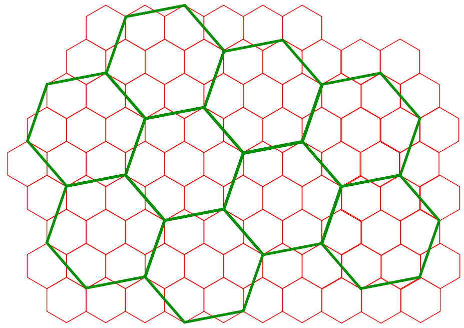

Figure 11: Visual explanation of the factor 1/7

3 A generalization

Consider a triangle ABC, just as in Figure 1, but now replace 1/3 by a factor α, that is, suppose that $AA'/BC = BB'/CA = CC'/AB = \alpha$, where $0 \le \alpha \le 1$.

$$\rho(\alpha) := \frac{\text{area}(A''B''C'')}{\text{area}(ABC)}.$$

So far, we have seen that $\rho(1/3) = 1/7$. We want to compute $\rho(\alpha)$ for all $0 \le \alpha \le 1$. The formula for $\rho(\alpha)$ is given by (8) in Section 4 below. But we want to explore the possibility of making a construction as that of Method 3 or Method 4, for which the value of $\rho(\alpha)$ is "obvious".

Let us then see if we can find some the values of $\rho(\alpha)$ by the method of tiling.

3.1 $\alpha = 1/M$ for positive integer M

Consider the hexagonal flower of Figure 10 and to it 12 additional hexagons to create the bigger flower depicted in the Figure 12. This consists of $7 + 12 = 19$ red hexagons. The

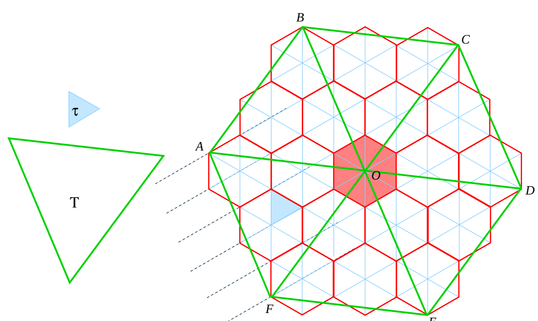

Figure 12: A bigger hexagonal flower; area of red hexagon =: R; area$(ABCDEF)$ =: G

bigger flower is a non-convex polygon with $6 \times 5 = 30$ vertices. We are going to pick every fifth vertex in order to create a bigger regular hexagon. However, unlike the previous case (Figure 10), the big hexagonal flower the starting vertex is important because there are fewer symmetries than before. Pick the starting vertex one which is farthest away from O, and call it A. Now, moving clockwise, pick every fifth vertex labeling them $B, \ldots, F$ to construct a large green regular hexagon of area, say, G. Observe that

$$G = 19R.$$

Observe also that (2) and (3) still hold (with obvious meaning for the symbols). Finally notice that (i) the dotted lines split side AF into 5 equal pieces; (ii) there is a side of the blue-shaded triangle which lies on a straight line through O; (iii) this line cuts AF at a point whose distance from F is 2/5 the length of AF. Combining these observations and the definition of ρ, we find

$$\rho(2/5) = 1/19.$$

It is now easy to generalize this observation. Recursively construct a sequence of "hexagonal flowers" in an obvious manner: having constructed the $(n-1)$-th flower (consisting, say, of κ_{n-1} red hexagons), we obtain a larger flower by surrounding the $(n-1)$-th one by $6n$ additional hexagons. So $\kappa_n = \kappa_{n-1} + 6n$. With $\kappa_0 = 1$, we find $\kappa_1 = 7$, $\kappa_2 = 19$, and, in general, $\kappa_n = 1 + \sum_{k=1}^{n} 6k = 1 + 3n(n+1)$. Forming again a big (green) regular hexagon with area, say, G, we have

$$G = (1 + 3n + 3n^2)R.$$

Upon inspecting the position of the blue triangle in relation to the green one, we find

$$\rho\left(\frac{n}{2n+1}\right) = \frac{1}{1+3n+3n^2}.$$

We summarize some of the values in the table below.

	n	1	2	3	4
number of red hexagons in a flower	$\kappa_n = G/R = 3n^2 + 3n + 1$	7	19	37	91
number of vertices of a flower	$v_n = 6(2n-1)$	6	18	30	42
	$\alpha_n = n/(2n+1)$	1/3	2/5	3/7	4/9

Since $\rho(1-\alpha) = \rho(\alpha)$ (this is obvious geometrically) we have computed $\rho(\alpha_n)$ for $\alpha = n/(2n+1)$ and $\rho(1-\alpha_n)$. In fact, we have computed $\rho(\alpha)$ for those α for which $\rho(\alpha) = 1/M$, where M is a positive integer:

Lemma 1. *If $\rho(\alpha) = 1/M$, for some positive integer M, then there is a positive integer n such that $M = 3n^2 + 3n + 1$ and $\alpha = n/(2n+1)$ or $\alpha = (n+1)/(2n+1)$.*

The proof of this lemma is in Section 4.

3.2 Another example

We can use our "hexagonal calculator" to compute $\rho(\alpha)$ for other values of α. For example, in the hexagonal flower of Figure 12 pick vertex A differently, as in Figure 13 below. The regular hexagon $ABCDEF$ has now moved to a new position and has smaller area: $G = 13R$. Consider the blue shaded triangle and let τ_0 be its area. We see that it consists of 4 smaller triangles: $\tau_0 = 4\tau$. As before, $G = 6T$ and $R = 6\tau$. Hence $\tau_0/T = 4/13$. Now the line containing O and one side of the blue triangle meets AF at a point whose distance from F is 1/4 the length of AF. We have thus shown that

$$\rho(1/4) = 4/13.$$

Can we do something similar for any rational α? The answer is in the next section.

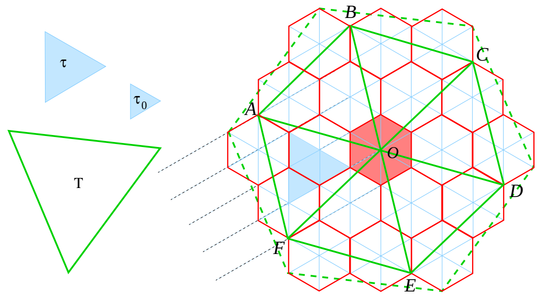

Figure 13: The hexagonal flower of Figure 12 with point A placed differently

3.3 Rational α by the tiling method

The question that naturally arises is whether we can compute $\rho(\alpha)$, for any rational α, by a tiling method, We will see that the answer is yes, as long as we are allowed to pick the vertices of $ABCDEF$ also at the centers of small hexagons.

Consider the standard regular triangular tiling $\mathbb{T}$ of the plane. That is, $\mathbb{T}$ is obtained by applying reflections to the sides of a unit-length equilateral triangle. Then $\mathbb{T}$ tessellates the plane into congruent equilateral triangles that are called faces of $\mathbb{T}$. The lattice $L(\mathbb{T})$ of $\mathbb{T}$ is the collection of the vertices of all of its faces. We say that another regular triangular tiling $\mathbb{T}'$ is embedded in $\mathbb{T}$ if $L(\mathbb{T}') \subset L(\mathbb{T})$.

Definition 1. Let $\Delta = ABC$ be a face of $\mathbb{T}'$ and $0 \leq \alpha \leq 1$ a rational number. An equilateral triangle δ contained in Δ is called *α-proper* for Δ if
(i) it is the union of faces of $\mathbb{T}$,
(ii) the lines ℓ_1, ℓ_2, ℓ_3 containing the sides of δ pass through the vertices A, B, C of Δ, and
(iii) line ℓ_1 meets AB at a point A' such that $A'B : A'C = \alpha : (1-\alpha)$, and, similarly, for the other sides.

Theorem 1. *Let $0 \leq \alpha \leq 1$ be a rational number. Then there is a triangular tiling $\mathbb{T}'$, embedded into $\mathbb{T}$, such that each face Δ of $\mathbb{T}'$ contains an α-proper triangle δ.*

We will prove this theorem by using complex numbers. Recall that the straight line $\ell[z_0, u]$ passing through $z_0 \in \mathbb{C}$ and having direction $u \in \mathbb{C}, u \neq 0$ is the set

$$\ell[z_0, u] = \{z \in \mathbb{C} : \overline{u}z + u\overline{z} = \overline{u}z_0 + u\overline{z}_0\}.$$

For $z_1, z_2, z_3 \in \mathbb{C}$, let $\mathsf{triangle}(z_1, z_2, z_3)$ be the triangle with vertices z_1, z_2, z_3. Recall that if $u \in C$, $|u| = 1$ then $z \mapsto uz$ is a rotation around the origin, so if

$$\sigma = e^{i\pi/3}, \quad \varphi = \sigma^2,$$

then $\mathsf{triangle}(z_0, (1+\sigma)z_0, (1+\varphi)z_0))$ is equilateral.

Proof of Theorem 1. Let $\mathbb{T}$ be the triangular tiling obtained by reflections on the sides of $\mathsf{triangle}(0, 1, \sigma))$. Its lattice is

$$L(\mathbb{T}) = \{m + n\varphi : m, n \in \mathbb{Z}\}. \tag{4}$$

Let $\sigma := 1 + \varphi = e^{i\pi/3}$. The triangle with vertices $0, 1, \sigma$ is a face of $\mathbb{T}$. If v is a vertex of $\mathbb{T}$, define $\Delta := \mathsf{triangle}(0, v, \sigma v))$. This is an equilateral triangle that generates, by reflections on its sides, a triangular tiling $\mathbb{T}'$ embedded in $\mathbb{T}$ with lattice

$$L(\mathbb{T}') = \{mv + n\varphi v : \, m, n \in \mathbb{Z}\}.$$

Suppose

$$v = a - b\varphi, \quad a, b \in \mathbb{Z}.$$

Then, since

$$1 + \varphi + \varphi^2 = 1,$$

we have

$$mv + nv\varphi = (ma + nb) + (na + nb - mb)\varphi,$$

so $L(\mathbb{T}') \subset L(\mathbb{T})$. We now define a triangle δ by letting its sides be segments contained in the lines

$$\begin{aligned} \ell[0,1] &= \{z \in \mathbb{C} : \, z = \overline{z}\} \\ \ell[v, \varphi] &= \{z \in \mathbb{C} : \, \sigma z - \overline{\sigma z} = \sigma v - \overline{\sigma v}\} \\ \ell[\sigma v, \varphi^2] &= \{z \in \mathbb{C} : \, \overline{\sigma} z - \sigma \overline{z} = v - \overline{v}\}. \end{aligned}$$

Since the second and third lines are obtained from the first by multiplication by σ and $\overline{\sigma}$, respectively, it follows that δ is equilateral. Let $P = \ell[0,1] \cap \ell[v, \varphi]$, $Q = \ell[0,1] \cap \ell[\sigma v, \varphi^2]$, $R = \ell[v, \varphi] \cap \ell[\sigma v, \varphi^2]$ be its vertices. By algebra, we find

$$P = \frac{\sigma v - \overline{\sigma v}}{\sigma - \overline{\sigma}}, \quad Q = \frac{v - \overline{v}}{\sigma - \overline{\sigma}}, \quad R = \frac{\sigma \overline{v} + 2\overline{\sigma} v}{\overline{\sigma} - \sigma}.$$

Since $v = a - b\varphi$, we further have

$$P = a, \quad Q = b, \quad R = a\sigma + b\overline{\sigma}.$$

Assuming that $0 < a < b$, we have that δ is nontrivial and that $\delta \subset \Delta$. We show that it is α-proper for some α. (i) The sides of δ have length $b - a$, that is $b - a$ times the length of the side of $\mathsf{triangle}(0, 1, \sigma)$. Hence δ is the union of $(b - a)^2$ congruent copies[3] of $\mathsf{triangle}(0, 1, \sigma)$. (ii) The lines containing the sides of δ pass through the vertices of Δ by the definition of δ. We next show that (iii) holds for some rational number α. Let w be the point of intersection of $\ell[0, 1]$ with the line $\ell[v, \sigma v - v]$ containing the side of Δ with endpoints v and σv. Since $\ell[v, \sigma v - v] = \ell[v, \varphi v] = \{z \in \mathbb{C} : \, \overline{\varphi v} z - \varphi v \overline{z} = \overline{\varphi v} v - \varphi v \overline{v}\}$ we have that

$$w = \frac{\overline{\varphi} - \varphi}{\overline{\varphi v} - \varphi v} = \frac{a^2 + ab + b^2}{a + b}.$$

By some further algebra, we have

$$\alpha = \frac{|w - v|}{\sigma v - v} = \frac{b}{a + b},$$

a rational number. Hence δ is $b/(a + b)$-proper for Δ. To finish the proof of the theorem, if $\alpha = m/n$ is a rational number with $m < n$, let $a = n - m$, $b = m$ and the above procedure constructs a m/n-proper triangle δ for Δ. □

[3] If the side of an equilateral triangle equals an integer N then the triangle is partitioned in $(2N - 1) + (2N - 3) + \cdots + 3 + 1 = N^2$ unit-side equilateral triangles.

The theorem hence gives us an algorithm for placing an $\alpha = m/n$–proper triangle δ for Δ. We start with a unit-side triangular lattice $\mathbb{T}$. We let

$$
\begin{aligned}
v &= (n-m) + m\varphi \\
\Delta &= \mathsf{triangle}(0, v, \sigma v) = \mathsf{triangle}(0, n - m\sigma, m + (n-m)\sigma) \qquad (5)\\
\delta &= \mathsf{triangle}(n-m, m, m + (n-2m)\sigma). \qquad (6)
\end{aligned}
$$

We can now compute $\rho(m/n)$ as the ratio of two areas:

$$
\begin{aligned}
\text{area}(\Delta) &= \frac{\sqrt{3}}{4}|n - m\sigma|^2 = \frac{\sqrt{3}}{4}(n^2 - nm + m^2),\\
\text{area}(\delta) &= \frac{\sqrt{3}}{4}|m - (n-m)|^2 = \frac{\sqrt{3}}{4}(n-2m)^2.
\end{aligned}
$$

Thus

$$
\rho(m/n) = \frac{(n-2m)^2}{n^2 - nm + m^2}. \qquad (7)
$$

So, for any rational $0 \le m/n \le 1$, we can construct a figure as in Figures 10, 12 and 13 and have a "visual" proof of the formula for $\rho(m/n)$. We mentioned earlier that we may have to pick the vertices not at the vertices of the hexagonal lattice but at their centers. To see this, note that $v = a - b\varphi$ is at the center of the hexagonal lattice which is the dual of $\mathbb{T}$ if and only if $a - b \equiv 0 \mod 3$ which translates into $m + n \equiv 0 \mod 3$. In the cases studied in Figures 10, 12 and 13 we had $m/n = 1/3, 2/5, 1/4$, respectively, and we could place the vertices at vertices of hexagons because $1+3, 2+5, 1+4 \not\equiv 0 \mod 3$. But, for example, if $m/n = 2/7$ then we need to place v at the center of a hexagon.

Rewriting (7) as

$$
\rho(m/n) = \frac{(1 - 2(m/n))^2}{1 - (m/n) + (m/n)^2},
$$

we have that

$$
\rho(\alpha) = \frac{(1-2\alpha)^2}{1 - \alpha + \alpha^2}, \quad \alpha \in \mathbb{R},\ 0 \le \alpha \le 1.
$$

This is by continuity.

4 Additional remarks

1. To find a general formula for $\rho(\alpha)$ we use Method 2. See Figure 3. The equations for the lines ℓ_0, ℓ_1, ℓ_2 become $y = \frac{\alpha}{1-\alpha}x$, $y = (1-\alpha)(1-x)$, $y = 1 - \frac{x}{\alpha}$, respectively. Then $A'' = \ell_0 \cap \ell_2$, $B'' = \ell_1 \cap \ell_0$, $C'' = \ell_2 \cap \ell_1$ have coordinates

$$
A'' = \frac{1}{N}\big(\alpha(1-\alpha),\ \alpha^2\big), \quad B'' = \frac{1}{N}\big((1-\alpha)^2,\ \alpha(1-\alpha)\big), \quad C'' = \frac{1}{N}\big(\alpha^2,\ (1-\alpha)^2\big),
$$

where

$$
N := \alpha^2 - \alpha + 1.
$$

We then have

$$
\rho(\alpha) = 2\text{area}(A''B''C'') = \frac{1}{N^2}\ \det\begin{pmatrix}(1-\alpha)(1-2\alpha) & \alpha(1-2\alpha)\\ \alpha(1-2\alpha) & 1-2\alpha\end{pmatrix} = \frac{(1-2\alpha)^2}{\alpha^2 - \alpha + 1}. \qquad (8)
$$

(In computing the determinant, we noticed that the matrix above equals $1-2\alpha$ times a matrix which has determinant equal to N.)

2. *Proof of Lemma 1.* The equation $\rho(\alpha) = 1/M$, where $\rho(\alpha)$ is as in (8), reduces is a quadratic and so $\alpha = (1/2)(1 \pm 3/\sqrt{12M-3})$. Suppose $\alpha = (1/2)(1 - 3/\sqrt{12M-3})$. But α must be rational. Therefore $12M - 3 = Z^2$ for some integer Z. Hence 3 divides Z. Write $Z = 3W$ to get $4M - 1 = 3W^2$, for some integer W. Since $4M - 1$ is odd, W must be an odd integer. So $W = 2n + 1$ for some integer n. Hence $4M - 1 = 3(2n + 1)^2$, which gives that $M = 3n^2 + 3n + 1$. The expression inside the radical is $4M - 3 = 9(2n + 1)^2$. Hence $\alpha = (1/2)(1 - 1/(2n + 1)) = n/(2n + 1)$. The other case, gives $\alpha = 1 - n/(2n + 1) = (n + 1)/(2n + 1)$. □

3. A further proof that $\rho(1/3) = 1/7$ can be given by using Pick's theorem [6] (see also Coxeter [4, 13.51]) that states that in a 2-dimensional lattice, the area of a (not necessarily convex) polygon whose vertices are lattice points equals $\frac{1}{2}b + c - 1$ where b is the number of lattice points on the boundary of the polygon and c the number of lattice points in the interior. For the proof, see [4, p. 211, Figure 13.5c]. On the same page, Coxeter claims that the formula for $\rho(\alpha)$ is obtainable for any α by means of Pick's theorem, but there is no proof of this statement. In order to use Pick's theorem, for $\alpha = m/n$, we have to construct triangles δ and Δ, as we did in (5) and (6), and then Pick's theorem is another way of computing their areas.

4. The oldest reference to the problem of computing $\rho(\alpha)$ is in [3] by Clausen. A more general case occurs when the ratios $\alpha := BA'/BC$, $\beta := CB'/CA$. $\gamma := AC'/AB$ (see Figure 1) are not identical. This was formulated and solved by Steiner [8, pp. 163-168]. A formula for the ratio of areas is given in [4, 13.55] and is proved using barycentric coordinates. A special case gives Ceva's theorem stating that the lines AA', BB' and CC' are concurrent if and only if

$$\frac{\alpha}{1-\alpha}\frac{\beta}{1-\beta}\frac{\gamma}{1-\gamma} = 1 \qquad (9)$$

Giovanni Ceva discovered this in 1678, but its dual theorem was one discovered by Menelaus of Alexandria (70-140 CE). See [4, p. 220] for proofs using barycentric coordinates and [1, p. 30] for a more elementary high-school level approach.

5. If we denote by $S_\alpha(\Delta)$ the triangle $A''B''C''$ from an arbitrary triangle $\Delta = ABC$ by using the same ratio α then we can show that, for a sequence α_n, $n \in \mathbb{N}$, of numbers between 0 and 1, the sequence of triangles $T_n := S_{\alpha_n}(S_{\alpha_{n-1}}(\cdots S_{\alpha_1}(\Delta)\cdots))$ converges to a single point, that point being the median of Δ, if and only if $\sum_{n=1}^{\infty} \frac{\alpha_n(1-\alpha_n)}{1-2\alpha_n)^2} = \infty$. This follows from the fact that Δ and $S_\alpha(\Delta)$ have the same median and that the distance of the boundary of T_n from the median is a constant times $\rho(\alpha_1)\cdots\rho(\alpha_n)$ and so this distance converges to 0 if and only if the sum of the logarithms of $\rho(a_k)$ is a divergent series. Consequently, T_n does not converges to the median only if α_n converges to 0 very fast.

6. The previous paragraph gives a way to create an infinite sequence of triangles inside a fixed triangle Δ. Other was of creating such sequences have been considered, even within the real of probability. See, for example, Diaconis and Miclo [5] and Volkov [9] for recent papers on this subject. Based on the concepts and problems discussed in our paper, one can study a sequence random subdivisions or random triangles of a fixed triangle in several ways. For example, by letting the triple (α, β, γ) be a random sequence $(\alpha_n, \beta_n, \gamma_n)$ taking values in $[0,1]^3$. This generalizes the idea of Diaconis and Miclo who work under the condition that all lines pass through the same point, that is if (9) is satisfied for all n. When such kind of randomness is imposed, geometry becomes harder. Therefore, it is essential that the correct point of view be chosen and, as we saw, there are several geometric approaches. The point of view of affine geometry, in particular, should not be underestimated. We hope that our paper, albeit entirely elementary, can form the basis for these kinds of random geometric problems.

7. In the Introduction, we referred to the statement that a planar affine transformation is completely determined by its effect on a triangle as the "fundamental theorem of affine geometry" and cited Coxeter [4, 13.32]. Calling this statement fundamental relies on the fact that affine geometry can be axiomatically developed as an extension of ordered geometry with the addition of two axioms, the parallelism axiom and Desargues' statement. In such a mathematical system, an affine transformation is defined as a bijective transformation that preserves collinearity. The "fundamental theorem of affine geometry" is becomes a theorem that can be proved. In other approaches to affine geometry, e.g., one based on coordinates (see, e.g., V.V. Prasolov and V.M. Tikhomirov [7, Chapter 2]), things are the other way round, that is, the preservation of collinearity is a theorem rather than an axiom [7, p. 39].

8. Algebraically, the lattice (4) is the set of integers in the field $\mathbb{Q}[\varphi] := \{x+y\varphi : x, y \in \mathbb{Q}\}$. They and are known as Eisenstein integers and, just like the Gaussian integers $m + n\sqrt{-1}$, they form a Euclidean domain under the norm $m + n\varphi \mapsto n^2 - mn + m^2$. In the preceding computations, we have been working, essentially, with Eisenstein integers.

References

[1] Ilka Agricola and Thomas Friedrich (2008). *Elementary Geometry.* Translated from the 2005 German original. American Mathematical Society, Providence.

[2] Kai Lai Chung (1995). *Green, Brown, and Probability.* World Scientific, Singapore.

[3] Thomas Clausen (1828). Geometrische Sätze. *J. reine angew. Mathematik (Crelle's Journal)* **3**, 197-198.

[4] H.S.M. Coxeter (1989). *Introduction to Geometry*, 2nd ed. Reprint of the 1969 edition. Wiley Classics Library. John Wiley, New York.

[5] Persi Diaconis and Laurent Miclo (2011), On barycentric subdivision. *Combin. Probab. Comput.* **20** No. 2, 213237.

[6] Georg Pick (1899). Geometrisches zur Zahlentheorie. *Sitzungsberichte des Deutschen Naturwissenschaftlich-Medicinischen Vereines für Böhmen "Lotos" in Prag.* (Neue Folge) **19**, 311-319.

[7] V.V. Prasolov and V.M. Tikhomirov (2001). *Geometry.* Translated from the 1997 Russian original. American Mathematical Society, Providence.

[8] Jacob Steiner (1881). *Gesammelte Werke*, Vol. I. Collected by K. Weierstrass. Druck Und Verlag Von G. Reimer. pp. 163-168.

[9] Stanislav Volkov (2013). Random geometric subdivisions. *Random Structures and Algorithms* **43**, No. 1, 115130.

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